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# A remark on the character ring of a compact Lie group

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## Introduction

Let $G$ be a compact topological group, $D(G)$ the set of equivalence classes of irreducible representations of $G$. (In this note the representation will mean always the continuous complex representation.) The character ring $R(G)$ of $G$ is the free abelian group generated by $D(G)$ with the ring structure induced by the tensor product of representations. In the present note we provide a method of finding a system of generators of the character ring $R(G)$ of a compact (not necessarily connected) Lie group $G$, assuming that the quotient group $G/G_0$ of $G$ modulo the connected component $G_0$ of $G$ is a cyclic group (Theorem 5). Our problem reduces to finding generators of a certain commutative semi-group in the similar way as for a compact *connected* Lie group.

By applying the theorem we can know the structure of the character ring of the orthogonal group $O(2l)$ of degree $2l$ or of the double covering group $\mathrm{Pin}(2l)$ of $O(2l)$. (See § 3 for the definition of $\mathrm{Pin}(2l)$.) Let $\lambda^i$ be the $i$-th exterior power of the standard representation of $O(2l)$, $\alpha$ the 1-dimensional representation of $O(2l)$ defined by $\alpha(x)=\det x$ for $x\in O(2l)$. Let $\mu^l$ be the irreducible representation of $\mathrm{Pin}(2l)$ such that its restriction to the connected component $\mathrm{Spin}(2l)$ of $\mathrm{Pin}(2l)$ splits into the direct sum of two half-spinor representations of $\mathrm{Spin}(2l)$ and $p: \mathrm{Pin}(2l)\to O(2l)$ denote the covering homomorphism. Then we have

$$R(O(2l))=\boldsymbol{Z}[\lambda^1, \lambda^2, \cdots, \lambda^l, \alpha] \text{ with relations } \alpha^2=1 \text{ and } \lambda^l\alpha=\lambda^l,$$

$$R(\mathrm{Pin}(2l))=\boldsymbol{Z}[\lambda^1\circ p, \lambda^2\circ p, \cdots, \lambda^{l-1}\circ p, \mu^l, \alpha\circ p]$$

$$\text{with relations } (\alpha\circ p)^2=1 \text{ and } \mu^l(\alpha\circ p)=\mu^l.$$

The character ring of $O(2l)$ was formerly presented by Minami [7] by different methods.

## § 1. Induced representations.

Let $G$ be a compact topological group. We consider the set of equivalence classes of representations of $G$ as a subset of the character ring $R(G)$ of $G$ and introduce an inner product $( , )$ on $R(G)$ in such a way that $D(G)$ is an orthonormal basis of $R(G)$. For an element $\chi \in R(G)$, an element $\rho \in D(G)$ such that the integer $(\chi, \rho)$, denoted by $m_\rho$, is not zero is called a *component* of $\chi$. We call $m_\rho$ the *multiplicity* of the component $\rho$ in $\chi$. For a representation $\rho$ of $G$, the equivalence class of $\rho$ will be denoted by $[\rho]$.

Let $h: H \rightarrow G$ be a continuous homomorphism from a compact group $H$ into a compact group $G$. Then $h$ induces a ring homomorphism $R(G) \rightarrow R(H)$ by the composition of $h$, denoted by $h^*$, and $R(H)$ becomes an $R(G)$-module by means of the homomorphism $h^*$.

Let $G$ be a compact group, $H$ a closed subgroup of $G$ with the finite index $[G:H]$, $i: H \rightarrow G$ the inclusion homomorphism. For a representation $\sigma: H \rightarrow GL(V)$ of $H$, the space

$$\Gamma(G, V)^H = \{f: G \rightarrow V \,;\, f(gh) = \sigma(h)^{-1} f(g) \ \text{for } g \in G,\ h \in H\}$$

is a complex vector space of dimension $[G, H] \dim V$. $G$ acts linearly on $\Gamma(G, V)^H$ by $(gf)(g') = f(g^{-1}g')$ for $g, g' \in G$ and we have a representation of $G$ on $\Gamma(G, V)^H$, which is called the *representation induced by* $\sigma$. The space $\Gamma(G, V)^H$ is naturally identified with the space of sections of the vector bundle $G \times_H V$ over $G/H$ associated with the representation $\sigma$ of $H$ and the action of $G$ on $\Gamma(G, V)^H$ is nothing but the one induced from the natural action of $G$ on $G \times_H V$. The equivalence class of this representation depends only on the equivalence class of $\sigma$ so that we have a map $i_*: D(H) \rightarrow R(G)$, which is linearly extended to an $R(G)$-homomorphism $i_*: R(H) \rightarrow R(G)$ (cf. Atiyah [1]). Then we have the Frobenius reciprocity:

$$(i^*\rho, \sigma) = (\rho, i_*\sigma) \qquad \text{for} \quad \rho \in R(G),\ \sigma \in R(H)\,.$$

Now we assume that *$H$ is a normal subgroup of $G$ with the finite index*. Then the quotient group $A = G/H$ of $G$ modulo $H$ is a finite group and the natural projection $\pi: G \rightarrow A$ is a homomorphism. $\hat{A}$ denotes the character group $\mathrm{Hom}(A, \boldsymbol{C}^*)$ of $A$. We imbed $\hat{A}$ into $D(G)$ by the product-preserving map $\alpha \mapsto \alpha \circ \pi$. Then $\hat{A}$ acts on $D(G)$, therefore on $R(G)$, by the multiplication of elements of $\hat{A}$. For a representation $\sigma: H \rightarrow GL(V)$ of $H$ and $g \in G$, another representation $\sigma': H \rightarrow GL(V)$ of $H$ is defined by

$$\sigma'(g') = \sigma(g^{-1}g'g) \qquad \text{for} \quad g' \in H\,.$$

The equivalence class of $\sigma'$ depends only on the equivalence class of $\sigma$ and on $\pi(g)$ so that $A$ acts on $D(H)$, therefore on $R(H)$, by conjugations. The followings are immediate consequences of definitions:

(1) $i^*(\alpha\cdot\rho)=i^*\rho$ for $\alpha\in\hat{A},\quad \rho\in R(G)$,

(2) $a\cdot(i^*\rho)=i^*\rho$ for $a\in A,\quad \rho\in R(G)$,

(3) $i_*(a\cdot\sigma)=i_*\sigma$ for $a\in A,\quad \sigma\in R(H)$,

(4) $\alpha\cdot(i_*\sigma)=i_*\sigma$ for $\alpha\in\hat{A},\quad \sigma\in R(H)$.

THEOREM 1. (Clifford) *Let $\rho\in D(G)$. Take $\sigma_1\in D(H)$ such that $(i^*\rho, \sigma_1)>0$ and put $m(\rho)=(i^*\rho, \sigma_1)$, $\Phi_\rho=A\cdot\sigma_1\subset D(H)$. Then both $m(\rho)$ and $\Phi_\rho$ depend only on $\rho$ and we have the decomposition*

$$i^*\rho=m(\rho)\sum_{\sigma\in\Phi_\rho}\sigma\,.$$

For the proof, see Feit [3].

