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Some improved upper and lower bounds are given for the excess of Hadamard matrices. The excess of orthogonal designs is defined and discussed.

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A REMARK ON THE EXCESS OF HADAMARD MATRICES AND ORTHOGONAL DESIGNS

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ABSTRACT. Some improved upper and lower bounds are given for the excess of Hadamard matrices. The excess of orthogonal designs is defined and discussed.

An *orthogonal design* $D = y_1Y_1 + y_2Y_2 + \dots + y_tY_t$ of order n and type $(s_1, \dots, s_t)$ on the commuting variables $y_1, \dots, y_t$ is a square matrix satisfying

$$DD^T = \sum_{j=1}^{t} s_j y_j^2 .$$

Each Y_j is a $(0,1,-1)$-matrix satisfying $Y_jY_j^T = s_jI_n$ and is called a *weighing matrix* of weight s_j. A weighing matrix of order n and weight n is called an *Hadamard matrix*.

We call the sum of the entries of an Hadamard matrix, H, the *excess* of H. We denote by $\sigma(n)$ the maximum possible sum of an Hadamard matrix of order n. We define the *excess of the orthogonal design* D as

$$\sigma(D) = \sigma(Y_1) + \dots + \sigma(Y_t) ,$$

where $\sigma(Y_i)$ is the sum of the entries of Y_i.

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Wallis [4] and Schmidt and Wang [2] defined the weight of an Hadamard matrix H, denoted $w(H)$, to be the number of ones in H. $W(n)$ is used for the maximum value of $w(H)$ for Hadamard matrices of order n. $W(n) = \frac{1}{2}\left(n^2 + \sigma(n)\right)$. The excess was studied by Best [1].

Wang and Schmidt found

$$(1) \qquad \sigma(n) \le 2\left[\tfrac{1}{2}n\sqrt{2n+1}\right] - n .$$

This was improved by Best who found

$$\sigma(n) \le n\sqrt{n} .$$

1. *A construction*.

Here are four slightly different ways to construct an orthogonal design of type (t,t,t,t) in order $4t$ using 4 circulant $(0,1,-1)$-matrices X_1, X_2, X_3, X_4 which satisfy

$$(2) \qquad \begin{cases} \text{(i)} & \sum_{i=1}^{4} X_i X_i^T = tI_t , \\ \text{(ii)} & X_i J = x_i J , \\ \text{(iii)} & X_i * X_j = 0 , \qquad i \ne j . \\ \text{(iv)} & \sum_{i=1}^{4} X_i \text{ is a } (1,-1)\text{-matrix} , \\ \text{(v)} & x_1^2 + x_2^2 + x_3^2 + x_4^2 = t . \end{cases}$$

Let y_1, y_2, y_3, y_4 be commuting variables and

$$U = \begin{bmatrix} -y_1 & y_2 & y_3 & y_4 \\ y_2 & y_1 & y_4 & -y_3 \\ y_3 & -y_4 & y_1 & y_2 \\ y_4 & y_3 & -y_2 & y_1 \end{bmatrix} = (u_{ij}) , \quad V = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & -y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_4 & -y_3 & y_2 & -y_1 \end{bmatrix} = (v_{ij}) .$$

Write

$$G = \begin{bmatrix} -A_1 & A_2R & A_3R & A_4R \\ A_2R & A_1 & A_4^TR & -A_3^TR \\ A_3R & -A_4^TR & A_1 & A_2^TR \\ A_4R & A_3^TR & -A_2^TR & A_1 \end{bmatrix} , \quad H = \begin{bmatrix} A_1 & A_2R & A_3R & A_4R \\ -A_2R & A_1 & A_4^TR & -A_3^TR \\ -A_3R & -A_4^TR & A_1 & A_2^TR \\ -A_4R & A_3^TR & -A_2^TR & A_1 \end{bmatrix}$$

where R is the back diagonal matrix and A_1, A_2, A_3, A_4 are circulant matrices.

Now we can form A_i by either choosing

$$A_i = \sum_{k=1}^{4} u_{ik}X_k \quad \text{or} \quad A_i = \sum_{k=1}^{4} v_{ik}X_k , \qquad i = 1,2,3,4 .$$

The different orthogonal designs now arise by using G and H.

