QUARTERLY OF APPLIED MATHEMATICS
VOLUME LX, NUMBER 2
JUNE 2002, PAGES 245-250

A REMARK ON THE EXISTENCE OF GLOBAL BV SOLUTIONS FOR A NONLINEAR HYPERBOLIC WAVE EQUATION

By

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Abstract. By means of a suitable change of variables we obtain, by application of a general result by Dafermos and Hsiao, cf. [2], an existence theorem in $L^\infty \cap BV_{\text{loc}}$ of a weak solution of the system corresponding to the quasilinear hyperbolic equation

$$\phi_{tt} - p'(\phi_x)\,\phi_{xx} + \phi_t + F(\phi) = 0 \quad \text{in } \mathbb{R} \times [0, +\infty[\ ,$$

for small initial data in BV. This theorem is a partial extension of Dafermos's result for the case with $F(\phi) \equiv 0$, proved in [1].

1. The auxiliary system. Let us consider the following Cauchy problem:

$$\overline{\phi}_{tt} - p'(\overline{\phi}_x)\,\overline{\phi}_{xx} + \overline{\phi}_t + \overline{F}(\overline{\phi}) = 0\ , \quad (x,t) \in \mathbb{R} \times [0, +\infty[\ , \tag{1.1}$$

$$\overline{\phi}(x,0) = \overline{\phi}_0(x), \quad \overline{\phi}_t(x,0) = \overline{v}_0(x)\ , \quad x \in \mathbb{R}\ , \tag{1.2}$$

where p is a given smooth function such that $p'(\xi) > 0$, $\forall\, \xi \in \mathbb{R}$, and $\overline{F} : \mathbb{R} \to \mathbb{R}$ is a smooth function verifying $\overline{F}(0) = 0$. We assume, to simplify, the condition $p'(0) = 1$. Putting $u = \phi_x$, $v = \phi_t$ we can write (1.1), (1.2) as a Cauchy problem for a hyperbolic system:

$$\begin{cases} \overline{\phi}_t = \overline{v} \\ \overline{u}_t = \overline{v}_x \\ \overline{v}_t - p'(\overline{u})\,\overline{u}_x + \overline{v} + \overline{F}(\overline{\phi}) = 0 \end{cases} \qquad (x,t) \in \mathbb{R} \times [0, +\infty[\ , \tag{1.3}$$

$$\overline{\phi}(x,0) = \overline{\phi}_0(x), \quad \overline{u}(x,0) = \overline{u}_0(x) = \overline{\phi}_{0x}(x), \quad \overline{v}(x,0) = \overline{v}_0(x)\ , \qquad x \in \mathbb{R}\ . \tag{1.4}$$

For technical reasons, if we choose $k > 1$, we can, by putting $u(x,t) = \overline{u}(kx, kt)$, $v(x,t) = \overline{v}(kx, kt)$, $\phi(x,t) = \frac{1}{k}\,\overline{\phi}(kx, kt)$, replace (1.3), (1.4) by

$$\begin{cases} \phi_t = v \\ u_t = v_x \\ v_t - p'(u)\,u_x + k\,v + F(\phi) = 0 \end{cases} \qquad (x,t) \in \mathbb{R} \times [0, +\infty[\ , \tag{1.3'}$$

Received March 14, 2000.
2000 *Mathematics Subject Classification.* Primary 35L70.

where $F(\phi) = k\,\overline{F}(k\phi)$,

$$\begin{aligned} \phi(x,0) &= \phi_0(x) = \frac{1}{k}\,\overline{\phi}_0(kx)\ , \\ u(x,0) &= u_0(x) = \overline{u}_0(kx) = \phi_{0x}(x)\ , \\ v(x,0) &= v_0(x) = \overline{v}_0(kx)\ , \qquad x \in \mathbb{R}\ . \end{aligned} \tag{1.4'}$$

