TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 299, Number 1, January 1987

A REMARK ON THE BLOWING-UP OF SOLUTIONS TO THE CAUCHY PROBLEM FOR NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We consider solutions to $iu_t = \Delta u + |u|^{p-1}u$, $u(0) = u_0$, where x belongs to a smooth domain $\Omega \subset \mathbf{R}^N$, and we prove that under suitable conditions on p, N and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\|\nabla u(t)\|_{L^2}$ blows up in finite time. The range of p's for which blowing-up occurs depends on whether Ω is star-shaped or not. Examples of blowing-up under Neuman or periodic boundary conditions are given.

RESUMÉ. On considère des solutions de $iu_t = \Delta u + |u|^{p-1}u$, $u(0) = u_0$, où la variable d'espace x appartient à un domaine régulier $\Omega \subset \mathbf{R}^N$, et on prouve que sous des conditions adéquates sur p, N et $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\|\nabla u(t)\|_{L^2}$ explose en temps fini. Les valeurs de p pour lesquelles l'explosion a lieu dépend de la forme de l'ouvert Ω (en fait Ω étoilé ou non). On donne également des exemples d'explosion sous des conditions de Neuman ou périodiques au bord.

1. Introduction and main results. Let $\Omega \subset \mathbf{R}^N$ (with $N \geq 1$) be a smooth domain and consider for $p > 1$ the nonlinear Schrödinger equation (NLS),

$$\begin{cases} iu_t = \Delta u + |u|^{p-1}u & \text{on } \Omega, \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

and

$$\begin{cases} iu_t = \Delta u - |u|^{p-1}u & \text{on } \Omega, \\ u(0,x) = u_0(x). \end{cases} \tag*{(1)$_-$}$$

When $\Omega = \mathbf{R}^N$, it is well known that, under appropriate conditions on the smoothness of the initial data u_0, there exists a local solution (in time) to (1.1) and (1.1)$_-$. For $(N-2)p < N+2$ and $u_0 \in H^1(\mathbf{R}^N)$, the corresponding solution of (1.1)$_-$ is unique and exists globally in time. For $1 < p < 1 + 4/N$ and $u_0 \in H^1(\mathbf{R}^N)$ the solution of (1.1) is global. (See e.g. Th. Cazenave [**1**], J. Ginibre and G. Velo [**2**], and references of these papers.) For $p \geq 1 + 4/N$ and initial data $u_0 \in \mathcal{S}(\mathbf{R}^N)$ such that the energy

$$E(u_0) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_0|^2\,dx - \frac{1}{p+1}\int_{\mathbf{R}^N} |u_0|^{p+1}\,dx \leq 0,$$

R. T. Glassey [**3**] proves that there exists a finite time $T_* > 0$ such that

$$\lim_{t \uparrow T_*} \|\nabla u(t)\|_{L^2} = \infty.$$

Received by the editors September 20, 1985 and, in revised form, January 24, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35B99, 35Q20; Secondary 81C05.

When $\Omega = \mathbf{R}^N$, other examples of initial data for which the solution $u(t)$ blows up in finite time are known. H. Berestycki and Th. Cazenave [**4**] prove that if $R > 0$ is a minimum-action solution (= ground state) of $-\Delta R + \omega R = R^p$ on $\mathbf{R}^N$ ($p \geq 1 + 4/N$) for some $\omega > 0$, then the solution $u(t)$ of (1.1) with initial data $u_0(x) = \lambda^{N/2} R(\lambda x)$ ($\lambda > 1$) blows up in finite time (note that $e^{-i\omega t} R(x)$ is a solitary wave solution of (NLS), and that $E(u_0) \geq 0$).

When $p = 1 + 4/N$ and $\Omega = \mathbf{R}^N$, Michael Wienstein has observed that if R satisfies $-\Delta R + R = R^p$ on $\mathbf{R}^N$, then for $(a,b,c,d) \in \mathbf{R}^4$, $ad - bc = 1$, the function

$$\psi(t,x) := (a+bt)^{-N/2} R\left(\frac{x}{a+bt}\right) \cdot \exp\left(i\frac{b|x|^2 + c + dt}{a+bt}\right),$$

is a solution of (NLS) which blows up in finite time (cf. M. Weinstein [**5**]).

When $\Omega \neq \mathbf{R}^N$, $N \leq 2$ and $p < 1 + 4/N$, H. Brézis and Th. Gallouët [**6**] prove that, for $u_0 \in H_0^1(\Omega)$, there exists a unique global solution to (1.1). But for $N \geq 3$, due to the fact that when $\Omega \neq \mathbf{R}^N$ the behavior of $e^{it\Delta}$ is not well known, much less can be said about the global existence or the blowing-up of solutions in finite time. However for $p = 1 + 4/N$, $N = 1$ or 2 and Ω being a ball, numerical computations made by A. Patera, C. Sulem, and P. L. Sulem [**7, 8**] suggest that there are solutions which blow up in finite time. In this paper we prove that depending on the shape of the domain Ω and the value of p there are solutions of (1.1) which blow up in finite time. (We do not study local existence and uniqueness of the Cauchy problem (1.1). We suppose that a local solution is given and we prove that it blows up in finite time—or rather it cannot exist globally in some appropriate space—, whenever the initial data satisfies a certain set of conditions.)