Let $A\backslash D(H)$ (resp. $\hat{A}\backslash D(G)$) denotes the set of $A$-orbits in $D(H)$ (resp. $\hat{A}$-orbits in $D(G)$). The map $\varphi: D(G)\to A\backslash D(H)$ defined by $\rho\mapsto\Phi_\rho$ is surjective from the Frobenius reciprocity and induces a surjective map

$$\Phi:\ \hat{A}\backslash D(G)\longrightarrow A\backslash D(H)$$

in view of (1). Note that $m(\rho)$ is constant on each $\hat{A}$-orbit in $D(G)$.

THEOREM 2. (Clifford-Iwahori) *If the quotient group $A=G/H$ is commutative, then the map $\Phi$ is bijective. The inverse map of $\Phi$ is given as follows. Let $\sigma\in D(H)$. Take $\rho_1\in D(G)$ such that $(i_*\sigma, \rho_1)>0$ and put $m(\sigma)=(i_*\sigma, \rho_1)$, $\Psi_\sigma=\hat{A}\cdot\rho_1\subset D(G)$. Then both $m(\sigma)$ and $\Psi_\sigma$ depend only on $\sigma$ and we have the decomposition*

$$i_*\sigma=m(\sigma)\sum_{\rho\in\Psi_\sigma}\rho\,.$$

*The map $\psi: D(H)\to\hat{A}\backslash D(G)$ defined by $\sigma\mapsto\Psi_\sigma$ induces a map*

$$\Psi:\ A\backslash D(H)\longrightarrow\hat{A}\backslash D(G)$$

*in view of* (3). *The map $\Psi$ is the inverse of $\Phi$. In particular:*

1) *If $A$ is a cyclic group, then $m(\rho)=m(\sigma)=1$ for any $\rho\in R(G)$ and $\sigma\in R(H)$.*

2) *If the order $|A|$ of $A$ is a prime number $p$, then for the orbits $\Phi_\rho$ and $\Psi_\sigma$ corresponding by the bijection $\Phi$ it happens one of following two cases:*

a) $|\Phi_\rho|=p$ *and* $|\Psi_\sigma|=1$,

b) $|\Phi_\rho|=1$ *and* $|\Psi_\sigma|=p$,

*where $|S|$ means the cardinality of the set $S$.*

PROOF. This theorem can be proved in the same way as the classical Clifford theorem for $A=\boldsymbol{Z}_2$ (Iwahori-Matsumoto [5]). But we give here another proof.

Let $\sigma\in D(H)$. For $\rho=\alpha\cdot\rho_1\in\Psi_\sigma$ we have $(i_*\sigma, \rho)=(\sigma, i^*(\alpha\rho_1))=(\sigma, i^*\rho_1)=(i_*\sigma, \rho_1)$. Therefore it suffices to show that if $\rho\in D(G)$ with $(i_*\sigma, \rho)>0$

then $\rho \in \Psi_\sigma$. Note that $i_* 1 = \sum_{\alpha \in \hat{A}} \alpha$ since $A$ is commutative. It follows that

$$i_*(i^* \rho_1) = i_*((i^* \rho_1) 1) = \rho_1(i_* 1) = \rho_1 \sum_{\alpha \in \hat{A}} \alpha = \sum_{\alpha \in \hat{A}} \alpha \cdot \rho_1 .$$

On the other hand, the Frobenius reciprocity yields that $(i^* \rho_1, \sigma) > 0$ so that $\rho$ is a component of $i_*(i^* \rho_1)$. Thus $\rho \in \hat{A} \cdot \rho_1 = \Psi_\sigma$. The above simple proof was communicated by Professor H. Nagao.

1) See Feit [3].

2) Recall (cf. Atiyah [1]) the general equality $\sum m_\rho^2 = |A_\sigma|$ for $\sigma \in D(H)$, $i_* \sigma = \sum_{\rho \in D(G)} m_\rho \rho$ and $A_\sigma = \{a \in A ; a \cdot \sigma = \sigma\}$. In our case we have $|\Psi_\sigma| = |A_\sigma| = |A| / |\Phi_\rho|$ by the above equality and 1), so that $|\Phi_\rho| |\Psi_\sigma| = p$, which yields the statement 2). q. e. d.

Note that $m(\sigma)$ is also constant on each $A$-orbit in $D(H)$ in view of (3) and that $m(\rho)$ and $m(\sigma)$ take the same value on the orbits corresponding by $\Phi$ from the Frobenius reciprocity.

REMARK. We denote by $R(G)^{\hat{A}}$ (resp. $R(H)^A$) the submodule of $R(G)$ (resp. $R(H)$) of elements fixed by $\hat{A}$ (resp. $A$). From (2) and (4) we have $i^* R(G) \subset R(H)^A$ and $i_* R(H) \subset R(G)^{\hat{A}}$. If $A$ is cyclic, then by Theorem 2, 1)

$$i_* R(H) = R(G)^{\hat{A}} .$$

It is also known (Atiyah [1]) that if the order $|A|$ of $A$ is square free ($A$ is not necessarily commutative), then

$$i^* R(G) = R(H)^A .$$

## § 2. Character ring of a compact Lie group.

Let $G$ be a compact Lie group, $G_0$ the connected component of $G$. Take a maximal torus $T_0$ of $G_0$. Note that $D(T_0)$ is a commutative group by the tensor product. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of $G_0$ and $T_0$. Take an $\mathrm{Ad}\, G$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Let $\Delta$ be the root system of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect to $\mathfrak{t}$, i. e. the set of non-zero elements $\alpha$ of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ such that

$$\mathfrak{g}_\alpha^C = \{X \in \mathfrak{g}^C ; [H, X] = 2\pi\sqrt{-1}\, \alpha(H) X \textit{ for any } H \in \mathfrak{t}\}$$

is not zero. Take a fundamental system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ of $\Delta$ and fix it once and for all. The duality defined by means of $\langle\ ,\ \rangle$ identifies $\mathfrak{t}$ with $\mathfrak{t}^*$ so that the root system $\Delta$ may be considered as a subset of $\mathfrak{t}$. Taking a basis $\{h_1, \cdots, h_m\}$ of the center of $\mathfrak{g}$, we introduce a lexicographic order $>$ on $\mathfrak{t}^*$ by the basis $\{\alpha_1, \cdots, \alpha_l, h_1, \cdots, h_m\}$ of $\mathfrak{t}$. Such order on $\mathfrak{t}^*$ will be called a *linear order associated with* $\Pi$. We put

$$Z_0 = \{\lambda \in \mathfrak{t}^* ;\ \lambda(H) \in \boldsymbol{Z} \textit{ for any } H \in \mathfrak{t} \textit{ such that } \exp H = 1\}$$

and

$$D_0 = \{\lambda \in Z_0 ;\ \lambda(\alpha_i) \geqq 0 \textit{ for any } \alpha_i \in \Pi\}\ .$$