For $m \in [1, +\infty[$, let us introduce

$$\begin{aligned} a &= a_m = (m^2 - 2m + 2)^{-1/2}\ , \\ b &= b_m = (m^2 + 2m + 2)^{-1/2}\ , \\ c_m &= 2 - \Big(\frac{a}{b} + \frac{b}{a}\Big)/2\ , \\ k_m &= 2\,c_m^{-1}\Big(a(m-1) + b(m+1)\Big)\ , \\ f_m &= c_m^2\left[2\Big(\frac{1}{a} + \frac{1}{b}\Big)\Big(a(m-1) + b(m+1)\Big)\right]^{-1} \end{aligned}$$

(notice that $c_1 > 0$ and $k_1 > 1$) and assume

$$|\overline{F}'(0)| < f_1\ . \tag{1.5}$$

We fix $m > 1$ such that $|\overline{F}'(0)| < f_m$, $c_m > 0$, and $k_m > 1$ and we put $k = k_m$. We obtain

$$\begin{aligned} &\left(2 - \Big(\frac{a}{b} + \frac{b}{a}\Big)\,/\,2\right) k - \Big(\frac{1}{2a} + \frac{1}{2b}\Big)\,|F'(0)| - a(m-1) - b(m+1) \\ &\quad = c_m\,k_m - \Big(\frac{1}{2a} + \frac{1}{2b}\Big)\,k_m^2\,|\overline{F}'(0)| - a(m-1) - b(m+1) > 0\ . \end{aligned} \tag{1.6}$$

Now, we consider the following nonsingular linear transformation $(\phi, u, v) \to (\phi, u, w)$, where $w = v + mu + k\phi$, m and k as above. The Cauchy problem (1.3′), (1.4′) takes the form

$$\begin{cases} \phi_t = w - m\,u - k\,\phi \\ u_t + k\,\phi_x + m\,u_x - w_x = 0 \qquad\qquad (x,t) \in \mathbb{R} \times [0, +\infty[\ , \\ w_t + m\,k\,\phi_x + (m^2 - p'(u))\,u_x - m\,w_x + F(\phi) = 0 \end{cases} \tag{1.3''}$$

$$\begin{aligned} \phi(x,0) &= \phi_0(x)\ , \\ u(x,0) &= u_0(x) = \phi_{0x}(x)\ , \\ w(x,0) &= w_0(x) = v_0(x) + m\,u_0(x) + k\,\phi_0(x)\ , \qquad x \in \mathbb{R}\ . \end{aligned} \tag{1.4''}$$

Now we introduce the following auxiliary Cauchy problem in (ϕ, u, w),

$$\begin{cases} \phi_t + m\,\phi_x - w + k\,\phi = 0 \\ u_t + m\,u_x - w_x + k\,u = 0 \\ w_t + (m^2 - p'(u))\,u_x - m\,w_x + m\,k\,u + F(\phi) = 0 \end{cases} \quad \text{in } \mathbb{R}\times[0,+\infty[\ , \tag{1.3'''}$$

with the initial data (1.4″).

If (ϕ, u, w) is a C^1 solution of (1.3‴), (1.4″) we easily derive, for $\varphi \in C_c^\infty(\mathbb{R}\times[0,+\infty[)$,

$$-\int_{\mathbb{R}_+}\int_{\mathbb{R}} \phi\,\varphi_{xt}\,dx\,dt - \int_{\mathbb{R}} \phi_0\,\varphi_x(\cdot,0)\,dx - m\int_{\mathbb{R}_+}\int_{\mathbb{R}} \phi\,\varphi_{xx}\,dx\,dt \\ - \int_{\mathbb{R}_+}\int_{\mathbb{R}} w\,\varphi_x\,dx\,dt + k\int_{\mathbb{R}_+}\int_{\mathbb{R}} \phi\,\varphi_x\,dx\,dt \;=\; 0$$