The main results are the following.

(1.2) PROPOSITION. *Let Ω be a smooth starshaped domain in $\mathbf{R}^N$ and $p \geq 1 + 4/N$. Let $T > 0$ and consider a solution $u(t)$ of* (1.1) (*with $u(0) = u_0$*) *such that*

$$u \in C^1([0,T], L^2(\Omega)) \cap C([0,T], H^2 \cap H_0^1 \cap L^{p+1}(\Omega)). \tag{1.3}$$

Then if $\int_\Omega |x|^2 |u_0(x)|^2\,dx < \infty$ and u_0 satisfies either of the following conditions (1.4) *or* (1.5), *then there exists T_** (*depending on u_0*) *such that $T < T_*$* (*i.e. the solution blows up in finite time*).

$$E(u_0) := \frac{1}{2}\int_\Omega |\nabla u_0|^2\,dx - \frac{1}{p+1}\int_\Omega |u_0|^{p+1}\,dx < 0, \tag{1.4}$$

$$\begin{cases} E(u_0) \geq 0, \quad \operatorname{Im}\int_\Omega (x\cdot\nabla u_0)\overline{u}_0(x)\,dx > 0 \quad \textit{and} \\ \left|\operatorname{Im}\displaystyle\int_\Omega (x\cdot\nabla u_0)u_0(x)\,dx\right|^2 \geq E(u_0)\cdot\displaystyle\int_\Omega |xu_0(x)|^2\,dx. \end{cases} \tag{1.5}$$

For a domain Ω which is not starshaped Proposition (1.6) holds ($N \geq 2$):

(1.6) PROPOSITION. *Let ω be a smooth domain in $\mathbf{R}^N$, starshaped with respect to some point $x_0 \in \omega$ and let $r > 0$ such that $B(x_0, r) \subset \omega$. Then if $\Omega := \omega \cap \overline{B(x_0,r)}^c$, $p \geq 5$ and u_0 is such that $\int_\Omega |x|^2|u_0(x)|^2\,dx < \infty$ and satisfies either of conditions* (1.4) *or* (1.5 bis), *then the solution of the* (*NLS*) (1.1) *satisfying* (1.3), *blows up in finite time.*

$$(1.5\text{ bis})\qquad \begin{cases} E(u_0) \geq 0, \quad \operatorname{Im} \int (\nabla\varphi \cdot \nabla u_0)\overline{u}_0(x)\,dx > 0 \quad \textit{and} \\ \left|\operatorname{Im} \int (\nabla\varphi \cdot \nabla u_0)\overline{u}_0(x)\,dx\right|^2 \geq 4NE(u_0) \int \varphi(x)|u_0(x)|^2\,dx \end{cases}$$

where

$$\varphi(x) := \frac{1}{2}|x - x_0|^2 + \frac{r^N}{(N-2)|x - x_0|^{N-2}} \quad \textit{when } N \geq 3$$

and

$$\varphi(x) := \frac{1}{2}|x - x_0|^2 - r^2 \log|x - x_0| \quad \textit{when } N = 2.$$

If one is interested in the blowing-up of solutions which satisfy other boundary value conditions than Dirichlet (i.e. $H_0^1(\Omega)$) such as Neuman or periodic conditions, one can construct such solutions using (1.2) (cf. Remark 4.9 below). Note that the condition $\int_\Omega |x|^2|u_0(x)|^2\,dx < \infty$ is satisfied when Ω is bounded and $u_0 \in H_0^1(\Omega)$; actually this condition seems technical when Ω is unbounded and one can give the following variant of Proposition (1.2).