Then $Z_0$ is a lattice of $\mathfrak{t}^*$ and isomorphic with $D(T_0)$ by the correspondence $\lambda \mapsto e^{2\pi\sqrt{-1}\lambda}$, where $e^{2\pi\sqrt{-1}\lambda}$ is the character of $T_0$ defined by $e^{2\pi\sqrt{-1}\lambda}(\exp H) = e^{2\pi\sqrt{-1}\lambda(H)}$ for $H \in \mathfrak{t}$. Thus we can introduce an order $>$ on $D(T_0)$ by means of the order $>$ on $Z_0$. $D_0$ is a commutative semi-group. We put

$$D_d(T_0) = \{e^{2\pi\sqrt{-1}\lambda} ;\ \lambda \in D_0\}\ .$$

An element of $D_d(T_0)$ will be called a *dominant* (with respect to $\Pi$) *irreducible representation* of $T_0$.

Now we define a closed subgroup $T$ of $G$ with the connected component $T_0$ by

$$T = \{g \in G ;\ \mathrm{Ad}\, g\mathfrak{t} = \mathfrak{t},\ \mathrm{Ad}\, g\Pi = \Pi\}\ .$$

The quotient group $T/T_0$ is naturally isomorphic with the quotient group $G/G_0$. This follows from $G_0 \cap T = T_0$, the conjugateness in $G_0$ of maximal tori of $G_0$ and that of fundamental systems of $\Delta$ under the normalizer of $T_0$ in $G_0$. We put $A = G/G_0 = T/T_0$. The adjoint representation Ad induces a homomorphism $\tau : A \to GL(\mathfrak{t})$. We define a finite subgroup $C$ of $GL(\mathfrak{t})$ by $C = \tau A$. It leaves $Z_0$ and $D_0$ invariant so that we can define the set $Z = C\backslash Z_0$ (resp. $D = C\backslash D_0$) of $C$-orbits in $Z_0$ (resp. in $D_0$). We introduce a linear order $>$ on $Z$ by defining that $\Lambda > \Lambda'$ for $\Lambda, \Lambda' \in Z$ if $\mathrm{Max}\, \Lambda > \mathrm{Max}\, \Lambda'$. We introduce also an operation $+$ on $Z$ by defining that for $\Lambda, \Lambda' \in Z$, $\Lambda + \Lambda'$ is the $C$-orbit through $\mathrm{Max}\, \Lambda + \mathrm{Max}\, \Lambda'$. Note that $\mathrm{Max}\, (\Lambda + \Lambda') = \mathrm{Max}\, \Lambda + \mathrm{Max}\, \Lambda'$. The operation $+$ induces a commutative semi-group structure on $D$.

Now we consider the following commutative diagram of inclusions:

$$\begin{array}{ccc} G & \xleftarrow{\ \ j\ \ } & T \\ {\scriptstyle i_{G_0}}\uparrow & & \uparrow{\scriptstyle i_{T_0}} \\ G_0 & \xleftarrow[\ \ j_0\ \ ]{} & T_0 \end{array}$$

Then we have the following commutative diagram of ring homomorphisms:

$$\begin{array}{ccc} R(G) & \xrightarrow{\ \ j^*\ \ } & R(T) \\ {\scriptstyle i^*_{G_0}}\downarrow & & \downarrow{\scriptstyle i^*_{T_0}} \\ R(G_0) & \xrightarrow[\ \ j^*_0\ \ ]{} & R(T_0) \end{array}$$

It is classical that $j^*_0$ is injective. The homomorphism $j^*$ is also injective since for any $g \in G$ there exists $g_0 \in G_0$ such that $g_0 g g_0^{-1} \in T$ (Gantmacher [4]). The inclusions $\hat{A} \subset D(G)$ and $\hat{A} \subset D(T)$ defined in §1 are compatible with the injection $j^*: R(G) \to R(T)$, i.e. $j^*\alpha = \alpha$ for $\alpha \in \hat{A}$. Let $\delta \in D(T)$. From Theorem 1 there exist a positive integer $m(\delta)$ and $\Lambda_\delta \in Z$ such that

$$i_{T_0}^*\delta = m(\delta) \sum_{\lambda \in \Lambda_\delta} e^{2\pi\sqrt{-1}\lambda}.$$

A surjective map $\varphi: D(T) \to Z$ is defined by the correspondence $\delta \mapsto \Lambda_\delta$. For $\delta, \delta' \in D(T)$, we say that $\delta$ is *strictly higher than* $\delta'$ if $\varphi(\delta) > \varphi(\delta')$ and it will be denoted by $\delta \gg \delta'$. We put

$$D_d(T) = \{\delta \in D(T)\,;\ \varphi(\delta) \in D\}.$$

An irreducible representation $\delta$ of $T$ is called *dominant* (with respect to $\Pi$) if the equivalence class of $\delta$ belongs to $D_d(T)$.

Let $\rho: G \to GL(V)$ be an irreducible representation of $G$. The holomorphic extension $G^C \to GL(V)$ of $\rho$ to the complexification $G^C$ of $G$ or its differential $\mathfrak{g}^C \to \mathfrak{gl}(V)$ will be denoted by the same letter $\rho$ and put

$$(5)\qquad \begin{aligned} V_0 &= \{v \in V\,;\ \rho(X)v = 0 \textit{ for any } X \in \mathfrak{m}\} \\ &= \{v \in V\,;\ \rho(\exp X)v = v \textit{ for any } X \in \mathfrak{m}\}, \end{aligned}$$

where $\mathfrak{m} = \sum_{\alpha \in \Delta, \alpha > 0} \mathfrak{g}_\alpha^C$. Then $V_0$ is $T$-invariant since $\mathrm{Ad}\, T$ leaves $\mathfrak{m}$ invariant, so that we have a representation $\delta_\rho: T \to GL(V_0)$ of $T$. We shall prove later that $\delta_\rho$ is a dominant irreducible representation of $T$. The equivalence class of $\delta_\rho$ depends only on the equivalence class of $\rho$. It is called the *Cartan component* of $\rho$ or of the equivalence class $[\rho]$ of $\rho$. The Cartan component of $\rho \in D(G)$ will be denoted by $\delta_\rho$ and the map $D(G) \to D_d(T)$ defined by $\rho \mapsto \delta_\rho$ will be denoted by $\gamma$. The classical representation theory of a compact *connected* Lie group yields the following

THEOREM 3. 1) *The Cartan component $\delta_\rho$ of $\rho \in D(G_0)$ belongs to $D_d(T_0)$ and the map $\gamma_0: D(G_0) \to D_d(T_0)$ defined by $\rho \mapsto \delta_\rho$ is an $A$-equivariant bijection.* (For $\lambda \in D_0$, $\gamma_0^{-1}(e^{2\pi\sqrt{-1}\lambda})$ will be denoted by $\rho_\lambda$.)

