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A REMARK ON SMALL DIVISORS PROBLEMS

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1. Introduction. In a recent series of investigations [4]–[8], V. Pták has developed a new theory of iterative existence proofs, the so called method of nondiscrete mathematical induction. The method is based on a simple abstract theorem about complete metric spaces, the induction theorem, and consists in reducing the problem to a system of functional inequalities to be satisfied by a certain function, called the rate of convergence.

In the present remark we apply this method to small divisors problems obtaining thereby an improvement of conditions and a considerable simplification of proofs. Problems of this type have been investigated previously by V. I. ARNOLD [1], J. MOSER [3], I. N. BLINOV [2] and E. ZEHNDER [9], [10]. The authors owe a debt of gratitude to V. PTÁK and E. ZEHNDER for the permission to use unpublished manuscripts [7], [10].

Let f be a mapping defined on a subset D of a Banach space Y with values in a normed space Z. Suppose that $u \in D$ and that the Fréchet derivative $f'(u)$ exists. It is natural to approximate the solution of $f(x) = 0$ by the element $u - (f'(u))^{-1} f(u)$ provided $f'(u)$ has a bounded inverse. In applications, this is not always the case so that it is necessary to replace $(f'(u))^{-1}$ by an approximate right inverse which maps, in general, the space Z into a larger space $Y' \supset Y$.

2. Preliminaries. We repeat here, for the reader's convenience, the essential facts about the method of nondiscrete induction (see [7]).

Definitions. Let T be an interval of the form $T = \{t; 0 < t < t_0\}$ for a positive t_0. A rate of convergence on T is a function ω defined on T which maps T into itself and

$$\sigma(t) = \sum_{n=0}^{\infty} \omega^n(t) < \infty$$

(here $\omega^n = \omega \circ \omega^{n-1}$, ω^0 is the identity function). As usual, given a metric space (E, d), a subset M of E and a positive number r, we denote $\mathrm{U}(M, r) = \{x \in E; \mathrm{d}(x, M) < r\}$. If we are given, for small t, a set $A(t) \subset E$, we define the limit $A(0)$ of the family $A(\cdot)$ as

$$A(0) = \bigcap_{s>0} \Big(\bigcup_{t \leqq s} A(t) \Big)^- .$$

Now we may state the induction theorem.

2.1. Theorem. *Let (E, d) be a complete metric space, let ω be a rate of convergence on $T = (0, t_0)$. For each $t \in T$ let $Z(t)$ be a subset of E. Suppose that*

$$W(t) \subset \mathrm{U}(W(\omega(t)), t)$$

for each $t \in T$. Then

$$W(t) \subset \mathrm{U}(W(0), \sigma(t))$$

for each $t \in T$.

Sometimes, it is more convenient to use the induction theorem in the following equivalent form.

2.2. Theorem. *Let (E, d) be a complete metric space, let ω be a positive function which maps $T = (0, t_0)$ into itself and such that $\lim_{n \to \infty} \omega^n(t) = 0$ for each $t \in T$. Let τ be a positive function defined on T such that $\sigma_\tau(t) = \sum_{n=0}^{\infty} (\tau \circ \omega^n)(t) < \infty$ for each $t \in T$. For each $t \in T$ let $W(t)$ be a subset of E. Suppose that*

$$W(t) \subset \mathrm{U}(W(\omega(t)), \tau(t))$$

for each $t \in T$. Then

$$W(t) \subset \mathrm{U}(W(0), \sigma_\tau(t)) .$$

This modification is obtained by setting $Z(t) = W(\tau^{-1}(t))$ and applying the induction theorem to the family $Z(\cdot)$ and the rate of convergence $\tilde{\omega} = \tau \circ \omega \circ \tau^{-1}$ under the assumption that the inverse τ^{-1} exists and is defined on an interval $T_1 = (0, t_1)$ for a positive t_1.

3.1. The main theorem. *For each number σ, $0 \leqq \sigma \leqq 1$, we are given a Banach space $(Y_\sigma, |\ |_\sigma)$ and a normed space $(Z_\sigma, |\ |_\sigma)$ with the following properties:*

1° *$Y_{\sigma'} \supset Y_\sigma$, $Z_{\sigma'} \supset Z_\sigma$ and $|\ |_{\sigma'} \leqq |\ |_\sigma$ for $\sigma' \leqq \sigma$;*

2° *each Y_σ is equipped with another norm $\|\ \|_\sigma$ such that $\|\ \|_{\sigma'} \leqq \|\ \|_\sigma$ for $\sigma' \leqq \sigma$ and $|\ |_\sigma \leqq \|\ \|_\sigma$.*

Let R be a positive number and set

$$R_\sigma = \{u \in Y_\sigma, |u|_\sigma < R\} , \quad \hat{R}_\sigma = \{u \in Y_\sigma, \|u\|_\sigma < R\}$$

so that $\hat{R}_\sigma \subset R_\sigma$.