(1.7) PROPOSITION. *Let Ω be a smooth domain in $\mathbf{R}^N$ such that there exists $k{}^{\geq 1}_{\leq N}$, $a \in \mathbf{R}^N$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq N$ such that if $\vec{n}(x)$ is the outer normal to the boundary $\partial\Omega$ one has*

$$\forall x \in \partial\Omega \quad (x_{j_1} - a_{j_1})n_{j_1}(x) + \cdots + (x_{j_k} - a_{j_k})n_{j_k}(x) \geq 0.$$

Then if $u_0 \in H_0^1(\Omega)$ satisfies

$$(1.8)\qquad \int_\Omega (|x_{j_1}|^2 + \cdots + |x_{j_k}|^2)|u_0(x)|^2\,dx < \infty$$

and either of conditions (1.4) *or* (1.9), *the solution u of* (1.1) *satisfying* (1.3) *blows up in finite time if $p \geq 1 + 4/k$.*

$$(1.9)\qquad \begin{cases} E(u_0) \geq 0, \quad \operatorname{Im} \int (\nabla\varphi \cdot \nabla u_0)\overline{u}_0(x)\,dx > 0 \quad \textit{and} \\ \left|\operatorname{Im} \int_\Omega (\nabla\varphi \cdot \nabla u_0)\overline{u}_0(x)\,dx\right|^2 \geq 2E(u_0) \int \varphi|u_0(x)|^2\,dx \end{cases}$$

where

$$\varphi(x) = \tfrac{1}{2}(|x_{j_1} - a_{j_1}|^2 + \cdots + |x_{j_k} - a_{j_k}|^2).$$

In particular if Ω is bounded in k directions and $p \geq 1 + 4/k$ the condition (1.8) is fulfilled and the only condition for the blowing-up is (1.4) (or (1.9)).

It is clear that the condition $p \geq 1 + 4/k$ in (1.7) is not optimal in the sense that $p < 1 + 4/k$ does not imply global existence. In §4.7 we prove that the condition $p \geq 5$ in (1.6) is optimal in the sense that there are global solutions for $p < 5$ and any value of $E(u_0)$; at the same time there are solutions which blow up when $1 + 4/N \leq p < 5$ (and $E(u_0) \leq 0$).

The proof of these results is a slight modification of the one given by R. T. Glassey [**3**] (see below).

The author wishes to thank Alain Haraux who brought his attention to this question.

2. Preliminary results. In what follows, we consider a sufficiently smooth solution of (1.1) for which the following hold for some $T > 0$:

(1.1) $iu_t = \Delta u + |u|^{p-1}u.$

(2.1) $u(0,x) = u_0(x) \not\equiv 0.$

(2.2) $u(t) \in H_0^1(\Omega)$ for $0 \le |t| \le T$ (Dirichlet boundary condition).

(2.3) $\int_\Omega |u(t,x)|^2\,dx = \int_\Omega |u_0(x)|^2\,dx$ for $0 \le |t| \le T$.

(2.4) $$\frac{1}{2}\int_\Omega |\nabla u(t,x)|^2\,dx - \frac{1}{p+1}\int_\Omega |u(t,x)|^{p+1}\,dx =: E(u(t)) = E(u_0)$$ for $0 \le |t| \le T$.

For instance any classical solution of (1.1) satisfies (2.3) and (2.4). (To obtain (2.3) multiply the equation (1.1) by $\overline{u}$, integrate over Ω and take the imaginary part; to obtain (2.4) multiply (1.1) by $\overline{u}_t$, take the real part and integrate over Ω.)

Following R. T. Glassey we consider the "variance" of u (in fact that of $|u|^2$) but we modify this variance according to the shape of Ω. More precisely let φ satisfy

(2.5) $$\varphi \ge 0, \quad \varphi \not\equiv 0, \quad \varphi \in C^4(\mathbf{R}^N)$$

and define for $t \in [-T, T]$

(2.6) $$V(t) := \frac{1}{2}\int_\Omega \varphi(x)|u(t,x)|^2\,dx.$$

Define also the Hessian of φ by

(2.7) $$H(\varphi)(x) := (\partial^2_{kj}\varphi(x))_{1\le k,j\le N}$$

and for $\xi \in \mathbf{C}^N$

(2.8) $$(H(\varphi)\xi|\xi) := \sum_{1\le k,j\le N} \partial^2_{kj}\varphi(x)\xi_j\overline{\xi}_k.$$

To prove the results about the blowing-up of solutions we prove first the following lemma and in the next sections we choose the function φ according to Ω.

(2.9) LEMMA. *Let $u \in C^1([0,T], L^2(\Omega)) \cap C([0,T], H^2 \cap H_0^1 \cap L^{p+1}(\Omega))$ be a solution of* (1.1) *with $u(0,x) = u_0(x)$, φ satisfying* (2.5) *with compact support and V defined as in* (2.6). *Then $V \in C^2([0,T])$ and for each t one has*

$$V'(t) = \operatorname{Im}\int_\Omega \varphi(x)\overline{u}(t,x)\Delta u(t,x)\,dx$$

$$\begin{aligned} V''(t) = {} & 2\int_\Omega (H(\varphi)\nabla u|\nabla u)(t,x)\,dx \\ & + \left(\frac{2}{p+1} - 1\right)\int_\Omega \Delta\varphi \cdot |u|^{p+1}(t,x)\,dx \\ & - \frac{1}{2}\int_\Omega \Delta^2\varphi \cdot |u|^2(t,x)\,dx \\ & - \int_{\partial\Omega} |\nabla u(t,x)\cdot\vec{n}(x)|^2 \nabla\varphi(x)\cdot\vec{n}(x)\,dx \end{aligned}$$

where $\vec{n}(x)$ is the outer normal at $x \in \partial\Omega$ (Ω is supposed to be smooth).