We further observe that if circulant X_1, X_2, X_3, X_4 of order t, exist with first rows $a_{i1}, a_{i2}, \ldots, a_{it}$ respectively, then the circulant matrices $\underline{X}_1, \underline{X}_2, \underline{X}_3, \underline{X}_4$ of order $2t$ with first rows

$$a_{11}a_{2t}a_{12}a_{2,t-1}a_{13} \cdots a_{22}a_{1t}a_{21} ,$$

$$\bar{a}_{21}a_{1t}\bar{a}_{22}a_{1,t-1}\bar{a}_{23} \cdots a_{12}\bar{a}_{2t}a_{11} ,$$

$$a_{31}a_{4t}a_{32}a_{4,t-1}a_{33} \cdots a_{42}a_{3t}a_{41} ,$$

$$\bar{a}_{41}a_{3t}\bar{a}_{42}a_{3,t-1}\bar{a}_{43} \cdots a_{32}\bar{a}_{4t}a_{31} ,$$

where $\bar{a}_{ij}$ denotes $-a_{ij}$, satisfy the similar equations

(i) $\sum_{i=1}^{4} \underline{X}_i\underline{X}_i^T = 2tI_{2t}$,

(ii) $\underline{X}_1J = (x_1 + x_2)J$, $\underline{X}_2J = (x_1 - x_2)J$,

$\underline{X}_3J = (x_3 + x_4)J$, $\underline{X}_4J = (x_3 - x_4)J$,

(iii) $\underline{X}_i * \underline{X}_j = 0$, $i \neq j$,

(iv) $\sum_{i=1}^{4} \underline{X}_i$ is a (1,-1)-matrix ,

(v) $(x_1 + x_2)^2 + (x_1 - x_2)^2 + (x_3 + x_4)^2 + (x_3 - x_4)^2 = 2t$.

We consider the excess of the different orthogonal designs D_1, D_2, D_3, D_4 and find for $D_i = y_1Y_{1i} + y_2Y_{2i} + y_3Y_{3i} + y_4Y_{4i}$

$$\sigma(D_i) = \sigma(Y_{1i}) + \sigma(Y_{2i}) + \sigma(Y_{3i}) + \sigma(Y_{4i}) .$$

Now $\sigma(G) = 2\sigma(A_1) + 2\sigma(A_2) + 2\sigma(A_3) + 2\sigma(A_4)$ and $\sigma(H) = 4\sigma(A_1)$.

Using the two methods for forming A_i we obtain

$$\sigma(D_1) = 4t(x_1 + x_2 + x_3 + x_4) ,$$

$$\sigma(D_2) = 8tx_1 ,$$

$$\sigma(D_3) = 8tx_1 ,$$

$$\sigma(D_4) = 4t(x_1 + x_2 + x_3 + x_4) .$$

Hence we obtain

(3) $$\sigma(4t) \geq 4t \max(2x_1, x_1+x_2+x_3+x_4) .$$

To see both are needed we consider

$t = 5 = 2^2 + 1^2 + 0 + 0$, $\sigma(20) \geq 20 \max(4,3) = 80$,

$t = 7 = 2^2 + 1^2 + 1^2 + 1^2$, $\sigma(28) \geq 28 \max(4,5) = 140$.

We note that for $t = s^2 + 0^2 + 0^2 + 0^2$

(4) $$\sigma(4s^2) \geq 8s^3 ,$$

and when the A_i are constructed using U and G or V and H we have

$$\sigma(A_1) = \sigma(A_2) = \sigma(A_3) = \sigma(A_4) = 2\sigma(X_1) .$$

LEMMA 1. *Suppose the matrices* X_1, X_2, X_3, X_4 *described above in (2) exist for order* t *with* $t = x_1^2 + x_2^2 + x_3^2 + x_4^2$. *Then there is an orthogonal design* D *of order* 4t *and type* (t,t,t,t) *with* $\sigma(4t) \ge \sigma(D) = 4t\max(2x_1, x_1+x_2+x_3+x_4)$, *where* $x_1 \ge x_2 \ge x_3 \ge x_4$. *Further if* $t = s^2$ *and* $x_1 = s$, $\sigma(D) = 8s^3$ *and* $\sigma(Y_1) = 2s^3$. *Also* $\sigma(8t) \ge 16t(x_1 + x_2)$.