2) *For $\rho \in D(G_0)$ the Cartan component $\delta_\rho$ of $\rho$ is the highest component among components of $j_0^*\rho \in R(T_0)$ and has the multiplicity* 1.

Now we shall prove that the former representation $\delta_\rho: T \to GL(V_0)$ of $T$ induced by an irreducible representation $\rho: G \to GL(V)$ of $G$ is irreducible and dominant. Let $W_0$ be a $T$-invariant subspace of $V_0$. Then the subspace $W = \{\rho(g_0)s\,;\ g_0 \in G_0,\ s \in W_0\}_C$ of $V$ spanned by $\rho(G_0)W_0$ is $G$-invariant because of $G = TG_0$. Decompose $W_0$ into the direct sum of 1-dimensional $T_0$-invariant subspaces: $W_0 = W_1 + \cdots + W_m$, where $T_0$ acts on $W_i$ by character $e^{2\pi\sqrt{-1}\lambda_i}$ of $T_0$ $(1 \leqq i \leqq m)$. Then the subspace $V_i$ of $W$ spanned by $\rho(G_0)W_i$ is a $G_0$-irreducible $G_0$-invariant subspace with the Cartan component $e^{2\pi\sqrt{-1}\lambda_i}$ and $W$

is the direct sum of the $V_i$'s. Therefore we have

$$W_0=\{v\in W;\ \rho(X)v=0 \text{ for any } X\in\mathfrak{m}\}=W\cap V_0.$$

It follows from the $G$-irreducibility of $V$ that $W_0=0$ or $V_0$. Thus $V_0$ is $T$-irreducible. We have

$$i_{T_0}^*[\delta_\rho]=m([\delta_\rho])\sum_{\lambda\in\Lambda_{[\delta_\rho]}} e^{2\pi\sqrt{-1}\lambda}.$$

It follows from Theorem 3 and (5) that

$$i_{G_0}^*[\rho]=m([\delta_\rho])\sum_{\lambda\in\Lambda_{[\delta_\rho]}}\rho_\lambda$$

and so $\Lambda_{[\delta_\rho]}\subset D_0$. Thus $\delta_\rho$ is dominant.

Theorem 3 is true also for a general compact Lie group $G$ in the following sense.

THEOREM 4. 1) (Kostant) *The map* $\gamma: D(G)\to D_d(T)$ *defined by* $\rho\mapsto\delta_\rho$ *is an* $\hat{A}$*-equivariant bijection such that*

$$i_{G_0}^*\rho=m(\delta_\rho)\sum_{\lambda\in\Lambda_{\delta_\rho}}\rho_\lambda.$$

*Therefore we have the following commutative diagram:*

$$\begin{array}{ccc}\hat{A}\backslash D(G) & \xrightarrow{\ \ \Gamma\ \ } & \hat{A}\backslash D_d(T)\\ \Phi_G\downarrow & & \downarrow\Phi_T\\ A\backslash D(G_0) & \xrightarrow[\ \ \Gamma_0\ \ ]{} & A\backslash D_d(T_0)\end{array}$$

*where vertical maps* $\Phi_G$ *and* $\Phi_T$ *are the surjective maps defined in* § 1, *horizontal maps* $\Gamma$ *and* $\Gamma_0$ *are the bijective maps induced by* $\gamma$ *and* $\gamma_0$.

2) *For* $\rho\in D(G)$ *the Cartan component* $\delta_\rho$ *of* $\rho$ *is the strictly highest component among components of* $j^*\rho\in R(T)$ *and has the multiplicity* 1.

PROOF. 1) This was stated in Kostant [6] without proof. We prove it here for the sake of completeness.

Let $\delta: T\to GL(V_0)$ be a dominant irreducible representation of $T$. Decompose $V_0$ into the direct sum of 1-dimensional $T_0$-invariant subspaces:

$$V_0=W_1+\cdots+W_m,$$

where $T_0$ acts on $W_i$ by character $e^{2\pi\sqrt{-1}\lambda_i}\in D_d(T_0)$ $(1\leqq i\leqq m)$. Take one of the $\lambda_i$'s, say $\lambda_1$, and let $\rho_1: G_0\to GL(V_1)$ be an irreducible representation of $G_0$ with the Cartan component $e^{2\pi\sqrt{-1}\lambda_1}$ (Theorem 3). We imbed $W_1$ into $V_1$ as a $T_0$-invariant subspace. The natural map $T\times_{T_0}W_1\to G\times_{G_0}V_1$ is a $T$-equivariant injective bundle map over $A=T/T_0=G/G_0$ so that we have a $T$-equivariant imbedding $\Gamma(T, W_1)^{T_0}\subset\Gamma(G, V_1)^{G_0}$. From the Frobenius reciprocity: $((i_{T_0})_* e^{2\pi\sqrt{-1}\lambda_1}, [\delta]) = (i^*_{T_0}[\delta], e^{2\pi\sqrt{-1}\lambda_1}) > 0$, we have a $T$-irreducible $T$-invariant subspace $V'_0$ of $\Gamma(T, W_1)^{T_0}$ and a $T$-equivariant isomorphism $\theta: V_0 \to V'_0$. Thus we have a $T$-equivariant injective homomorphism $\theta: V_0 \to \Gamma(G, V_1)^{G_0}$. The subspace $V'$ of $\Gamma(G, V_1)^{G_0}$ spanned by $G_0\theta(V_0)$ is $G$-invariant because of $G = TG_0$ so that we have a representation $\rho': G \to GL(V')$ of $G$. For $g_0 \in G_0$, $s \in V_0$ and $t \in T$ we have

$$(\rho'(g_0)\theta(s))(t) = \theta(s)(g_0^{-1}t) = \theta(s)(t(t^{-1}g_0^{-1}t)) = \rho_1(t^{-1}g_0^{-1}t)^{-1}\theta(s)(t)\,.$$

It follows seeing $\theta(s)(t) \in W_1$ and $\mathrm{Ad}\, t^{-1}\mathfrak{m} = \mathfrak{m}$ that

$$\rho'(X)\theta(s) = 0 \qquad \textit{for any} \quad X \in \mathfrak{m} \textit{ and } s \in V_0\,.$$

Thus we have

$$(6) \qquad V'_0 = \{f \in V'\,;\ \rho'(X)f = 0 \textit{ for any } X \in \mathfrak{m}\}\,,$$

$$i^*_{G_0}[\rho'] = \rho_{\lambda_1} + \cdots + \rho_{\lambda_m}\,.$$

The representation $\rho'$ is $G$-irreducible since for a non-trivial $G$-invariant subspace $W'$ of $V'$, $W' \cap V'_0$ is also a non-trivial $T$-invariant subspace of $V'_0$ in view of (6). The equivalence class of $\rho'$ depends only on the equivalence class of $\delta$ since $\lambda_i \in A\cdot\lambda_1$ for any $i$ (Theorem 1).