and

$$-\int_{\mathbb{R}_+}\int_{\mathbb{R}} u\,\varphi_t\,dx\,dt - \int_{\mathbb{R}} u_0\,\varphi(\cdot,0)\,dx - m\int_{\mathbb{R}_+}\int_{\mathbb{R}} u\,\varphi_x\,dx\,dt \\ + \int_{\mathbb{R}_+}\int_{\mathbb{R}} w\,\varphi_x\,dx\,dt + k\int_{\mathbb{R}_+}\int_{\mathbb{R}} u\,\varphi\,dx\,dt \;=\; 0\ ,$$

and so, by addition, we obtain

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \phi\Big[\varphi_t + m\,\varphi_x - k\,\varphi\Big]_x\,dx\,dt + \int_{\mathbb{R}_+}\int_{\mathbb{R}} u\Big[\varphi_t + m\,\varphi_x - k\,\varphi\Big]\,dx\,dt = 0\ .$$

Now, given $\psi \in \mathcal{D}(\mathbb{R}\times\,]0,+\infty[)$, it is easy to find $\varphi \in C_c^\infty(\mathbb{R}\times[0,+\infty[)$ such that $\varphi_t + m\,\varphi_x - k\,\varphi = \psi$: first, with $\psi_1 = e^{-kt}\,\psi$, $\varphi_1 = e^{-k\varphi}\,\varphi$, we reduce to $\varphi_{1t} + m\,\varphi_{1x} = \psi_1$. We put

$$\varphi_1(m\,t + c, t) = \int_0^t \psi_1(m\,\tau + c, \tau)\,d\tau - \widetilde{\varphi}_1(c)\ ,$$

where

$$\widetilde{\varphi}_1(c) = \int_0^{+\infty} \psi_1(m\,t + c, t)\,dt\ , \qquad \forall\, c \in \mathbb{R}\ .$$

Hence,

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \phi\,\psi_x\,dx\,dt + \int_{\mathbb{R}_+}\int_{\mathbb{R}} u\,\psi\,dx\,dt = 0\ , \qquad \forall\,\psi \in \mathcal{D}(\mathbb{R}\times\,]0,+\infty[)\ .$$

We derive $u = \phi_x$. It is now easy to prove

PROPOSITION 1.1. For given initial data $(\phi_0, u_0 = \phi_{0x}, w_0)$ in $C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ the systems (1.3″) and (1.3‴) have the same (local in time) C^1 solutions.

Given (ϕ_0, u_0, w_0) in $L^\infty(\mathbb{R})$, we say, as usually, that $(\phi, u, w) \in (L^\infty_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3$ is a weak (global) solution of the Cauchy problem (1.3″), (1.4″) if we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \phi\, \varphi_t \,dx\,dt + \int_{\mathbb{R}} \phi_0\, \varphi(\cdot, 0)\,dx + \int_{\mathbb{R}_+}\int_{\mathbb{R}} (w - m\,u - k\,\phi)\,\varphi \,dx\,dt \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}} u\,\psi_t \,dx\,dt + \int_{\mathbb{R}} u_0\,\psi(\cdot, 0)\,dx + \int_{\mathbb{R}_+}\int_{\mathbb{R}} (k\,\phi + m\,u - w)\,\psi_x \,dx\,dt \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}} w\,\theta_t \,dx\,dt + \int_{\mathbb{R}} w_0\,\theta(\cdot, 0)\,dx \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}} \Big[m^2\,u - p(u) - m\,w\Big]\,\theta_x \,dx\,dt - \int_{\mathbb{R}_+}\int_{\mathbb{R}} F(\phi)\,\theta \,dx\,dt = 0 , \\
&\qquad\qquad\qquad\qquad \forall\, \varphi, \psi, \theta \in C^\infty_c(\mathbb{R} \times [0, +\infty[) ,
\end{aligned} \tag{1.7}$$

and a similar definition for the Cauchy problem (1.3‴), (1.4″).