A similar process, using 4 circulant (1,-1)-matrices A, B, C, D of order t with row sums x_1, x_2, x_3, x_4 allows us to say:

LEMMA 2. *Let* $n = 4t = x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv 4 \pmod 8$ *be the order of an Hadamard matrix,* H, *constructed using four circulant matrices with row sums* x_1, x_2, x_3, x_4 *respectively,* $x_1 \ge x_2 \ge x_3 \ge x_4$. *Then*

$$\sigma(H) = 2t(x_1 + x_2 + x_3 + x_4) \quad \textit{and} \quad \sigma(2H) = 8t(x_1 + x_2),$$

$$\sigma(2^{2r}H) = 2^{3r-1}n(x_1 + x_2 + x_3 + x_4) \quad \textit{and}$$

$$\sigma(2^{2r+1}H) = 2^{3r+1}n(x_1 + x_2).$$

Some suitable first rows to use in Lemmas 1 and 2 are given in Table 1, others can be found in J. Wallis [3]. So we have $\sigma(12) \ge 36$, $\sigma(16) \ge 64$, $\sigma(20) \ge 80$ and $\sigma(24) \ge 96$.

t	W(H)	σ(H)	A	B	C	D
3	90	36	111	1--	1--	1--
4	160	64	-111	-111	-111	-111
5	240	80	1-11-	11--1	-1111	-1111
6	336	96	111-1-	1-1111	11----	1--1-1

Table 1

In particular, using $a_1 = a_2 = a_3 = a_4 = 2$ and H of order $n = 4$ we obtain

(5) $$\begin{cases} \sigma(2^{2r}) \geq 2^{3r-1} , \\ \sigma(2^{2r+1}) \geq 2^{3r} . \end{cases}$$

We also see

(6) $$\begin{cases} \sigma(2^{2r}.3) \geq 2^{3r-1}.9 , \\ \sigma(2^{2r+1}.3) \geq 2^{3r+2}.9 . \end{cases}$$

(7) $$\begin{cases} \sigma(2^{2r}.5) \geq 2^{3r+1}.5 , \\ \sigma(2^{2r+1}.5) \geq 2^{3r+1}.15 . \end{cases}$$

If $p \equiv 1 \pmod 4$ is a prime power, then $p = x^2 + y^2$. Moreover, a suitable Hadamard matrix can be found [3, p.315] of order $2(p+1)$ with row sums

$$a_1 = x + 1, \quad a_2 = x - 1, \quad a_3 = a_4 = y .$$

Hence

$$\sigma\big(2(p+1)\big) \geq 2(p+1)(x+y) ,$$

which gives $\sigma(12) \geq 36$, $\sigma(28) \geq 140$, $\sigma(52) \geq 364$ and $\sigma(60) \geq 420$.

2. *An Upper Bound.*

Let $a_i \geq 0$ be the column sum of the ith column of a weighing matrix, W, of order n and weight k. Then $WW^T = kI_n$. Let e be the $1 \times n$ matrix of ones. Then

$$eWW^Te^T = ekI_ne^T = nk$$

$$= (a_1 \ \dots \ a_n)\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} a_i^{\,2} .$$

Now to find the maximum excess $\sigma(W)$ we wish to maximize $\sum_{i=1}^{n} a_i$ subject to $\sum_{i=1}^{n} a_i^2 = nk$. We use Schwarz' inequality to see that the maximum is for

$$a_1 = a_2 = \ldots = a_n = \sqrt{k},$$

so the maximum excess is

$$\sigma(W) = \sum_{i=1}^{n} a_i = n\sqrt{k}.$$

Since this is the maximum, we have for any weighing matrix and Hadamard matrix

$$(8) \qquad \sigma(W) \le n\sqrt{k} \qquad \text{and} \qquad \sigma(n) \le n\sqrt{n}.$$

This improves Schmidt and Wang's bound (1) and is that given by Best. In particular, we have for Hadamard matrices of order $4s^2$

$$\sigma(4s^2) \le 8s^3.$$

Compare this with (4).

Now one of us (Seberry) has shown:

THEOREM 3. *Given any natural number* $q > 3$ *there exists a regular symmetric Hadamard matrix with constant diagonal of order* $2^{2s}q^2$ *for every* $s \ge \lceil 2 \log_2(q-3) \rceil$.

So we have, using a result of Best:

COROLLARY 4. *Let* $q > 3$ *be any natural number and* $s \geq 2\lceil \log_2(q-3) \rceil$ *be an integer. Then*

$$\sigma(2^{2s}q^2) = 2^{3s}q^4 .$$

If $\sqrt{k}$ is not an integer we use a lattice point nearest $(a_1, \ldots, a_n) = \sqrt{k}(1, \ldots, 1)$. For the next discussion we confine ourselves to $k = n$. Then, with x the greatest even integer $< \sqrt{n}$,