Next we prove that if a dominant irreducible representation $\delta: T \to GL(V_0)$ of $T$ is obtained from an irreducible representation $\rho: G \to GL(V)$ of $G$ by (5), then the above obtained representation $\rho'$ is equivalent to $\rho$. Let

$$i^*_{T_0}[\delta] = e^{2\pi\sqrt{-1}\lambda_1} + \cdots + e^{2\pi\sqrt{-1}\lambda_m}\,.$$

Then (5) implies

$$i^*_{G_0}[\rho] = \rho_{\lambda_1} + \cdots + \rho_{\lambda_m}\,.$$

Together with (6) we have a $G_0$-equivariant isomorphism $\theta: V \to V'$ which is an extension of the $T$-equivariant isomorphism $\theta: V_0 \to V'_0$. For $t \in T$, $g_0 \in G_0$ and $s \in V_0$ we have

$$\begin{aligned}\theta(\rho(t)\rho(g_0)s) &= \theta(\rho(tg_0t^{-1})\rho(t)s) = \rho'(tg_0t^{-1})\theta(\rho(t)s)\\ &= \rho'(tg_0t^{-1})\rho'(t)\theta(s) = \rho'(tg_0)\theta(s)\\ &= \rho'(t)\rho'(g_0)\theta(s) = \rho'(t)\theta(\rho(g_0)s)\,.\end{aligned}$$

It follows that $\theta$ is a $G$-equivariant isomorphism since $V$ is spanned by $\rho(G_0)V_0$ and $G = TG_0$. Thus we have proved that the map $\gamma$ is bijective.

The other statements are clear from the construction.

2) Let

$$j^*\rho = \sum_{\delta \in D(T)} m_\delta \delta$$

and

$$i_{T_0}^* \delta = m(\delta) \sum_{\lambda \in \Lambda_\delta} e^{2\pi\sqrt{-1}\lambda}.$$

Then

$$i_{T_0}^* j^* \rho = \sum_\delta m_\delta m(\delta) \sum_{\lambda \in \Lambda_\delta} e^{2\pi\sqrt{-1}\lambda}.$$

On the other hand, we have by 1)

$$i_{G_0}^* \rho = m(\delta_\rho) \sum_{\lambda \in \Lambda_\delta} \rho_\lambda$$

so that

$$j_0^* i_{G_0}^* \rho = m(\delta_\rho) \sum_{\lambda \in \Lambda_\delta} j_0^* \rho_\lambda.$$

It follows from Theorem 3 that the highest component of $j_0^* i_{G_0}^* \rho \in R(T_0)$ is $e^{2\pi\sqrt{-1}\lambda_0}$, where $\lambda_0 = \operatorname{Max} \Lambda_{\delta_\rho}$, with the multiplicity $m(\delta_\rho)$. Comparing the highest components of $i_{T_0}^* j^* \rho$ and $j_0^* i_{G_0}^* \rho$, we know that $m_{\delta_\rho} = 1$ and that $m_\delta \neq 0$, $\delta \neq \delta_\rho$ imply $\Lambda_\delta < \Lambda_{\delta_\rho}$. This completes the proof of 2). q. e. d.

For each $\Lambda \in Z$ the complete inverse $\varphi^{-1}(\Lambda)$ of $\Lambda$ for the map $\varphi: D(T) \to Z$ defined by $\delta \mapsto \Lambda_\delta$ is a finite subset of $D(T)$ from the Frobenius reciprocity. We introduce on each $\varphi^{-1}(\Lambda)$ an arbitrary linear order and fix it once and for all. We introduce a linear order $>$ on $D(T)$ as follows: For $\delta$, $\delta' \in D(T)$, $\delta > \delta'$ if and only if $\delta \gg \delta'$ or $\varphi(\delta) = \varphi(\delta')$, $\delta > \delta'$.

LEMMA 1. *The highest* (with respect to the above order) *component of an element* $\chi \in j^* R(G) \subset R(T)$ *belongs to* $D_d(T)$.

PROOF. Let

$$\chi = j^* \sum_{\rho \in D(G)} m_\rho \rho, \qquad \delta_{\rho_0} = \operatorname*{Max}_{m_\rho \neq 0} \delta_\rho$$

and

$$j^* \rho = \sum_{\delta \in D(T)} m_{\rho,\delta} \delta,$$

so that

$$\chi = \sum_\rho m_\rho \sum_\delta m_{\rho,\delta} \delta.$$

It follows from Theorem 4 that the highest component of $\chi$ is $\delta_{\rho_0}$, which belongs to $D_d(T)$, with the multiplicity $m_{\rho_0}$. q. e. d.

Henceforth we assume that *the quotient group* $A = G/G_0$ *is a cyclic group*. Let $\alpha$ be a generator of the character group $\hat{A}$ of $A$. Then Theorem 2 implies that $m(\delta) = 1$ for any $\delta \in D(T)$ and that $\Phi_T$ and $\Phi_G$ are bijections so that $\varphi(\delta) = \varphi(\delta')$ for $\delta$, $\delta' \in D(T)$ if and only if there exists a non-negative integer $n$ such that $\alpha^n \delta = \delta'$.

LEMMA 2. *Let* $\delta$, $\delta' \in D(T)$. *Then* $\delta\delta' \in R(T)$ *has the strictly highest component* $\delta''$ *with the multiplicity* 1 *such that* $\varphi(\delta) + \varphi(\delta') = \varphi(\delta'')$.

PROOF. Let $\lambda_0 = \operatorname{Max} \Lambda_\delta$ and $\lambda_0' = \operatorname{Max} \Lambda_{\delta'}$. Then the highest component of

$$(i_{T_0}^*\delta)(i_{T_0}^*\delta')=\sum_{(\lambda,\lambda')\in\Lambda_\delta\times\Lambda_{\delta'}} e^{2\pi\sqrt{-1}(\lambda+\lambda')}\in R(T)$$

is $e^{2\pi\sqrt{-1}(\lambda_0+\lambda'_0)}$ with the multiplicity 1. On the other hand, if $\delta\delta'=\sum_{\varepsilon\in D(T)} m_\varepsilon \varepsilon$ we have

$$i_{T_0}^*(\delta\delta')=\sum_\varepsilon m_\varepsilon \sum_{\mu\in\Lambda_\varepsilon} e^{2\pi\sqrt{-1}\mu}.$$

Comparing the highest components of $(i_{T_0}^*\delta)(i_{T_0}^*\delta')$ and $i_{T_0}^*(\delta\delta')$ we know that there exists $\delta''\in D(T)$ such that $m_{\delta''}=1$ and $\Lambda_\delta+\Lambda_{\delta'}=\Lambda_{\delta''}$ and that $\delta''$ is strictly highest in $\delta\delta'$. q. e. d.