PROOF. The fact that $V \in C^2$ is straightforward, but for the sake of completeness we sketch here the proof. As $\varphi \in C_c^4(\mathbf{R}^N)$ and $u \in C^1([-T,T], L^2(\Omega))$ it is clear that $V \in C^1([-T,T])$ and

$$V'(t) = \operatorname{Re}\int_\Omega \varphi\overline{u}(t,x)u_t(t,x)\,dx$$

but by (1.1) $u_t = -i\Delta u - i|u|^{p-1}u$ and hence (denoting by $\langle\cdot,\cdot\rangle$ the duality H^{-1}, H_0^1):

$$V'(t) = \operatorname{Im}\langle \Delta u(t), \varphi\overline{u}(t)\rangle$$

Now, for $h \in \mathbf{R}$, $h \neq 0$, we have

$$\begin{aligned} V'(t+h) - V'(t) &= \operatorname{Im}\langle \Delta u(t+h), \varphi[\overline{u}(t+h) - \overline{u}(t)]\rangle \\ &\quad + \operatorname{Im}\langle \Delta(u(t+h) - u(t)), \varphi\cdot\overline{u}(t)\rangle \end{aligned} \tag{2.10}$$

or, equivalently,

$$\begin{aligned} V'(t+h) - V'(t) &= \operatorname{Im}\langle \Delta(u(t+h) - u(t)), \varphi[\overline{u}(t+h) - \overline{u}(t)]\rangle \\ &\quad + 2\operatorname{Im}\int_\Omega \nabla\varphi\cdot\nabla\overline{u}(t)[u(t+h) - u(t)]\,dx \\ &\quad + \operatorname{Im}\int_\Omega \Delta\varphi\cdot\overline{u}(t)[u(t+h) - u(t)]\,dx. \end{aligned} \tag{2.11}$$

The first term in (2.11) can be written as

$$\begin{aligned} &\operatorname{Im}\langle \Delta(u(t+h) - u(t)), \varphi[\overline{u}(t+h) - \overline{u}(t)]\rangle \\ &\quad = -\operatorname{Im}\int_\Omega [\overline{u}(t+h) - \overline{u}(t)]\nabla\varphi\cdot\nabla(u(t+h) - u(t))\,dx, \end{aligned} \tag{2.12}$$

and using the fact that $u \in C^1([-T,T], L^2(\Omega))$ and $u \in C([-T,T], H_0^1(\Omega))$, one sees that by (2.11), (2.12) $\lim_{h\to 0} \frac{1}{h}[V'(t+h) - V'(t)]$ exists and

$$V''(t) = 2\operatorname{Im}\int_\Omega \nabla\varphi\cdot\nabla\overline{u}(t)\cdot u_t(t)\,dx + \operatorname{Im}\int_\Omega \Delta\varphi\cdot\overline{u}(t)u_t(t)\,dx. \tag{2.13}$$

This identity proves that $V \in C^2([-T,T])$.

In the sequel, for the sake of simplicity we drop the subscript Ω, the variable t and set

$$A_1 := \operatorname{Im}\int \nabla\varphi\cdot\nabla\overline{u}\cdot u_t\,dx, \tag{2.14}$$

$$A_2 := \operatorname{Im}\int \Delta\varphi\cdot\overline{u}\cdot u_t\,dx \tag{2.15}$$

(so $V'' = 2A_1 + A_2$). By (1.1) one has $u_t = -i\Delta u - i|u|^{p-1}u$, and we can study A_1, A_2.

For A_2: an integration by parts give

$$A_2 = -\operatorname{Re}\int \Delta\varphi\cdot|u|^{p+1}\,dx + \operatorname{Re}\int \Delta\varphi\cdot|\nabla u|^2\,dx + \operatorname{Re}\int \overline{u}\nabla u\cdot\nabla(\Delta\varphi)\,dx.$$

But $\operatorname{Re}\overline{u}\nabla u = \frac{1}{2}\nabla(|u|^2)$ and hence

$$A_2 = -\operatorname{Re}\int \Delta\varphi|u|^{p+1}\,dX + \operatorname{Re}\int \Delta\varphi\cdot|\nabla u|^2\,dx - \frac{1}{2}\int \Delta^2\varphi\cdot|u|^2\,dx. \tag{2.16}$$