Let f be a mapping defined on R_0 with values in Z_0 such that f maps each R_σ into Z_σ. Suppose that the following conditions are satisfied:

3° *f is continuous as a mapping from $(R_\sigma, |\ |_\sigma)$ into $(Z_0, |\ |_0)$ for each $\sigma \in [0, 1]$;*

4° *for each $u \in \bigcup_{0<\sigma} \hat{R}_\sigma$ there exists a mapping $f'(u) : \bigcup_{\sigma>0} Y_\sigma \to \bigcup_{\sigma>0} Z_\sigma$ such that, for each $\sigma' < \sigma$, $u \in \hat{R}_\sigma$ implies $f'(u)\, Y_\sigma \subset Z_{\sigma'}$, and*

$$|f(u + v) - f(u) - f'(u)\, v|_{\sigma'} \leqq K_1(\sigma - \sigma') \,.\, |v|_\sigma^2$$

whenever u and $u + v$ belong to $\hat{R}_\sigma$;

5° *if $u \in \hat{R}_\sigma$, there exists $v \in \bigcap_{\sigma'<\sigma} Y_{\sigma'}$ such that, for each $\sigma' < \sigma$,*

$$(1) \qquad |f'(u)\, v - f(u)|_\sigma \leqq K_2(\sigma - \sigma')\, |f(u)|_\sigma^2\,, \quad |v|_{\sigma'} \leqq K_3(\sigma - \sigma')\, |f(u)|_\sigma\,,$$
$$\|v\|_{\sigma'} \leqq K_4(\sigma - \sigma')\, |f(u)|_\sigma$$

where K_i $(i = 1, 2, 3, 4)$ are positive nonincreasing functions defined on the interval $(0, 1]$, $\inf K_4 > 0$.

Let K be any function defined on $(0, 1]$ such that $K \geqq \max\,(K_1 K_3^2 + K_2, K_4)$. Suppose that there exist positive increasing functions ω, φ and g defined on $[0, 1]$ such that $\varphi \leqq 1$, $\omega, g < 1$ and

$$(2) \qquad (K \circ \alpha)\,(r)^{-1}\,(K \circ \alpha \circ \omega)\,(r) \leqq g(r)^{-2}\,(g \circ \omega)\,(r)$$

for each $r \in (0, t)$, $0 < t \leqq 1$ (here $\alpha = 2^{-1}(\varphi - \varphi \circ \omega)$). Then there exists $u \in \hat{R}_{\varphi(0)}$ such that $f(u) = 0$, whenever $0 \leqq r_0 < t$, $u_0 \in \hat{R}_{\varphi(r_0)}$,

$$(3) \qquad \sum_{n=0}^{\infty} (g \circ \omega^n)\,(r)\,(K_4 \circ \alpha \circ \omega^n)\,(r)\,(K \circ \alpha \circ \omega^n)\,(r)^{-1} < R - \|u_0\|_{\varphi(r_0)}$$

for $r \leqq r_0$

and

$$(4) \qquad |f(u_0)|_{\varphi(r_0)} \leqq g(r_0)\,(K \circ \alpha)\,(r_0)^{-1}\,.$$

Proof. We set, for a fixed $u_0 \in \hat{R}_{\varphi(r_0)}$, $W(r) = \{u \in \hat{R}_{\varphi(r)}, |f(u)|_{\varphi(r)} \leqq S(r), \|u - u_0\|_{\varphi(r)} < R - \|u_0\|_{\varphi(r_0)} - k(r)\}$ for $0 < r < t$ and suitable positive increasing functions S, k defined on $(0, t)$ and such that $\lim_{r\to 0+} S(r) = 0$.

Now let $u \in W(r)$. According to 5° there exists $v \in \bigcap_{\sigma<\varphi(r)} Y_\sigma$ such that, for each $\sigma < \varphi(r)$,

$$(5) \qquad |f'(u)\, v - f(u)|_\sigma \leqq K_2(\varphi(r) - \sigma)\, |f(u)|_{\varphi(r)}^2\,,$$
$$|v|_\sigma \leqq K_3(\varphi(r) - \sigma)\, |f(u)|_{\varphi(r)}\,, \quad \|v\|_\sigma \leqq K_4(\varphi(r) - \sigma)\, |f(u)|_{\varphi(r)}\,.$$

Set now $u' = u - v$. Given σ, τ such that $\sigma < \tau < \varphi(r)$, we have the following estimate

$$R - \|u_0\|_{\varphi(r_0)} - \|u' - u_0\|_\sigma \geqq R - \|u_0\|_{\varphi(r_0)} - \|u - u_0\|_\sigma - \|v\|_\sigma \geqq$$
$$\geqq R - \|u_0\|_{\varphi(r_0)} - \|u - u_0\|_{\varphi(r)} - \|v\|_\tau \geqq$$
$$\geqq k(r) - K_4(\varphi(r) - \tau) |f(u)|_{\varphi(r)} \geqq k(r) - K_4(\varphi(r) - \tau) S(r) .$$