We can repeat the calculations made to prove Proposition 1.1 for a $(\phi, u, w) \in (L^\infty_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3$ weak solution of (1.3‴), (1.4″) and we obtain $\phi_x = u$ in $\mathcal{D}'(\mathbb{R} \times]0, +\infty[)$. We derive $\phi \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \times [0, +\infty[)$ and it is now easy to prove that (ϕ, u, w) is a weak solution of (1.3″), (1.4″). The converse is also true, by similar considerations. Hence, we have

Proposition 1.2. For a given initial data $(\phi_0, u_0 = \phi_{0x}, w_0)$ in $L^\infty(\mathbb{R})$, the systems (1.3″) and (1.3‴) have the same weak solutions.

Now let (η, q) be a pair of smooth convex entropy/entropy flux for the system (1.3″) (cf. [4]).

$$\left(\text{Example:} \quad \eta_1(\phi, u, w) = \frac{1}{2}\,\phi^2 + \int_0^u p(\xi)\,d\xi + \frac{1}{2}\,(w - m\,u - k\,\phi)^2 , \right.$$

$$\left. q_1(\phi, u, w) = -(w - m\,u - k\,\phi)\,p(u) \quad \right) .$$

A weak solution (ϕ, u, w) of (1.3″), (1.4″) is called an entropy weak solution if, in $\mathcal{D}'(\mathbb{R} \times]0, +\infty[)$,

$$\eta(\phi, u, w)_t + q(\phi, u, w)_x + \nabla\eta \cdot \Big(-w + m\,u + k\,\phi,\, 0,\, F(\phi)\Big) \le 0 \tag{1.8}$$

for all pairs (η, q), η convex.

The system (1.3‴) admits the entropy/entropy flux pair $(\widetilde{\eta}_1, \widetilde{q}_1)$, $\widetilde{\eta}_1$ strictly convex, defined by

$$\widetilde{\eta}_1(\phi, u, w) = \frac{1}{2}\,\phi^2 + \int_0^u p(\xi)\,d\xi + \frac{1}{2}(w - m\,u)^2 ,$$

$$\widetilde{q}_1(\phi, u, w) = \frac{1}{2}\,m\,\phi^2 - (w - m\,u)\,p(u) .$$

If $(\phi, u, w) \in (L^\infty_{\text{loc}} \cap BV_{\text{loc}})^3$ is a weak solution of (1.3″), (1.4″) with initial data in $BV(\mathbb{R})$, we can prove (with some tedious computations, taking in mind that $\phi_x =$

u, cf. Proposition 1.2, and applying the theorem in section 13.2 of [5] concerning the differentiation of the composition) that we have, in $\mathcal{D}'(\mathbb{R}\times]0,+\infty[)$,

$$\widetilde{\eta}_1(\phi,u,w)_t+\widetilde{q}_1(\phi,u,w)_x+\nabla\widetilde{\eta}_1\cdot\Big(-w+k\,\phi,\;k\,u,\;m\,k\,u+F(\phi)\Big)$$
$$=\eta_1(\phi,u,w)_t+q_1(\phi,u,w)_x+\nabla\eta_1\cdot\Big(-w+m\,u+k\,\phi,\;0,\;F(\phi)\Big)\;.$$

Hence, by Proposition 1.2, we conclude

THEOREM 1.3. Assume that $(\phi_0,u_0=\phi_{0x},w_0)\in BV(\mathbb{R})^3$ and let $(\phi,u,w)\in(L^\infty_{\rm loc}\cap BV_{\rm loc}(\mathbb{R}\times[0,+\infty[))^3$ be an entropy weak solution of (1.3‴), (1.4″). Then, (ϕ,u,w) is also a weak solution of (1.3″), (1.4″) verifying (1.8) for the pair (η_1,q_1) defined above.