For A_1: using (1.1) we have by integration by parts

$$
\begin{aligned}
A_1 &= -\operatorname{Re}\int(\nabla\varphi\cdot\nabla\overline{u})\Delta u\,dX - \operatorname{Re}\int\nabla\varphi\cdot\nabla u\cdot|u|^{p-1}u\,dx,\\
A_1 &= -\operatorname{Re}\int(\nabla\varphi\cdot\nabla\overline{u})\Delta u\,dx - \frac{1}{p+1}\int\nabla\varphi\cdot\nabla(|u|^{p+1})\,dx, \qquad (2.17)\\
A_1 &= -\operatorname{Re}\int(\nabla\varphi\cdot\nabla\overline{u})\Delta u\,dx + \frac{1}{p+1}\int\Delta\varphi\cdot|u|^{p+1}\,dx.
\end{aligned}
$$

On the other hand

$$-\operatorname{Re}\int(\nabla\varphi,\nabla\overline{u})\Delta u\,dx = B_1+B_2+B_3$$

where

$$B_1 := -\operatorname{Re}\int_{\partial\Omega}(\nabla\varphi\cdot\nabla u)(\nabla\overline{u}\cdot\vec{n})\,dx, \tag{2.18}$$

$$B_2 := \operatorname{Re}\sum_{1\le k\le N}\int_\Omega\partial_k\varphi(\nabla\overline{u}\cdot\partial_k\nabla u)\,dx, \tag{2.19}$$

$$B_3 := \operatorname{Re}\sum_{1\le k\le N}\int_\Omega\partial_k u(\nabla\overline{u}\cdot\partial_k\nabla\varphi)\,dx.$$

First note that by (2.8), B_3 can be written

$$B_3 = \int_\Omega(H(\varphi)\nabla u|\nabla u)\,dx. \tag{2.20}$$

We remark also that $\operatorname{Re}\nabla\overline{u}\cdot\partial_k\nabla u = \frac{1}{2}\partial_k|\nabla u|^2$ and hence

$$
\begin{aligned}
B_2 &= \int_\Omega\nabla\varphi\cdot\nabla\left(\frac{1}{2}|\nabla u|^2\right)\,dx, \qquad (2.21)\\
B_2 &= \frac{1}{2}\int_{\partial\Omega}(\nabla\varphi\cdot\vec{n})|\nabla u|^2\,dx - \frac{1}{2}\int_\Omega\Delta\varphi|\nabla u|^2\,dx.
\end{aligned}
$$

Concerning B_1, note that $u|_{\partial\Omega}=0$ and hence on $\partial\Omega$ $\nabla u = (\nabla u\cdot\vec{n})\vec{n}$: this yields

$$B_1 = -\int_{\partial\Omega}(\nabla\varphi\cdot\vec{n})|\nabla u|^2\,dx.$$

Finally using this and (2.21), (2.20) we get

$$
\begin{aligned}
A_1 = &-\frac{1}{2}\int_{\partial\Omega}(\nabla\varphi\cdot\vec{n})|\nabla u|^2\,dx - \frac{1}{2}\int_\Omega\Delta\varphi|\nabla u|^2\,dx\\
&+\int_\Omega(H(\varphi)\nabla u|\nabla u)\,dx + \frac{1}{p+1}\int_\Omega\Delta\varphi|u|^{p+1}\,dx
\end{aligned}
$$

and this, together with (2.16), gives the lemma. □

Now for the proof of the propositions of §1, we have to choose an appropriate function φ.

3. Proof of Proposition (1.2) and (1.7). Without loss of generality we may assume that Ω is starshaped with respect to $0 \in \Omega$, i.e.

$$\forall x \in \partial\Omega, \qquad x \cdot \vec{n}(x) \geq 0. \tag{3.1}$$

First let $\psi \in C_c^\infty(\mathbf{R})$ be such that

$$\begin{gathered} \psi(-y) = \psi(y) \qquad \forall y \in \mathbf{R}, \\ \psi(y) = 1 \quad \text{for } |y| \leq 1, \\ \psi(y) = 0 \quad \text{for } |y| \geq 2, \\ \psi'(y) \leq 0 \qquad \forall y \in \mathbf{R}_+, \end{gathered}$$

and define $f_m(x) := \psi(|x|/m)$ for $x \in \mathbf{R}^N$ and $m \geq 1$. Next, for a solution u such as in Proposition (1.2), define

$$V(t) := \frac{1}{4}\int_\Omega |x|^2 |u(t,x)|^2\,dx$$

and

$$V_m(t) := \frac{1}{4}\int_\Omega |x|^2 f_m(x) |u(t,x)|^2\,dx.$$

By Lemma 2.9 we know that

$$\begin{aligned} V_m''(t) = 2&\int_\Omega (H(\varphi_m)\nabla u|\nabla u)(t,x)\,dx \\ &+ \left(\frac{2}{p+1} - 1\right)\int_\Omega \Delta\varphi_m |u|^{p+1}(t,x)\,dx \\ &- \frac{1}{2}\int_\Omega \Delta^2\varphi_m |u|^2(t,x)\,dx \\ &- \int_{\partial\Omega} |\nabla u(t,x)|^2 \nabla\varphi_m(x)\cdot\vec{n}(x)\,dx \end{aligned}$$