$$a_1 = a_2 = \ldots = a_i = t ,$$

and

$$a_{i+1} = a_{i+2} = \ldots = a_n = t + 4 ,$$

where $t = x$ if $|n-x^2| < |(x+2)^2-n|$ and $t = x-2$ otherwise. Now

$$a_1{}^2 = a_2{}^2 = \ldots = a_i{}^2 = t^2 < n$$

and

$$a_{i+1}{}^2 = \ldots = a_n{}^2 = (t+4)^2 > n .$$

So
$$\sum_{j=1}^{n} a_j{}^2 = it^2 + (n-i)(t+4)^2 = n^2 ,$$

that is
$$i = \frac{n\left((t+4)^2 - n\right)}{8(t+2)} .$$

This value of i may not be integral and must be an integer, so we choose the integer part of $\frac{n((t+4)^2 - n)}{8(t+2)}$ for i. We note we increase $\sum a_j$ by choosing i smaller. Thus

$$\sigma(n) = \sum_{j=1}^{n} a_j = ti + (t+4)(n-i) ,$$
$$= n(t+4) - 4\left[\frac{n((t+4)^2 - n)}{8(t+2)}\right] .$$

Hence

$$(9) \qquad W(n) \le \tfrac{1}{2}n(t+n+4) - 2\left[\frac{n((t+4)^2 - n)}{8(t+2)}\right] ,$$

where x is the greatest integer $\equiv 0 \pmod 2$ and $\le \sqrt{n}$, $t = x$ if $|n-x^2| < |(x+2)^2-n|$ and $t = x-2$ otherwise.

3. *A lower Bound.*

An examination of Best's lower bound shows that we can exclude the rows of the Hadamard matrix of order n and so obtain

$$\sigma(n) \ge \frac{n^2\left(\binom{n}{\frac{1}{2}n} - 2\right)}{2^n - 2n} .$$

While we were always able to improve on this bound constructively, it is, nevertheless, the best general lower bound available.

4. *The Powers of 2.*

Schmidt and Wang noted that if H and G are Hadamard matrices of orders h and g, respectively, and maximal weights, then

$$(10) \quad W(hg) \geq w(H\times G) = h^2g^2 + 2W(g)W(h) - h^2W(g) - g^2W(h) .$$

THEOREM 5.

$$\sigma(2^{2r}) = 2^{3r} \qquad \text{i.e.} \qquad W(2^{2r}) = 2^{2r-1}(2^{2r} + 2^r) .$$

Proof. Let $X_1=[1]$, $X_2 = X_3 = X_4 = [0]$ be circulant matrices of order 1. Then by repeated use of the doubling construction of section 1 we obtain four circulant matrices of order $t = 2^{2r-2}$ with row sums 2^{r-1}, 0, 0, 0 respectively and by (3) we have

$$\sigma(2^{2r}) \geq 2^{3r} .$$

By equation (8) we have

$$\sigma(2^{2r}) \leq 2^{3r} .$$

So we have the result.

LEMMA 6.

$$2^{3r-1} + 2^{4r+1} \le W(2^{2r+1})$$

$$\le 2^{2r}(t + 4 + 2^{2r+1}) - 2\left[\frac{2^{2r-2}((t+4)^2 - 2^{2r+1})}{t+2}\right],$$

where $t = [2^r\sqrt{2}]$ *or* $[2^r\sqrt{2}] - 3$ *according as* $[2^r\sqrt{2}]$ *is even or odd.*

Proof. The right-hand-side follows immediately from (9). Also we have

$$W(2^{2r+1}) = 2W(2^a)(W(2^{2r+1-a}) - 2^{4r+1-2a})$$

$$+ 2^{2a}(2^{4r+2-2a} - W(2^{2r+1-a})) .$$

Since we know $W(2^{2s})$ exactly, we use $a = 2p\pm1$ when $2r+1 = 4p\pm1$ in (10) to obtain

(11) $$W(2^{2r+1}) \ge 2^{3[\frac{1}{2}(r+1)]}W(2^{2[r/2]+1}) + 2^{4r+1}$$

$$- 2^{4r+1-[\frac{1}{2}(r+1)]} .$$

Now we use (5) to obtain the quoted lower bound.

COROLLARY 7. *Bounds on* $W(2^{2r+1})$ *are given by the table*

r	$x \le W(2^{2r+1}) \le y$				
1	42	=	W(8)		
2	592	≤	W(32)	≤	600
3	8832	≤	W(128)	≤	8910
4	136192	≤	W(512)	≤	136854
5	2138112	≤	W(2048)	≤	2143466

Proof. The lower bounds for W(32) and W(128) are found using W(8) and (11), while those for W(512) and W(2048) are found using W(32) and (11).

5. *The Excess of Orthogonal Designs.*

From the discussion at the beginning of section 2 we see that a weighing matrix of order n with k non-zero entries per row has excess $n\sqrt{k}$.