Assume for a moment that $k(r) - K_4(\varphi(r) - \tau) S(r)$ is positive. Then $u' \in \hat{R}_\sigma$ and

$$|f(u')|_\sigma \leqq |f(u') - f(u) + f'(u) v|_\sigma + |f'(u) v - f(u)|_\sigma \leqq$$
$$\leqq K_1(\tau - \sigma) |v|_\tau^2 + |f'(u) v - f(u)|_\tau \leqq$$
$$\leqq K_1(\tau - \sigma) (K_3(\varphi(r) - \tau))^2 |f(u)|_{\varphi(r)}^2 + K_2(\varphi(r) - \tau) |f(u)|_{\varphi(r)}^2 \leqq$$
$$\leqq [K_1(\tau - \sigma) (K_3(\varphi(r) - \tau))^2 + K_2(\varphi(r) - \tau)] S(r)^2 .$$

It is natural to take τ so that $\tau - \sigma = \varphi(r) - \tau$. Then

$$|f(u')|_\sigma \leqq (K_1 K_3^2 + K_2) (2^{-1}(\varphi(r) - \sigma)) S(r)^2 .$$

Clearly, it is desirable to find functions ω, k and S so that, for $\sigma = (\varphi \circ \omega)(r)$ and $\alpha = 2^{-1}(\varphi(r) - (\varphi \circ \omega)(r))$,

$$k(r) - (K_4 \circ \alpha)(r) S(r) \geqq (k \circ \omega)(r) \tag{6}$$

and

$$((K_1 K_3^2 + K_2) \circ \alpha)(r) S(r)^2 \leqq (S \circ \omega)(r) . \tag{7}$$

The inequality (7) is equivalent to

$$((K_1 K_3^2 + K_2) \circ \alpha)(r) S(r)^2 (S \circ \omega)(r)^{-1} \leqq 1 . \tag{8}$$

Since $S(r)(S \circ \omega)(r)^{-1} > 1$ it follows that S should be majorized by $1/((K_1 K_3^2 + K_2) \circ \alpha)$. As the inequality (6) is obviously satisfied for $k(r) = \sum (K_4 \circ \alpha \circ \omega^n)(r) . (S \circ \omega^n)(r)$ if the series converges, it is convenient to set $S(r) = g(r) (K \circ \alpha)(r)^{-1}$ for a positive g, $g < 1$. If g satisfies (2) then (8) is fulfilled. Moreover, if g satisfies also (3) then $k(r) < R - \|u_0\|_{\varphi(r_0)}$.

It follows from (5) that

$$W(r) \subset \mathrm{U}(W(\omega(r)), S(r) (K_3 \circ \alpha)(r)) \tag{8,1}$$

in the space $(Y_{(\varphi\circ\omega)(r)}, | \, |_{(\varphi\circ\omega)(r)})$ and, obviously, in the space $(Y_{\varphi(0)}, | \, |_{\varphi(0)})$ as well.

If $|f(u_0)|_{\varphi(r_0)} \leqq g(r_0) (K \circ \alpha)(r_0)^{-1}$ then the set $W(r_0)$ as well as $W(0)$ is nonempty. Since $\lim_{n \to \infty} (S \circ \omega^n)(r) = 0$ for each $r \leqq r_0$ it follows from 3° that each $u \in W(0)$ satisfies $f(u) = 0$. The proof is complete.

3.2. Remark. We can also estimate the distance between the initial point $u_0 \in \hat{R}_{\varphi(r_0)}$ we are starting with and a solution. Assume that (3) and (4) are fulfilled. Then there exists a solution of $f(u) = 0$ in the space $Y_{\varphi(0)}$ satisfying

$$(9) \qquad |u - u_0|_{\varphi(0)} \leqq |f(u_0)|_{\varphi(r_0)}\, g(r_0)^{-1} (K \circ \alpha)(r_0) \sum_{n=0}^{\infty} (K_3 \circ \alpha \circ \omega^n)(r_0) \,. \\ . (g \circ \omega^n)(r_0) (K \circ \alpha \circ \omega^n)(r_0)^{-1}$$

and

$$(10) \qquad \|u - u_0\|_{\varphi(0)} \leqq |f(u_0)|_{\varphi(r_0)}\, g(r_0)^{-1} (K \circ \alpha)(r_0) \,. \\ . \sum_{n=0}^{\infty} (K_4 \circ \alpha \circ \omega^n)(r_0) (g \circ \omega^n)(r_0) (K \circ \alpha \circ \omega^n)(r_0)^{-1} \,.$$

Proof. The reasoning in the preceding proof remains valid if g is replaced by any function of the form $v \,.\, g$, $0 < v \leqq 1$. Denote $S' = vS$ and

$$W'(r) = \{u \in \hat{R}_{\varphi(r)},\ |f(u)|_{\varphi(r)} \leqq S'(r),\ \|u\|_{\varphi(r)} < R - \|u_0\|_{\varphi(r_0)} - k(r)\} \,.$$

Then the inclusion (8,1) has the form $W'(r) \subset \mathrm{U}(W'(\omega(r)), S'(r)(K_3 \circ \alpha)(r))$ in the space $(Y_{\varphi(0)}, |\ |_{\varphi(0)})$ and, in virtue of (5), $W'(r) \subset \mathrm{U}(W'(\omega(r)), S'(r)(K_4 \circ \alpha)(r))$ in the space $(Y_{\varphi(0)}, \|\ \|_{\varphi(0)})$.