2. Application of Theorem 2 in [2]. Now, in order to apply to the Cauchy problem (1.3‴), (1.4″) Theorem 2 in [2], we give initial data $(\phi_0,u_0=\phi_{0x},w_0)$ in $BV(\mathbb{R})$. The system (1.3‴) can be written as follows (recall that $p'(0)=1$ and $m>1$):

$$\frac{\partial}{\partial t}\begin{pmatrix}\phi\\ u\\ w\end{pmatrix}+A(u)\,\frac{\partial}{\partial x}\begin{pmatrix}\phi\\ u\\ w\end{pmatrix}+g(\phi,u,w)=0\;,$$

where

$$A(u)=\begin{pmatrix}m&0&0\\0&m&-1\\0&m^2-p'(u)&-m\end{pmatrix}\;,\qquad g(\phi,u,w)=\begin{pmatrix}-w+k\,\phi\\ k\,u\\ m\,k\,u+F(\phi)\end{pmatrix}\;.$$

The eigenvalues of $A(u)$ are $(m,\sqrt{p'(u)},-\sqrt{p'(u)})$. The matrix of the corresponding (independent) normalized right eigenvectors for $u=0$ is

$$B=\begin{pmatrix}1&0&0\\0&a&b\\0&a(m-1)&b(m+1)\end{pmatrix}\quad\text{and}\quad\nabla g(0,0,0)=\begin{pmatrix}k&0&-1\\0&k&0\\F'(0)&m\,k&0\end{pmatrix}\;,$$

where $a=(m^2-2m+2)^{-1/2}$ and $b=(m^2+2m+2)^{-1/2}$.

Hence, $R=\{r_{ij}\}=B^{-1}\,\nabla g(0,0,0)\,B$ is given by

$$R=\begin{pmatrix}k&a(-m+1)&b(-m-1)\\[4pt] -\dfrac{1}{2\,a}\,F'(0)&\dfrac{k}{2}&\dfrac{b}{a}\,\dfrac{k}{2}\\[4pt] \dfrac{1}{2\,b}\,F'(0)&\dfrac{a}{b}\,\dfrac{k}{2}&\dfrac{k}{2}\end{pmatrix}$$

and verifies

$$\sum_i r_{ii}-\sum_{i\neq j}|r_{ij}|=\Big(2-\Big(\frac{a}{b}+\frac{b}{a}\Big)/2\Big)\,k-\Big(\frac{1}{2a}+\frac{1}{2b}\Big)\,|F'(0)|$$
$$-\,a(m-1)-b(m+1)\;>\;0\qquad\text{by (1.6)}$$

and so R is diagonal dominant.

By applying Theorem 2 in [2] we derive

THEOREM 2.1. Let us assume (1.5). Then, there exist two positive constants $a_0, b_0 > 0$ such that, if

$$(\phi_0, u_0 = \phi_{0x}, w_0) \in (BV(\mathbb{R}))^3$$

and

$$\|(\phi_0, u_0, w_0)\|_{L^\infty(\mathbb{R})} \leq a_0 \ , \qquad TV_x(\phi_0, u_0, w_0) \leq b_0 \ ,$$

then there exists a weak entropy solution $(\phi, u, w) \in (L^\infty \cap BV_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3$ of (1.3‴), (1.4″). Moreover $(\phi(\cdot, t), u(\cdot, t), w(\cdot, t)) \in (BV(\mathbb{R}))^3$ for each $t \geq 0$, with a uniformly bounded (in t) total variation TV_x.

Hence, by Theorem 1.3, we can derive a similar result for the Cauchy problem (1.3), (1.4) if we replace the general entropy condition (1.8) by the following particular one:

$$\overline{\eta}(\overline{\phi}, \overline{u}, \overline{v})_t + \overline{q}(\overline{\phi}, \overline{u}, \overline{v})_x + \nabla\overline{\eta} \cdot \Big(-\overline{v},\ 0,\ \overline{v} + \overline{F}(\overline{\phi})\Big) \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R} \times]0, +\infty[) \ ,$$

where

$$(\overline{\eta}, \overline{q}) = \left(\frac{1}{2}\overline{\phi}^2 + \int_0^{\overline{u}} p(\xi)\, d\xi + \frac{1}{2}\overline{v}^2,\ -\overline{v}\, p(\overline{u})\right) .$$

See [1] for the case with $\overline{F} \equiv 0$ and [3] for related results.

Acknowledgments. This work was supported by FCT, PRAXIS XXI, FEDER and Project PRAXIS/2/2.1/MAT/125/94.

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