LEMMA 3. 1) *Let $\delta_1, \delta_2, \delta'\in D(T)$ such that $\delta_1\ll\delta_2$ and $\delta_1''$ (resp. $\delta_2''$) the strictly highest component of $\delta_1\delta'$ (resp. of $\delta_2\delta'$). Then $\delta_1''\ll\delta_2''$.*

2) *Let $\delta_1, \delta_2, \delta_1', \delta_2'\in D(T)$ such that $\delta_1\ll\delta_2$, $\delta_1'\ll\delta_2'$ and $\delta_1''$ (resp. $\delta_2''$) be the strictly highest component of $\delta_1\delta_1'$ (resp. of $\delta_2\delta_2'$). Then $\delta_1''\ll\delta_2''$.*

PROOF. 1) We have by Lemma 2 $\varphi(\delta_1'')=\varphi(\delta_1)+\varphi(\delta')$ and $\varphi(\delta_2'')=\varphi(\delta_2)+\varphi(\delta')$. Together with $\varphi(\delta_1)<\varphi(\delta_2)$ we have $\varphi(\delta_1'')<\varphi(\delta_2'')$.

2) We have by Lemma 2 $\varphi(\delta_1'')=\varphi(\delta_1)+\varphi(\delta_1')$ and $\varphi(\delta_2'')=\varphi(\delta_2)+\varphi(\delta_2')$. Together with the inequalities $\varphi(\delta_1)<\varphi(\delta_2)$ and $\varphi(\delta_1')<\varphi(\delta_2')$ we have $\varphi(\delta_1'')<\varphi(\delta_2'')$. q. e. d.

LEMMA 4. *Let $\chi_i$ $(1\leqq i\leqq m)$ be an element of $R(G)$ such that $j^*\chi_i$ has the strictly highest component $\delta_i\in D_d(T)$ with the multiplicity 1 and $n_i$ $(1\leqq i\leqq m)$ be a non-negative integer. Then $j^*(\chi_1^{n_1}\cdots\chi_m^{n_m})$ has the strictly highest component $\delta\in D_d(T)$ with the multiplicity 1 such that $\varphi(\delta)=n_1\varphi(\delta_1)+\cdots+n_m\varphi(\delta_m)$.*

PROOF. The existence of the strictly highest component $\delta$ follows from Lemma 3. The other statements follow from Lemma 2. q. e. d.

THEOREM 5. *Assume that $A=G/G_0$ is a cyclic group. Let $\{\Lambda_1,\cdots,\Lambda_m\}$ be a system of generators of the semigroup $D$, $\delta_i$ $(1\leqq i\leqq m)$ an element of $D_d(T)$ with $\varphi(\delta_i)=\Lambda_i$, $\chi_i$ $(1\leqq i\leqq m)$ an element of $R(G)$ such that $j^*\chi_i$ has the strictly highest component $\delta_i$ with the multiplicity* 1. (Existence of such $\chi_i$ is assured by Theorem 4.) *Let $\alpha$ be a generator of the character group $\hat{A}$ of $A$. Then the character ring $R(G)$ of $G$ is generated by $\chi_1,\cdots,\chi_m,\alpha$.*

PROOF. Take any element $\chi\in R(G)$. Let $\delta_0$ be the highest component of $j^*\chi$. $m_{\delta_0}$ denotes the multiplicity of $\delta_0$ in $j^*\chi$. Since $\delta_0\in D_d(T)$ (Lemma 1), we have non-negative integers $n_1,\cdots,n_m$ such that $n_1\Lambda_1+\cdots+n_m\Lambda_m=\varphi(\delta_0)$. On the other hand, Lemma 4 implies that $j^*(\chi_1^{n_1}\cdots\chi_m^{n_m})$ has the strictly highest component, say $\delta$, with the multiplicity 1 such that $n_1\Lambda_1+\cdots+n_m\Lambda_m=\varphi(\delta)$. Thus there exists a non-negative integer $n$ such that $\alpha^n\delta=\delta_0$. It follows that the highest component of $j^*(m_{\delta_0}\alpha^n\chi_1^{n_1}\cdots\chi_m^{n_m})=m_{\delta_0}\alpha^n j^*(\chi_1^{n_1}\cdots\chi_m^{n_m})$ is $\delta_0$ with the multiplicity $m_{\delta_0}$. Therefore the highest component of $j^*(\chi-m_{\delta_0}\alpha^n\chi_1^{n_1}\cdots\chi_m^{n_m})$ is lower than $\delta_0$. Thus we can show inductively that $\chi$ is a polynomial of $\chi_1,\cdots,\chi_m,\alpha$ with coefficients in $\boldsymbol{Z}$, recalling that $j^*$ is injective. q. e. d.

THEOREM 6. *Assume that* $A=G/G_0$ *is a cyclic group. Let* $\{\Lambda_1, \cdots, \Lambda_m\}$ *be an independent system of* $D$, *i. e.* $\sum_{i=1}^{m} n_i\Lambda_i = \sum_{i=1}^{m} n'_i\Lambda_i$, *where the* $n_i$, $n'_i$ *are non-negative integers, implies* $n_i = n'_i$ *for any* $i$. *Let* $\delta_i$ $(1 \leqq i \leqq m)$ *be an element of* $D_d(T)$ *with* $\varphi(\delta_i) = \Lambda_i$, $\chi_i$ *an element of* $R(G)$ *such that* $j^*\chi_i$ *has the strictly highest component* $\delta_i$ *with the multiplicity* 1. *Then the system* $\{\chi_1, \cdots, \chi_m\}$ *has no relations in* $R(G)$.

PROOF. Let $F \in \boldsymbol{Z}[X_1, \cdots, X_m]$ be a relation for $\{\chi_1, \cdots, \chi_m\}$, i. e. $F = \sum a_{n_1\cdots n_m} X_1^{n_1} \cdots X_m^{n_m}$ $(a_{n_1\cdots n_m} \in \boldsymbol{Z})$ satisfies $F(\chi_1, \cdots, \chi_m) = 0$. Suppose that $F \neq 0$. Let $\sum n_i^0 \Lambda_i$ be the highest among the $\sum n_i\Lambda_i$ such that $a_{n_1\cdots n_m} \neq 0$. The assumption for $\{\Lambda_1, \cdots, \Lambda_m\}$ implies the uniqueness of such $(n_1^0, \cdots, n_m^0)$, so that $a_{n_1\cdots n_m} \neq 0$, $(n_1, \cdots, n_m) \neq (n_1^0, \cdots, n_m^0)$ imply $\sum n_i\Lambda_i < \sum n_i^0\Lambda_i$. It follows from Lemma 4 that $j^*F(\chi_1, \cdots, \chi_m)$ has the strictly highest component with the multiplicity $a_{n_1^0\cdots n_m^0}$, which contradicts $F(\chi_1, \cdots, \chi_m) = 0$. q. e. d.