Suppose that $u_0 \in W(r_0)$ and take v so that $|f(u_0)|_{\varphi(r_0)} = v\, S(r_0)$. Then $u_0 \in W'(r_0)$ as well. It follows from the induction theorem that there exists $u \in \hat{R}_{\varphi(0)}$ satisfying (9), (10) and $f(u) = 0$.

4. Remarks and applications. The above theorem generalizes the results of [2] and [9]. First, we shall show how to find functions ω, φ and g under certain growth conditions on the functions K_i.

4.1. Lemma. *Suppose that K from Theorem 2.1 is a decreasing continuous function defined on the interval $(0, 1)$ such that $\lim_{r \to 0^+} K(r) = \infty$. Suppose further that there exist numbers $1 < a \leqq 2$, $0 < d < 1$, $b, w > 0$ and a positive decreasing continuous function h defined on the interval $(0, 1)$ such that $\lim_{r \to 0^+} h(r) = \infty$,*

$$(11) \qquad h(r^a)\, h(r)^{-1} \leqq b\, r^{w(a-2)}$$

$$(12) \qquad (K^{-1} \circ h)(r^a) \leqq \mathrm{d}(K^{-1} \circ h)(r)$$

for each $r \in (0, 1)$.

Then the functions

$$\omega(r) = r^a \,,$$
$$\alpha(r) = K^{-1}(h(r)) \,,$$
$$\varphi(r) = \sigma_0 + 2 \sum_{n=0}^{\infty} (\alpha \circ \omega^n)(r) \quad \text{with a fixed} \quad \sigma_0 \in [0, 1) \,,$$
$$g(r) = b^{-1} r^w$$

satisfy (2) for small r.

Proof. If $\omega(r) = r^a$, we are to find $\varphi, \alpha = 2^{-1}(\varphi - \varphi \circ \omega)$ and g so that (2) be satisfied.

It is natural to take $\alpha = K^{-1} \circ h$ for a positive decreasing function h such that $\lim_{r \to 0^+} h(r) = \infty$. As h satisfies (11) we have

$$\alpha(r^a) = (K^{-1} \circ h)(r^a) \leqq d(K^{-1} \circ h)(r) = d \, . \, \alpha(r)$$

and

$$(13) \qquad \sum_{n=0}^{\infty} (\alpha \circ \omega^n)(r) = \sum_{n=0}^{\infty} (K^{-1} \circ h)(r^{a^n}) \leqq (1 - d)^{-1} (K^{-1} \circ h)(r) \, .$$

With respect to the equality $\alpha = 2^{-1}(\varphi - \varphi \circ \omega)$ one possible choice of φ is to set $\varphi(r) = \sigma_0 + 2 \sum_{n=0}^{\infty} (\alpha \circ \omega^n)(r)$ for some σ_0, $0 \leqq \sigma_0 < 1$. Because of continuity of α and with respect to $\lim_{r \to 0^+} h(r) = \infty$ there exists r_0 such that $\varphi \leqq 1$ for $r \leqq r_0$.

The condition (2) of Theorem 3.1 turns out to be

$$h(r^a) \, h(r)^{-1} \leqq g(r^a) \, g(r)^{-2}$$

for small r.

It is convenient to have g commuting with ω in the sense of superposition, so we set $g(r) = b^{-1} r^w$ for some positive b, w.

4.2. Lemma. *Suppose that the assumptions of* 4.1 *are satisfied and replace the inequality* (12) *by*

$$(12') \qquad (K^{-1} \circ h)(r^a) = d(K^{-1} \circ h)(r)$$

for each $r \in (0, 1)$.

Let $\sigma \in (0, 1]$ *and* $u_0 \in \hat{R}_\sigma$ *be given. Denote by* q_a *the solution of the equation* $(1 - q) \, b(R - \|u_0\|_\sigma) = q^{1/(a-1)}$.

If

$$(14) \qquad q^{1/(a-1)w} > (h^{-1} \circ K)(\sigma(1 - d)/4)$$

and

$$(15) \qquad |f(u_0)|_\sigma < \frac{(h^{-1} \circ K)(\sigma(1 - d)/4)^w}{bK(\sigma(1 - d)/4)}$$

then there exists $u \in \hat{R}_{\sigma/2}$ *such that* $f(u) = 0$.

Proof. Given $\sigma \in (0, 1]$, we set $\varphi(r) = \sigma/2 + 2 \sum_{n=0}^{\infty} (\alpha \circ \omega^n)(r) = \sigma/2 + 2 \sum_{n=0}^{\infty} (K^{-1} \circ h)(r^{a^n}) = \sigma/2 + 2(1 - d)^{-1} (K^{-1} \circ h)(r)$ according to (12'). For $r_\sigma = (h^{-1} \circ K)(\sigma(1 - d)/4)$ we have $\varphi(r_\sigma) = \sigma$.

As $K_4 \leqq K$, the inequality (3) will be satisfied if $\sum_{n=0}^{\infty} (g \circ \omega^n)(r_\sigma) = b^{-1} \sum_{n=0}^{\infty} r^{aw} <$ $< R - \|u_0\|_\sigma$. The last series is majorized by the geometric series $r_\sigma^w \sum_{n=0}^{\infty} q^n <$ $< b(R - \|u_0\|_\sigma)$ for $r_\sigma < \min((1-q)\, b(R - \|u_0\|_\sigma), q^{1/(a-1)})^{1/w}$. In order to ensure the best estimate for r_σ we shall suppose that q_a is taken so that $(1 - q_a) \,.$ $. \, b(R - \|u_0\|_\sigma) = q_a^{1/(a-1)}$.