where

$$\varphi_m := \tfrac{1}{2}|x|^2 f_m(x)$$

and by the above hypotheses on φ_m and u one sees easily that $(V_m'')_m$ converges in $L^1([-T,T])$ to

$$\begin{aligned} W(t) := 2\int_\Omega |\nabla u|^2(t,x)\,dx + \left(\frac{2}{p+1} - 1\right) N \int_\Omega |u|^{p+1}(t,x)\,dx \\ - \int_{\partial\Omega} |\nabla u(t,x)|^2 x\cdot\vec{n}(x)\,dx \end{aligned} \tag{3.2}$$

(here we use the fact that if $\varphi(x) := \frac{1}{2}|x|^2$, $(H(\varphi)\nabla u|\nabla u) = |\nabla u|^2$ and $\Delta\varphi = N$). On the other hand $V_m(t) \uparrow V(t)$ as $m \to \infty$ and

$$\begin{aligned} V_m'(t) &= \operatorname{Im} \int_\Omega \varphi_m \overline{u}(t,x)\Delta u(t,x)\,dx \\ &= \operatorname{Im} \int_\Omega \overline{u}(t,x)\nabla\varphi_m(x)\cdot\nabla u(t,x)\,dx. \end{aligned}$$

$$V_m(t) = V_m(0) + V_m'(0)\cdot t + \int_0^t (t-s)V_m''(s)\,ds, \tag{3.3}$$

$$V_m'(0) \to -\operatorname{Im}\int_\Omega \overline{u}_0(x)x\cdot\nabla u_0(x)\,dx,$$

$$V(t) = V(0) - \left(\operatorname{Im}\int_\Omega \overline{u}_0(x)x\cdot\nabla u_0(x)\,dx\right)t + \int_0^t (t-s)W(s)\,ds.$$

But by (2.4) and (3.1) we have

$$\begin{aligned} W(t) &\le 4E(u_0) + \left(\frac{2N+4}{p+1} - N\right)\int_\Omega |u(t,x)|^{p+1} \\ &\le 4E(u_0) \le 0 \quad \text{if } p \ge 1 + 4/N \end{aligned}$$

and hence

$$0 < V(t) \le V(0) - \left(\operatorname{Im}\int_\Omega \overline{u}_0 x\cdot\nabla u_0(x)\,dx\right)t + 2E(u_0)\cdot t^2. \tag{3.4}$$

Now it is clear that if u_0 satisfies (1.4) or (1.5) the solution $u(t)$ cannot exist globally (notice that if $E(u_0) < 0$, the blow-up occurs for some $T_* > 0$ and also for some $T_{**} < 0$). This proves Proposition (1.2).

The proof of Proposition (1.7) is the same as above by choosing (we may suppose $a = 0$)

$$\varphi(x) := \tfrac{1}{2}(|x_{j_1}|^2 + \cdots + |x_{j_k}|^2)$$

and then $\Delta\varphi = k$, $\Delta^2\varphi = 0$

$$\begin{gathered} (H(\varphi)\nabla u|\nabla u) = |\partial_{j_1}u|^2 + \cdots + |\partial_{j_k}u|^2 \le |\nabla u|^2, \\ \nabla\varphi\cdot\vec{n} = x_{j_1}\cdot n_{j_1}(x) + \cdots + x_{j_k}\cdot n_{j_k}(x) \ge 0, \\ W(t) \le 4E(u_0) + \left(\frac{2k+4}{p+1} - k\right)\int_\Omega |u(t,x)|^{p+1}\,dx \end{gathered}$$

(W is defined in 3.2). Now if $p \ge 1 + 4/k$ one has $W(t) \le 4E(u_0)$ and hence one observes that (3.4) holds and the proof of Proposition 1.7 is over.