Thus an orthogonal design $D = y_1Y_1 + y_2Y_2 + \dots + y_tY_t$ of type $(s_1, s_2, \dots, s_t)$ would have maximal excess

$$\sigma(D) = \sigma(Y_1) + \dots + \sigma(Y_t) \le n\sqrt{s_1} + n\sqrt{s_2} + \dots + n\sqrt{s_t},$$

while the orthogonal design $\underline{D} = y(Y_1 + \dots + Y_t)$ obtained by equating all the variables has maximal excess

$$\sigma(\underline{D}) = n\sqrt{s_1 + \dots + s_t} \ .$$

Clearly, in general,

$$\sigma(\underline{D}) = n\sqrt{s_1 + \dots + s_t} \le n\sqrt{s_1} + \dots + n\sqrt{s_t} \ .$$

So the question arises:

QUESTION. *Let* $D = y_1 Y_1 + \dots + y_t Y_t$ *be an orthogonal design of order* n *and type* $(s_1, \dots, s_t)$ *on the commuting variables* $y_1, \dots, y_t$. *For what types and orders of orthogonal design is the excess the same as the excess of the weighing matrix of order* n *and weight* $s_1 + \dots + s_t$ *?*

As a partial answer we see the following orthogonal designs of orders 4 and 8 and types (1,1,1,1) and (1,1,1,1,1,1,1,1), respectively, have the same excess as the underlying Hadamard matrix:

$$\begin{bmatrix} -a & b & c & d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{cccc|cccc} -a & b & c & d & e & f & g & h \\ b & a & d & -c & f & -e & h & -g \\ c & -d & a & b & -g & h & e & -f \\ d & c & -b & a & h & g & -f & -e \\ \hline e & -f & g & -h & a & b & -c & d \\ f & e & -h & -g & -b & a & d & c \\ g & -h & -e & f & c & -d & a & b \\ h & g & f & e & -d & -c & -b & a \end{array}\right]$$

Replacing the variables of this orthogonal design of order eight by circulant matrices with first rows 111, for a and b and -11 for c, d, e, f, g, h gives an Hadamard matrix of order 24 with $\sigma(24) = 108$. Replacing the variables by the circulant matrices with first rows 1111111 for a and 111-1-- for b, c, d, e, f, g, h gives an Hadamard matrix of order 56 with $\sigma(56) = 392$). Also the orthogonal designs of order 4t, t odd and square free, and type (t,t,t,t) constructed via Lemma 1 have the same excess as the underlying Hadamard matrix for t = 3, 5, 7 and 13 and give the best known lower bound for t = 11 and 15.

This suggests another possible conjecture:-

$$\sigma(4t) = 4t \max(2x_1, x_1+x_2+x_3+x_4) ,$$

where $t = x_1^2 + x_2^2 + x_3^2 + x_4^2$, $x_1 \ge x_2 \ge x_3 \ge x_4$, t odd and square-free.

6. *Numerical Results*.

Previously Wallis had found $W(8) = 42$, $W(12) = 90$ and $W(20) = 240$. Theorem 5 of Wallis [2] gave $W(16) \ge 154$. The results of Wallis [4] and Schmidt and Wang [2] gave

$$334 \le W(24) \le 360$$
$$756 \le W(36)$$
$$866 \le W(40) \le 960 .$$

Best found $W(24) = 344$. We find

n	Best's conjecture for $\sigma(n)$	$W(n)$	$\sigma(n)$	Calculated $W(n)$
4			2	3
8	20		20	42
12	36		36	90
16	64		64	160
20	80		80	240
24	108	342	112	344
28	140	462	140	462
32	176	600		$592 \le W(32) \le 600$
36	216	756	216	756
40	240	920	240	920
44	264	1100		$1100 \le W(44) \le 1108$
48	312	1308		$1296 \le W(48) \le 1312$
52	364	1534	364	1534
56	392	1764	392	$1764 \le W(56) \le 1774$
60	420	2010		$2010 \le W(60) \le 2030$
64	512	2304	512	2304

Best conjectured

$$W(n) = \tfrac{1}{4}\left(n^2 + \sigma(n)\right) \quad \begin{aligned} &= \tfrac{1}{4}\left(n^2 + \tfrac{1}{2}n[2\sqrt{n}]\right), & n \equiv 0 \pmod 8, \\ &= \tfrac{1}{4}\left(n^2 + n[\sqrt{n}]\right), & n \equiv 4 \pmod 8. \end{aligned}$$

Comparison with the above results shows this is an excellent conjecture.

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