Finally, the initial condition (4) has the form

$$g(r_\sigma)(K \circ \alpha)(r_\sigma)^{-1} = g(r_\sigma)\, h(r_\sigma)^{-1} = b^{-1} r_\sigma^w\, h(r_\sigma)^{-1} =$$
$$= ((h^{-1} \circ K)(\sigma(1-d)/4))^w (bK(\sigma(1-d)/4))^{-1}\,.$$

4.3. Corollary. (Theorem 1 of [9].) *Consider the same situation as in Theorem* 3.1 *with* $K_1(r) = K_1 r^{-\alpha}$, $K_2(r) = K_2 r^{-(\alpha+\gamma)}$, $K_3(r) = K_3 r^{-\gamma}$, $K_4(r) = K_3 r^{-(\gamma+\beta)}$ *for* $r \in (0, 1]$, K_1, K_2, K_3, α, β, γ *being positive numbers. Denote* $\delta = \max(\alpha + 2\gamma, \gamma + \beta)$.

Then there exists a constant c depending on K_i, α, β, γ *such that, whenever*

$$|f(u_0)|_\sigma \leqq c \frac{(R - \|u_0\|_\sigma)}{1 + 2(R - \|u_0\|_\sigma)} \sigma^\delta \tag{16}$$

for some $u_0 \in \hat{R}_\sigma$ *and some* $0 < \sigma \leqq 1$, *then there exists* $u \in Y_{\sigma/2}$ *such that*

1° $f(u) = 0$,

2° $|u - u_0|_{\sigma/2} \leqq c^{-1} |f(u_0)|_\sigma (1 + 2(R - \|u_0\|_\sigma)\, \sigma^{-\gamma} \leqq (R - \|u_0\|_\sigma)\, \sigma^{\delta-\gamma}$,

3° $\|u - u_0\|_{\sigma/2} \leqq c^{-1} |f(u_0)|_\sigma (1 + 2(R - \|u_0\|_\sigma))\, \sigma^{-\gamma-\beta} \leqq (R - \|u_0\|_\sigma)\, \sigma^{\delta-\gamma-\beta}$.

Proof. Set $K(r) = M r^{-\delta}$ where $M = \max(K_3, K_1 K_3^2 + K_2)$, then $K^{-1}(r) =$ $= (M^{-1} r)^{-1/\delta}$.

Given $\sigma \in (0, 1]$ and $u_0 \in \hat{R}_\sigma$, we are to find, according to 4.1 and 4.2, $1 < a \leqq 2$, $0 < d < 1$, $b > 0$, $w > 0$ and a function h satisfying

$$h(r^a)\, h(r)^{-1} \leqq b r^{w(a-2)}$$

and

$$(M^{-1} h(r^a))^{-1/\delta} = d(M^{-1}\, h(r))^{-1/\delta}$$

for small r, or equivalently,

$$d^{-\delta} = h(r^a)\, h(r)^{-1} \leqq b r^{w(a-2)}\,. \tag{17}$$

Further, the function h should satisfy

$$(h^{-1} \circ K)(\sigma(1-d)/4) < q_a^{1/(a-1)w}\,.$$

Since the function $q_a^{1/(a-1)}$ increases in the interval $(1, 2]$ the best choice, with respect to the initial condition, is $a = 2$; then $q_2 = b(R - \|u_0\|_\sigma)(1 + b(R - \|u_0\|_\sigma))^{-1}$.

Going back to the inequality (17) we see that $h(r^2)\, h(r)^{-1}$ is to be a bounded function, so we set $d = 2^{-1/\delta}$, $b = 2$, $w = 1$, $h(r) = -N_\sigma \log r$ for $0 < r < 1$ with N_σ such that

$$r_\sigma = (h^{-1} \circ K)(\sigma(1 - d)/4) = \exp(-N_\sigma^{-1} M \sigma^{-\delta}(1 - d)^{-\delta} 4^\delta) = \eta q_2$$

for arbitrary fixed $0 < \eta < 1$.

Finally, set c to satisfy $cK(\sigma(1 - d)/4) = \sigma^{-\delta}$. According to what has been said above and according to 4.2 it follows that the following implication holds: whenever

$$|f(u_0)|_\sigma < c \frac{R - \|u_0\|_\sigma}{1 + 2(R - \|u_0\|_\sigma)} \sigma^\delta$$

for some $u_0 \in \hat{R}_\sigma$ then there exists an element $u \in \hat{R}_{\sigma/2}$ with $f(u) = 0$.

The proof of the first part is complete.