4. Proof of Proposition (1.6). Without loss of generality one can assume that $x_0 = 0$ and $r = 1$. Thus

$$\partial\Omega = \{x \in \mathbf{R}^N; |x| = 1\} \cup \partial\omega$$

(note that $\partial\omega \cap \{x; |x| = 1\} = \varnothing$), and denoting by $\vec{n}(x)$ the outward normal at $x \in \partial\Omega$ on has

$$\begin{cases} \text{if } |x| = 1 & \vec{n}(x) = -x, \\ \text{if } x \in \partial\omega & \vec{n}(x)\cdot x \ge 0. \end{cases} \tag{4.1}$$

Now define for $x \in \overline{\Omega}$

$$\begin{cases} \varphi(x) := \frac{1}{2}|x|^2 + \frac{1}{(N-2)|x|^{N-2}} & \text{if } N \geq 3, \\ \varphi(x) := \frac{1}{2}|x|^2 - \log|x| & \text{if } N = 2, \end{cases} \tag{4.2}$$

(the case $N = 1$ is already contained in Proposition (1.2)), and

$$V(t) := \frac{1}{2}\int_\Omega \varphi(x)|u(t,x)|^2\,dx. \tag{4.3}$$

As in §3, consider $f_m(x) := \psi(|x|/m)$ where $\psi \in C_c^\infty(\mathbf{R})$ and

$$V_m(t) := \frac{1}{2}\int_\Omega \varphi(x) f_m(x)|u(t,x)|^2\,dx.$$

In the same fashion, one can check easily that $(V_m'')_m$ converges in $L^1([-T,T])$ to

$$\begin{aligned} W(t) := 2\int_\Omega (H(\varphi)\nabla u|\nabla u)(t,x)\,dx \\ + \left(\frac{2}{p+1} - 1\right) N \int |u|^{p+1}(t,x)\,dx \\ - \int_{\partial\Omega} |\nabla u(t,x)\cdot\vec{n}(x)|^2 \nabla\varphi\cdot\vec{n}(x)\,dx \end{aligned} \tag{4.4}$$

(here we use the fact that $\Delta\varphi = N$).

But $\nabla\varphi = (1 - |x|^{-N})X$ and

if $x \in \partial\Omega$ $|x| = 1$ then $\nabla\varphi\cdot\vec{n}(x) = 0$;

if $x \in \partial\omega$ then $\nabla\varphi(x)\cdot\vec{n}(x) \geq 0$.

This means that

$$\forall x \in \partial\Omega \qquad \nabla\varphi(x)\cdot\vec{n}(x) \geq 0. \tag{4.5}$$

On the other hand

$$\partial^2_{kj}\varphi = (1 - |x|^{-N})\delta_{kj} + N|x|^{-(N+2)}x_j x_k$$

and

$$\begin{aligned} (H(\varphi)\nabla u|\nabla u) &\leq |\nabla u|^2 + (N-1)|x|^{-N}|\nabla u|^2 \\ &\leq N|\nabla u|^2 \quad \text{since } |x| \geq 1. \end{aligned} \tag{4.6}$$

Hence (4.5) and (4.6) yield

$$\begin{aligned} W(t) &\leq N\left(2\int_\Omega |\nabla u|^2 + \left(\frac{2}{p+1} - 1\right)\int |u|^{p+1}\right) \\ &\leq N\left(4E(u_0) + \left(\frac{6}{p+1} - 1\right)\int |u|^{p+1}\right), \\ W(t) &\leq 4NE(u_0) \quad (\text{since } p \geq 5). \end{aligned}$$

So we get

$$0 < V(t) \leq V(0) + V'(0)t + 2NE(u_0)t^2$$

and again this proves Proposition (1.6), noting that (when $E(u_0) \geq 0$)

$$V'(0) = \operatorname{Im}\int \varphi\overline{u}_0\Delta u_0 = -\operatorname{Im}\int_\Omega (\nabla\varphi\cdot\nabla u_0)\overline{u}_0(x)\,dx. \quad \square$$

(4.7) REMARK. When Ω is starshaped and $N \leq 2$, $p < 1 + 4/N$ the solution of (1.1) satisfying (1.3) is global in time, no matter what the sign of $E(u_0)$ is. When Ω is not starshaped and $1 + 4/N \leq p < 5$ the situation is somewhat complicated.

Consider for instance $\Omega = B(0,1)^c$ where $B(0,1) = \{x \in \mathbf{R}^N, |x| \leq 1\}$, $N \geq 2$.

If u_0 is spherically symmetric with respect to the origin 0, it is clear that the solution $u(t)$ is spherically symmetric for each t. On the other hand if $\varphi \in C_c^1(\Omega)$ is spherically symmetric, then for any $\sigma \in \mathbf{R}^N$ with $|\sigma| = 1$

$$\begin{aligned} |\varphi(r\sigma)|^2 &= -2 \int_r^\infty \varphi(z\sigma)\sigma \cdot \nabla\varphi(z\sigma)\, dx \\ &\leq 2 \left(\int_1^\infty z^{-(N-1)} |\varphi(z\sigma)|^2\, dz \right)^{1/2} \left(\int_1^\infty z^{N-1} |\nabla\varphi(z\sigma)|^2\, dz \right)^{1/2} \end{aligned}$$

and this yields

$$\|\varphi\|_{L^\infty(\Omega)} \leq C \|\varphi\|_{L^2}^{1/2} \|\nabla\varphi\|_{L^2}^{1/2}. \tag{4.8}$$

