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A REMARK ON C_σ SPACES

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ABSTRACT. We give a simple new proof of the following result, conjectured by Effros and proved by Fakhoury: Let E be a C_σ space and Z the set of extreme points of the unit ball of E^*. Then $Z \cup \{0\} = \{p \in E^*: \langle fgh, p\rangle = \langle f, p\rangle\langle g, p\rangle\langle h, p\rangle$ for all f, g, h in $E\}$.

Let $C(X)$ be the Banach space of all continuous real valued functions on a compact Hausdorff space X, equipped with the supremum norm. If $\sigma: X \to X$ is an involutory homeomorphism, then $C_\sigma(X)$ is the subspace of $C(X)$ consisting of those f in $C(X)$ which satisfy $f(\sigma x) = -f(x)$ for all x in X. e_x will stand for the point functional corresponding to a point x in X. If E is a Banach space, then we shall denote its unit ball by $B(E)$ and its conjugate space by E^*. The set of extreme points of a subset Q of E will be denoted by ext Q.

The following result was conjectured by Effros [**2**, Remark 8.4] and proved by Fakhoury [**4**, Theorem 15].

THEOREM. *If E is a C_σ space, then* ext $B(E^*) \cup \{0\} = \{p \in E^*: \langle fgh, p\rangle = \langle f, p\rangle\langle g, p\rangle\langle h, p\rangle$ *for all f, g, h in E*$\}$.

Fakhoury's proof is measure-theoretic in nature. Since this theorem appears to be useful (see, for instance, [**4**, Theorem 25] and [**3**, Theorem 11]), a simple different proof might perhaps be of some interest. The purpose of this note is to present such a proof. We shall need a few auxiliary propositions.

LEMMA 1 [**1**, p. 89]. *If $E = C_\sigma(X)$, then* ext $B(E^*) = \{e_x: x \in X$ and $x \neq \sigma x\}$.

LEMMA 2 [**5**, PROPOSITION 3.5]. *Let E be a Banach space and J^* the adjoint of the canonical map $J: E \to E^{**}$. If $y \in$* ext $B(E^{***})$, *then J^*y belongs to the weak star closure of* ext $B(E^*)$.

Since the dual of a C_σ space is isometric to an L space, its second dual is isometric to $C(Y)$ for some (extremally disconnected) compact Hausdorff space Y.

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Lemma 3. *Let $E=C_\sigma(X)$ and $E^{**}=C(Y)$. Then $J(fgh)=J(f)J(g)J(h)$ for all f, g, h in E.*

Proof. Let y belong to Y. The previous lemmas imply that J^*e_y belongs to $\{e_x : x\in X \text{ and } x\neq\sigma x\}\cup\{0\}$. Therefore $J(fgh)(y)=\langle J(fgh), e_y\rangle=\langle fgh, J^*e_y\rangle=\langle f, J^*e_y\rangle\langle g, J^*e_y\rangle\langle h, J^*e_y\rangle=J(f)(y)J(g)(y)J(h)(y)$.

Lemma 4. *If $E=C(X)$, then the Theorem is true.*

Proof. If $p\neq 0$, then the equality $\langle f, p\rangle=\langle f, p\rangle\langle 1, p\rangle^2$, valid for each f in E, implies that either $\langle 1, p\rangle=1$, or $\langle 1, p\rangle=-1$. Clearly we may assume that the former possibility holds. Then p is multiplicative on $C(X)$, hence an extreme point of the positive face of $B(E^*)$.

Proof of the Theorem. By Lemma 3, $\langle p, J(f)J(g)J(h)\rangle=\langle p, J(fgh)\rangle=\langle fgh, p\rangle=\langle f, p\rangle\langle g, p\rangle\langle h, p\rangle=\langle p, J(f)\rangle\langle p, J(g)\rangle\langle p, J(h)\rangle$ for all f, g, h in E. Let $K: E^*\to E^{***}$ be the canonical map. For each r in $E^{**}=C(Y)$, the operator defined on $C(Y)$ by $s\to rs$ is weak star continuous. Therefore $\langle p, rst\rangle=\langle p, r\rangle\langle p, s\rangle\langle p, t\rangle$ for all r, s, t in $C(Y)$. By Lemma 4, Kp belongs to ext $B(C(Y)^*)\cup\{0\}$. Since $p=J^*Kp$, an appeal to Lemmas 1 and 2 concludes the proof.

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