Using the inequalities (9) and (10) of Remark 3.2, the relations $r_\sigma = (h^{-1} \circ K) . . (\sigma(1 - d)/4) < q_2$ and $cK(\sigma(1 - d)/4) = \sigma^{-\delta}$, we can estimate the distance between a solution u and the initial point $u_0 \in Y_\sigma$ satisfying (16) as follows

$$\begin{aligned}
|u - u_0|_{\sigma/2} &\leqq |f(u_0)|_\sigma\, r_\sigma^{-1}\, h(r_\sigma) \sum_{n=0}^{\infty} K_3 M^{-1} (K^{-1} \circ h)(r_\sigma^{2^n})^{\delta - \gamma}\, r_\sigma^{2^n} \leqq \\
&\leqq |f(u_0)|_\sigma\, r_\sigma^{-1}\, h(r_\sigma)\, K_3 M^{-1} \sum_{n=0}^{\infty} (K^{-1} \circ h)(r_\sigma)^{\delta - \gamma}\, r_\sigma^{2^n} \leqq \\
&\leqq |f(u_0)|_\sigma\, r_\sigma^{-1}\, h(r_\sigma)(K^{-1} \circ h)(r_\sigma)^{\delta - \gamma} \sum_{n=0}^{\infty} r_\sigma^{2^n} \leqq \\
&\leqq |f(u_0)|_\sigma\, r_\sigma^{-1} M^{1 - \gamma/\delta}\, h(r_\sigma)^{\gamma/\delta} \sum_{n=0}^{\infty} r_\sigma q_2^n = \\
&= |f(u_0)|_\sigma\, M^{1 - \gamma/\delta}\, K(\sigma(1 - d)/4)^{\gamma/\delta} (1 - q_2)^{-1} \leqq \\
&\leqq |f(u_0)|_\sigma\, K(\sigma(1 - d)/4)\, \sigma^{\delta - \gamma} (1 + 2(R - \|u_0\|_\sigma)) = \\
&= |f(u_0)|_\sigma\, c^{-1}(1 + 2(R - \|u_0\|_\sigma))\, \sigma^{-\gamma}
\end{aligned}$$

and, using the substitution $\gamma + \beta$ for γ,

$$\|u - u_0\|_{\sigma/2} \leqq c^{-1} |f(u_0)|_\sigma (1 + 2(R - \|u_0\|_\sigma))\, \sigma^{-\gamma - \beta} \leqq (R - \|u_0\|_\sigma)\, \sigma^{\delta - \gamma - \beta} .$$

Remark. We intend now to estimate the rate of convergence $\tilde{\omega}$ associated by the induction theorem with the above mentioned process. According to (8,1) the function τ of 2.2 has the form

$$\tau(r) = (K_3 \circ \alpha)(r)\, g(r)\, (K \circ \alpha)(r)^{-1} .$$

In our case $g(r) = 2^{-1} r$, $K_3(r) = K_3 r^{-\gamma}$, $K(r) = M r^{-\delta}$ and $\alpha = (K^{-1} \circ h)(r) = M^{1/\delta}\, h(r)^{-1/\delta} = M^{1/\delta}(-N_\sigma \log r)^{-1/\delta}$ so that

$$\tau(r) = 2^{-1} K_3 M^{-\gamma/\delta} r h(r)^{\gamma/\delta - 1} .$$

We have, for $s = \tau^{-1}(r)$,

$$\tilde{\omega}(r) = (\tau \circ \omega)(s) = \tau(s^2) = 2^{-1}K_3M^{-\gamma/\delta}s^2\, h(s^2)^{\gamma/\delta - 1} = \\ = 2K_3^{-1}M^{\gamma/\delta}\, 2^{\gamma/\delta-1}r^2\, h(s)^{1-\gamma/\delta}\,.$$

We intend to show that there exists, for each $\sigma \in (0, 1]$, a constant Q_σ such that

$$\tilde{\omega}(r) \leqq Q_\sigma(-\log r)^{1-\gamma/\delta}\, r^2$$

for $r \in (0, r_\sigma]$.

Obviously, it suffices to show that $h(s) \leqq B_\sigma(-\log r)$ for suitable positive B_σ and $r \in (0, r_\sigma]$, or equivalently, that there exists a constant C_σ such that $\tau^{-1}(r) = s \geqq r^{C_\sigma}$ for $r \leqq r_\sigma$. Since $\tau(r) \leqq K_\sigma r$ in $(0, r_\sigma]$ it suffices to take C_σ so that $\tau(r^{C_\sigma}) \leqq K_\sigma r^{C_\sigma} \leqq$ $\leqq r = \tau(s)$ for $r \leqq r_\sigma$.

We shall turn now our attention to the paper [2]. It is not difficult to prove that the main theorem of the above mentioned paper is a discrete case of our Theorem 3.1. More interesting is the illustrative example in which the author proves the existence of solutions of a nonlinear differential equation with odd quasi-periodic coefficients. In this case there exists an exact right inverse, however, its growth is of exponential type.

Consider the Banach space E_σ of all compositions $x = f \circ q$ where f is a 2π-periodic scalar function of n complex variables, bounded for $|\mathrm{Im}\, z| \leqq \sigma$ ($|z| = \sum_{i=1}^{n} |z_i|$) and holomorfic inside, and q is an n-tuple $(q_1, \ldots, q_n)$, $q_j = i\alpha_j + \omega_j t$ (ω_j are linearly independent real algebraic numbers of degree v and $|\alpha| < \sigma$, $\alpha = (\alpha_1, \ldots, \alpha_n)$), equipped with the norm $|x|_\sigma = \sup_{|\mathrm{Im} z| \leqq \sigma} |f(z)|$.