## § 3. Character rings of $O(n)$ and Pin $(n)$.

We recall the notion of the group Pin $(n)$ (Atiyah-Bott-Shapiro [2]). Let $C_n$ be the Clifford algebra over $\boldsymbol{R}$ of degree $n$ associated with the positive definite quadratic form, i. e. the associative algebra over $\boldsymbol{R}$ generated by $1, e_1, \cdots, e_n$ with relations $e_i^2 = 1$ $(1 \leqq i \leqq n)$ and $e_ie_j + e_je_i = 0$ $(1 \leqq i < j \leqq n)$. $C_n^*$ denotes the group of invertible elements of $C_n$. For $i = 0, 1$, $C_n^i$ denotes the subspace of $C_n$ spanned by the $e_{i_1}e_{i_2}\cdots e_{i_r}$ $(1 \leqq i_1 < i_2 < \cdots < i_r \leqq n,\ r \equiv i \bmod 2)$. Then $C_n^0$ is a subalgebra of $C_n$ and $C_n = C_n^0 + C_n^1$ (direct sum). Let $\iota$ be an automorphism of $C_n$ defined by $x^0 + x^1 \mapsto x^0 - x^1$ for $x^i \in C_n^i$ $(i = 0, 1)$, $x \mapsto x^t$ be an anti-automorphism of $C_n$ defined by $e_{i_1}e_{i_2}\cdots e_{i_r} \to e_{i_r}\cdots e_{i_2}e_{i_1}$ $(1 \leqq i_1 < i_2 < \cdots < i_r \leqq n)$. Put $\bar{x} = \iota(x^t)$ for $x \in C_n$ and define the *norm* $\nu$ of $x \in C_n$ by $\nu(x) = \bar{x}x$. We shall identify $\boldsymbol{R}^n$ with the subspace of $C_n$ spanned by $e_1, \cdots, e_n$ and $\boldsymbol{R}$ with that spanned by 1. Then we have a natural homomorphism $p$ from the *twisted Clifford group* $\Gamma_n$ defined by

$$\Gamma_n = \{s \in C_n^* \ ;\ \iota(s)\boldsymbol{R}^n s^{-1} \subset \boldsymbol{R}^n\}$$

into $GL(n, \boldsymbol{R})$. It is known that the norm $\nu$ induces a homomorphism $\nu : \Gamma_n \to \boldsymbol{R}^*$. We put

$$\mathrm{Pin}\,(n) = \{s \in \Gamma_n \ ;\ |\nu(s)| = 1\}$$

and

$$\mathrm{Spin}\,(n) = \mathrm{Pin}\,(n) \cap C_n^0 .$$

Both Pin $(n)$ and Spin $(n)$ are compact Lie groups with respect to the topology induced by that of $C_n$. For $n \geqq 2$, Spin $(n)$ is the connected component of Pin $(n)$ and Pin $(n)$/Spin $(n) \cong \boldsymbol{Z}_2$. For $n \geqq 3$, Spin $(n)$ is simply connected. We have the following exact sequences:

$$1 \longrightarrow \boldsymbol{R}^* \longrightarrow \Gamma_n \xrightarrow{p} O(n) \longrightarrow 1$$

$$1 \longrightarrow \boldsymbol{Z}_2 \longrightarrow \mathrm{Pin}\,(n) \xrightarrow{p} O(n) \longrightarrow 1$$

$$1 \longrightarrow \boldsymbol{Z}_2 \longrightarrow \mathrm{Spin}\,(n) \xrightarrow{p} SO(n) \longrightarrow 1$$

where $\boldsymbol{Z}_2=\{\pm 1\}\subset \boldsymbol{R}^*$. Our $\mathrm{Pin}\,(n)$ is slightly different from $\mathrm{Pin}\,(n)$ in [2], which was defined by the Clifford algebra associated with the *negative* definite quadratic form. For example, our $\mathrm{Pin}\,(1)$ is isomorphic with $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$, while $\mathrm{Pin}\,(1)$ in [2] is isomorphic with $\boldsymbol{Z}_4$.

If $n=2l+1$ is odd, we have isomorphisms

$$O(2l+1)\cong SO(2l+1)\times \boldsymbol{Z}_2$$

and

$$\mathrm{Pin}\,(2l+1)\cong \mathrm{Spin}\,(2l+1)\times \boldsymbol{Z}_2$$

so that we shall confine ourselves to consider $O(2l)$ or $\mathrm{Pin}\,(2l)$.

Let first $G=O(2l)$ $(l\geqq 1)$. Then $G_0=SO(2l)$ and $A\cong \boldsymbol{Z}_2$.

$$T_0=\left\{\begin{pmatrix} r(t_1) & & \\ & \ddots & \\ & & r(t_l)\end{pmatrix};\quad t_i\in \boldsymbol{R}\right\}$$

where

$$r(t_i)=\begin{pmatrix} \cos 2\pi t_i & -\sin 2\pi t_i \\ \sin 2\pi t_i & \cos 2\pi t_i\end{pmatrix}$$

is a maximal torus of $G_0$. The Lie algebra $\mathfrak{t}$ of $T_0$ is identified with

$$\mathfrak{t}=\left\{H(x_1,\cdots,x_l)=\begin{pmatrix} R(x_1) & & \\ & \ddots & \\ & & R(x_l)\end{pmatrix};\quad x_i\in \boldsymbol{R}\right\}$$

where

$$R(x_i)=\begin{pmatrix} 0 & -2\pi x_i \\ 2\pi x_i & 0\end{pmatrix}.$$

The linear form on $\mathfrak{t}$ taking value $x_i$ at $H(x_1,\cdots,x_l)$ will be denoted by $x_i$ $(1\leqq i\leqq l)$. Then the root system is

$$\Delta=\{\pm(x_i\pm x_j)\,;\ 1\leqq i<j\leqq l\}$$

and

$$\Pi=\{\alpha_i=x_i-x_{i+1}\ (1\leqq i\leqq l-1),\ \alpha_l=x_{l-1}+x_l\}$$

is a fundamental system of $\Delta$. The order $x_1>\cdots>x_l>0$ is a linear order associated with $\Pi$. We have

$$D_0 = \{\sum m_i x_i\,;\ m_i \in \boldsymbol{Z},\ m_1 \geqq m_2 \geqq \cdots \geqq m_{l-1} \geqq |m_l|\}\,.$$

$\tau: A \to C$ is an isomorphism and $C \cong \boldsymbol{Z}_2$ is generated by the transformation $\tau_0$ of $\mathfrak{t}^*$ defined by $\tau_0 x_i = x_i$ $(1 \leqq i \leqq l-1)$ and $\tau_0 x_l = -x_l$. Thus the set

$$\{\sum m_i x_i\,;\ m_i \in \boldsymbol{Z},\ m_1 \geqq m_2 \geqq \cdots \geqq m_{l-1} \geqq m_l \geqq 0\}$$

is a complete set of representatives of $D = C \backslash D_0$. If we put $\Lambda_i = C(x_1 + \cdots + x_i)$ $(1 \leqq i \leqq l)$, i.e.