Now if $u(t)$ is a spherically symmetric solution of (1.1)

$$\begin{aligned} \|\nabla u(t)\|^2 &\leq 2E(u_0) + \frac{2}{p+1} \int |u(t)|^{p+1} \\ &\leq 2E(u_0) + C \|u(t)\|_{L^\infty}^{p+1} \|u(t)\|_{L^2}^2. \end{aligned}$$

But $\|u(t)\|_{L^2} = \|u_0\|_{L^2} = C^{te}$ and hence by (4.8) we get

$$\|\nabla u(t)\|_{L^2}^2 \leq C + C \|\nabla u(t)\|_{L^2}^{(p-1)/2}.$$

So if $p < 5$ then $\|\nabla u(t)\|_{L^2} \leq C^{te}$ and using again (4.8) we get a uniform estimate for $\|u(t)\|_{L^\infty}$ and this proves that the spherically symmetric solutions of (1.1) are global in time, whatever $E(u_0)$ can be.

Consider now, for the sake of simplicity, the case where $N = 2$, $\Omega = B(0,1)^c$ and $1 + 4/N \leq p < 5$. We are going to construct a solution of (1.1) which blows up in finite time.

Let $\Omega_+ := \{(x,y) \in \mathbf{R}^2,\ x > 0,\ y > 0,\ x^2 + y^2 > 1\}$.

It is clear that Ω_+ is starshaped with respect to the point $(1,1)$. Now let $v_0 \in C_0^\infty(\Omega_+)$ be such that $E(v_0) < 0$. By Proposition (1.2) the solution $v(t)$ of (1.1) with $v(t) \in H_0^1(\Omega_+)$ and $v(0) = v_0$ blows up in finite time. If one considers $u(t)$ defined as

$$u(t,x,y) = \begin{cases} v(t,x,y) & \text{if } x \geq 0,\ y \geq 0, \\ -v(t,x,y) & \text{if } x \geq 0,\ y \leq 0, \\ v(t,-x,-y) & \text{if } x \leq 0,\ y \leq 0, \\ -v(t,-x,y) & \text{if } x \leq 0,\ y \geq 0, \end{cases}$$

then $u(t)$ is a solution of (1.1) $u(t) \in H_0^1(\Omega)$ and blows up in finite time.

(4.9) REMARK. If one considers other boundary conditions than Dirichlet (that is other than $u(t) \in H_0^1(\Omega)$), using (1.2) one can construct solutions of (1.1) which blow up in finite time. Indeed consider, for example, the case where $N = 1$, and the periodic boundary condition on $\Omega =]-1,+1[$, i.e.

$$u(t,-1) = u(t,+1), \qquad u_x(t,-1) = u_x(t,+1).$$

If one takes an initial data u_0 such that

$$u_0(-x) = -u_0(x) \quad \forall x \in [-1,+1], \qquad u_0 \in H_0^1(]-1,+1[)$$

then the solution of (1.1) with $u(0,x) = u_0(x)$ satisfies

$$u(t,-x) = -u(t,x) \quad \forall x \in [-1,+1],$$
$$u(t,-1) = u(t,+1) = 0, \quad u_x(t,-1) = u_x(t,1).$$

So if $p \geq 5$ and

$$\frac{1}{2}\int_{-1}^{+1} |u_{0x}|^2 - \frac{1}{p+1}\int_{-1}^{+1} |u_0|^{p+1} < 0,$$

the periodic solution $u(t)$ blows up in finite time (because it does so in $H_0^1([-1,+1])$ by Proposition (1.2)).

For the Neuman boundary condition (i.e. for instance $\Omega =]0,2[$ and $u_x(t,0) = u_x(t,2) = 0$) consider an initial data $v_0 \in H_0^1(-1,+1)$ such that

$$\forall x \in [-1,+1] \quad v_0(x) = v_0(-x)$$

$p \geq 5$ and

$$\frac{1}{2}\int_0^1 |v_{0x}|^2\,dx - \frac{1}{p+1}\int_0^1 |v_0|^{p+1}\,dx < 0.$$

Then the solution $v(t) \in H_0^1(]-1,+1[)$ with $v(0) = v_0$ blows up in finite time and satisfies

$$v(t,x) = v(t,-x) \quad \forall x \in [-1,+1].$$

Hence $v_x(t,0) = 0$, and if $u(t)$ is defined as

$$u(t,x) = \begin{cases} v(t,x) & \text{for } 0 \leq x \leq 1, \\ -v(t,2-x) & \text{for } 1 \leq x \leq 2, \end{cases}$$

$u(t)$ is a solution of (1.1) with $u(t) \in H^1(]0,2[)$, $u_x(t,0) = u_x(t,2) = 0$, and $u(t)$ blows up in finite time.

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