It follows that $f(z) = \sum_k f_k e^{i(k,z)}$ for $|\mathrm{Im}\, z| < \sigma$ and $|f_k| \leqq \sup_{|\mathrm{Im} z| \leqq \sigma} |f(z)|\, e^{-|k|\sigma}$. On the other hand, any sequence (f_k) such that $|f_k| \leqq Me^{-|k|\sigma}$ defines a holomorphic function f for $|\mathrm{Im}\, z| < \sigma$ and $\sup_{|\mathrm{Im} z| \leqq \sigma'} |f(z)| \leqq (4/(\sigma - \sigma'))^n M$ for each $0 < \sigma' < \sigma$ (see [1], p. 168).

Let F_σ be the subspace of E_σ consisting of all functions $x = f \circ q \in E_\sigma$ such that $\dot{x} \in E_\sigma$ as well ($\dot{x} = \mathrm{d}(f \circ q)(t)/\mathrm{d}t$) with the norm $\|x\|_\sigma = |x|_\sigma + |\dot{x}|_\sigma$.

Consider the operator

$$P(x) = \dot{x} + F(x, q(\cdot)) + f \circ q$$

where $F(x, z) = \sum_{k=1}^{\infty} f_k(z)\, x^k$, $f_k \circ q$, $f \circ q \in E_\sigma$, $x \in \hat{R}_\sigma = \{u \in F_\sigma, \|u\|_\sigma < R\}$ and $\sum |f_k \circ q|_\sigma\, |u|^k < \infty$ for $|u| \leqq R + \varepsilon$ ($\varepsilon > 0$).

The operator P maps each F_σ into E_σ and has bounded first and second derivatives

$$P'(u)\, x = \dot{x} + \frac{\partial F(u, q(\cdot))}{\partial u}\, x\,, \quad P''(u)\,(x, z) = \frac{\partial^2 F(u, (q(\cdot))}{\partial u^2}\, xz$$

for $u \in \hat{R}_\sigma$, $x, z \in F_\sigma$. Note that $P'(u)$ maps each F_σ into E_σ for all $0 < \sigma \leqq 1$.

The boundedness of P'' yields

$$|P(u + v) - P(u) - P'(u)\, v|_\sigma \leqq M_1 |v|_\sigma^2$$

whenever $u,\ u + v \in R_\sigma$.

Take $u \in \hat{R}_\sigma$. We shall show that there exists an exact right inverse to $P'(u)$, i.e. we can find, for each $x = f \circ q \in E_\sigma$, an element $v \in F_{\sigma'}$ $(\sigma/2 < \sigma' < \sigma)$ such that $P'(u)\, v = x$.

Indeed, denote $a(u, z) = \partial F(u, z)/\partial u$. Then the function v defined by the formula

$$(18) \qquad v(t) = \exp\left(-\int_0^t a(u, q(y))\, \mathrm{d}y\right) \int_0^t (f \circ q)(w) \exp\left(\int_0^w a(u, q(y))\, \mathrm{d}y\right) \mathrm{d}w$$

satisfies $\dot{v}(t) = -a(u, q(t))\, v(t) + (f \circ q)(t)$ for each t.

To prove that $v \in F_{\sigma'}$, $(\sigma/2 < \sigma' < \sigma)$ we shall use Lemma 2 from [2]:

There exists a constant $b(v, n)$ such that, given a function $g \circ q \in E_\sigma$ with $g(z) = \sum_{k \neq 0} g_k e^{i(k,z)}$, the function h defined by

$$h(t) = \int_0^t (g \circ q)(y)\, \mathrm{d}y = \sum_{k \neq 0} \frac{g_k}{i(k, \omega)} e^{i(k, q(t))} \Big|_0^t$$

belongs to $F_{\sigma'}$ and $|h|_{\sigma'} \leqq |g \circ q|_\sigma\, b(v, n)\, (\sigma - \sigma')^{-(n+v)}$.

It follows that we have the estimate

$$(19) \qquad |v|_{\sigma'} \leqq \exp\left(2|a(u, q)|_\sigma\, b(v, n)\, (\sigma - \sigma')^{-(n+v)}\right) |f \circ q|_\sigma\, b(v, n)\, . \\ . (\sigma - \sigma')^{-(n+v)} \leqq M(v, n)^{(\sigma-\sigma')^{-(v+n)}}\, |f \circ q|_\sigma$$

for any function v defined by (18) such that

$$(20) \qquad \int_0^{2\pi} a(u, q(y))\, \mathrm{d}y = 0$$

and

$$\int_0^{2\pi} (f \circ q)(w) \exp\left(\int_0^w a(u, q(y))\, \mathrm{d}y\right) \mathrm{d}w = 0\, .$$

It follows that

$$\|v\|_{\sigma'} = |v|_{\sigma'} + |\dot{v}|_{\sigma'} \leqq \left(|a(u)|_{\sigma'} + 1\right) |v|_{\sigma'} + |f|_{\sigma'} \leqq \\ \leqq \left(\left(|a(u)|_\sigma + 1\right) M(v, n)^{(\sigma-\sigma')^{-(v+n)}} + 1\right) |f|_\sigma \leqq M_2^{(\sigma-\sigma')^{-p}} |f|_\sigma$$

where $p = v + n$.