$$\Lambda_i = \begin{cases} \{x_1 + \cdots + x_i\} & 1 \leqq i \leqq l-1 \\ \{x_1 + \cdots + x_{l-1} + x_l,\ x_1 + \cdots + x_{l-1} - x_l\} & i = l\,, \end{cases}$$

then $\{\Lambda_1, \cdots, \Lambda_m\}$ is a system of generators of the semigroup $D$. Let $\rho_0 \in D(O(2l))$ be the equivalence class of the standard representation of $O(2l)$, $\lambda^i(\rho_0) \in D(O(2l))$ the $i$-th exterior power of $\rho_0$. Then

$$i^*_{SO(2l)} \lambda^i(\rho_0) = \begin{cases} \rho_{x_1 + \cdots + x_i} & 1 \leqq i \leqq l-1 \\ \rho_{x_1 + \cdots + x_{l-1} + x_l} + \rho_{x_1 + \cdots + x_{l-1} - x_l} & i = l \end{cases}$$

so that $\varphi(\delta_{\lambda^i(\rho_0)}) = \Lambda_i$ $(1 \leqq i \leqq l)$. Moreover the determinant representation $\alpha \in D(O(2l))$ generates $\hat{A}$. It follows from Theorem 5 that $R(O(2l))$ is generated by $\lambda^1(\rho_0), \cdots, \lambda^l(\rho_0), \alpha$. More precisely we have

THEOREM 7.

$$R(O(2l)) = \boldsymbol{Z}[\lambda^1(\rho_0), \cdots, \lambda^l(\rho_0), \alpha]$$

*with relations* $\alpha^2 = 1$ *and* $\lambda^l(\rho_0)\alpha = \lambda^l(\rho_0)$.

PROOF. Let $R = \boldsymbol{Z}[\lambda^1(\rho_0), \cdots, \lambda^l(\rho_0)]$ be the subring of $R(O(2l))$ generated by $\{\lambda^i(\rho_0)\,;\ 1 \leqq i \leqq l\}$. Then $R(O(2l))$ is generated by $\alpha$ over $R$.

The highest components of $j_0^* i^*_{SO(2l)} \lambda^i(\rho_0)$ are $e^{2\pi\sqrt{-1}\lambda}$ for $\lambda = x_1 + \cdots + x_i$ $(1 \leqq i \leqq l)$, which are linearly independent. It follows from Theorem 6 that $\{i^*_{SO(2l)} \lambda^i(\rho_0)\,;\ 1 \leqq i \leqq l\}$ has no relations in $R(SO(2l))$. Therefore $\{\lambda^i(\rho_0)\,;\ 1 \leqq i \leqq l\}$ has no relations and the homomorphism $i^*_{SO(2l)}$ is injective on $R$.

Thus it remains to prove that the ideal

$$I = \{F \in R[X]\,;\ F(\alpha) = 0\}$$

of $R[X]$ is generated by $X^2 - 1$ and $\lambda^l(\rho_0)X - \lambda^l(\rho_0)$. Since the first polynomial clearly belongs to $I$ and the second belongs to $I$ in view of Theorem 2, 2), it suffices to show that if $F = fX + g$ $(f, g \in R)$ is a polynomial in $I$ with degree 1, then $g = -f$ and $f$ is divisible by $\lambda^l(\rho_0)$. From $0 = i^*_{SO(2l)} F(\alpha) = i^*_{SO(2l)} f + i^*_{SO(2l)} g = i^*_{SO(2l)}(f + g)$ and that $i^*_{SO(2l)}$ is injective on $R$, we have $g = -f$ and thus $f\alpha = f$. Let $f = h + k\lambda^l(\rho_0)$, where $h \in \boldsymbol{Z}[\lambda^1(\rho_0), \cdots, \lambda^{l-1}(\rho_0)]$ and $k \in R$. $f\alpha = f$ implies $h\alpha = h$. We shall show that $h = 0$. Suppose that

$$h=\sum a_{n_1\cdots n_{l-1}}\lambda^1(\rho_0)^{n_1}\cdots\lambda^{l-1}(\rho_0)^{n_{l-1}} \qquad (a_{n_1\cdots n_{l-1}}\in \boldsymbol{Z})$$

is not zero. Let $\sum_{i=0}^{l-1} n_i^0(x_1+\cdots+x_i)$ be the highest among the $\sum_{i=1}^{l-1} n_i(x_1+\cdots+x_i)$ such that $a_{n_1\cdots n_{l-1}}\neq 0$. It follows from Lemma 4 that $j^*h$ has the strictly highest component, say $\delta$, with the multiplicity 1 such that $\Lambda_\delta=\left\{\sum_{i=1}^{l-1} n_i^0(x_1+\cdots+x_i)\right\}$. On the other hand, $h\alpha=h$ implies that $\alpha\delta=\delta$, which contradicts $|\Lambda_\delta|=1$ in view of Theorem 2, 2). q. e. d.

Now let $G=\mathrm{Pin}(2l)$ $(l\geqq 1)$. Then $G_0=\mathrm{Spin}(2l)$ and $A=G/G_0\cong \boldsymbol{Z}_2$. We have $\nu=\alpha\circ p$ and $\hat{A}$ is generated by $\nu$. Let $\hat{\rho}_0\in D(\mathrm{Pin}(2l))$ be the equivalence class of the covering homomorphism $p:\mathrm{Pin}(2l)\to O(2l)$ and $\lambda^i(\hat{\rho}_0)\in D(\mathrm{Pin}(2l))$ the $i$-th exterior power of $\hat{\rho}_0$. Take an irreducible $C_{2l}^{0\boldsymbol{C}}$-module $M_0$ and let $M=C_{2l}^{\boldsymbol{C}}\otimes_{C_{2l}^{0\boldsymbol{C}}} M_0$, where $C_{2l}^{0\boldsymbol{C}}$ (resp. $C_{2l}^{\boldsymbol{C}}$) denotes the complexification of $C_{2l}^0$ (resp. of $C_{2l}$). Then $M$ is an irreducible $\mathrm{Pin}(2l)$-module, whose equivalence class will be denoted by $\mu^l$. The restriction $i_{\mathrm{Spin}(2l)}^*\mu^l$ is the sum of two half-spinor representations of $\mathrm{Spin}(2l)$. Then we have the following theorem in the same way as for $O(2l)$, but replacing $x_1+\cdots+x_{l-1}\pm x_l$ by $\frac{1}{2}(x_1+\cdots+x_{l-1}\pm x_l)$.

THEOREM 8. *$R(\mathrm{Pin}(2l))=\boldsymbol{Z}[\lambda^1(\hat{\rho}_0),\cdots,\lambda^{l-1}(\hat{\rho}_0),\mu^l,\nu]$ with relations $\nu^2=1$ and $\mu^l\nu=\nu$.*

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