The conditions (20) are fulfilled if u is even and all $f_k \circ q$, $f \circ g$ odd functions.

We are led to the following definitions:

Let Y_σ be the Banach space consisting of all even functions from F_σ and let Z_σ be the Banach space consisting of all odd functions from E_σ.

Then the operator P maps Y_σ into Z_σ and satisfies $1°-5°$ of Theorem 3.1 (here norms on Y_σ coincide). Hence we shall apply Lemmas 4.1 and 4.2 with

$$K_1(r) = M_1 \geqq 1\,, \quad K_2(r) = 0\,, \quad K_3(r) = K_4(r) = M_2^{r^{-p}}$$

for $0 < r \leqq 1$.

4.3. Corollary. *Let $u_0 \in \hat{R}_\sigma$ be given, $0 < \sigma \leqq 1$. There exists a positive m depending on M_2, p and $R - \|u_0\|_\sigma$ such that $|P(u_0)|_\sigma < M_1^{-1} m^{-\sigma^{-p}}$ implies the existence of an element $u \in \hat{R}_{\sigma/2}$ such that $P(u) = 0$.*

Proof. Set $K(r) = M_1 M_2^{2r^{-p}}$ for $0 < r < 1$. Then $K^{-1}(s) = (2^{-1} \log_{M_2} M_1^{-1} s)^{-1/p}$ for $s > M_1 M_2^2$. Let $\sigma \in (0, 1]$, $u_0 \in \hat{R}_\sigma$ be given. According to 4.1 it is sufficient to find constants $w > 0$, $b > 0$, $0 < d < 1$, $1 < a \leqq 2$ and a function h such that

$$h(r^a) \leqq b r^{w(a-2)}\, h(r)$$

and

$$(2^{-1} \log_{M_2} M_1^{-1}\, h(r^a))^{-1/p} \leqq d(2^{-1} \log_{M_2} M_1^{-1}\, h(r))^{-1/p}$$

or equivalently,

$$M_1^{1-d^{-p}}\, h(r)^{d^{-p}} \leqq h(r^a) \leqq b r^{w(a-2)}\, h(r)\,. \tag{21}$$

Since $d^{-p} > 1$ we set $h(r) = M_1 r^{-z}$ for a positive z. To satisfy (21) it is sufficient to take $1 < a < 2$, $w = z(a-1)(2-a)^{-1}$, $d = a^{-1/p}$ and $b = 1$. As $(h^{-1} \circ K)(r) = M_2^{-2r^{-p}z^{-1}}$ the condition (14) has the form

$$M_2^{-2(a-1)(2-a)^{-1}(1-a^{-1/p})^{-p}4^p\sigma^{-p}} < q_a^{1/(a-1)} \tag{22}$$

where q_a is the solution of the equation

$$(1 - q)(R - \|u_0\|_\sigma) = q^{1/(a-1)}\,.$$

Since $\lim\limits_{a \to 2^-} q_a^{1/(a-1)} > 0$ it follows that there exists $a_0 \in (1, 2)$ such that the inequality (22) holds for each $\sigma \in (0, 1]$.

Finally, if

$$|P(u_0)|_\sigma < \frac{(h^{-1} \circ K)(\sigma(1-d)/4)^w}{bK(\sigma(1-d)/4)} = \frac{1}{M_1} M_2^{-2(2-a_0)^{-1}(1-a_0^{-1/p})^{-p}4^p\sigma^{-p}}$$

then, according to 4.2, there exists $u \in \hat{R}_{\sigma/2}$ such that $P(u) = 0$.

References

[1] *В. И. Арнольд:* Малые знаменатели и проблемы устойчивости движения в классической и небеской механике, УМН *18*, вып. 6 (1963), 91—192.

[2] *И. Н. Блинов:* Об одном итерационном процессе Ньютона, Изв. АН, сер. мат. *33* (1969), 3—14.
[3] *J. Moser:* A new technique for the construction of solutions of nonlinear differential equations, Proc. Nat. Acad. Sci. *47* (1961), 1824—1831.
[4] *V. Pták:* Deux théorèmes de factorisation, Comptes Rendus Ac. Sci. Paris *278* (1974), 1091—1094.
[5] *V. Pták:* A theorem of the closed graph type, Manuscripta Math. *13* (1974), 109—130.
[6] *V. Pták:* A quantitative refinement of the closed graph theorem, Czech. Math. Journal *99* (1974), 503—506.
[7] *V. Pták:* Nondiscrete mathematical induction and iterative existence proofs, Linear Algebra and its Applications *13* (1976), 223—238.
[8] *V. Pták:* A modification of Newton's method, Čas. pěst. mat. *101* (1976), 188—194.
[9] *E. Zehnder:* An implicit function theorem for small divisor problems, Bull. Amer. Math. Soc. *80* (1974), 174—179.
[10] *E. Zehnder:* Generalized implicit function theorems with applications to some small divisor problems (preprint).

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