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A Remark on the Existence of Steady Navier-Stokes Flows in a Certain Two-Dimensional Infinite Channel

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Abstract. We consider the steady Navier-Stokes equations

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p & \text{in } \Omega\,, \\ \operatorname{div}\mathbf{u} \quad = 0 & \text{in } \Omega\,, \end{cases}$$

in a 2-dimensional unbounded multiply-connected domain Ω contained in an infinite straight channel $\mathbf{R} \times (-1, 1)$, under general outflow condition. We look for a solution which tends to a Poiseuille flow at infinity.

In this note, we shall show the existence of solution to this problem under the assumption of symmetry with respect to the axis for the domain and the boundary value, and for small Poiseuille flow. We do not assume that the boundary value is small. The regularity and the asymptotic behavior of the solution are also discussed.

1. Introduction.

The problem of existence of solutions to the stationary Navier-Stokes equations in channels which are cylindrical outside some compact set was suggested in the nineteen fifties by J. Leray when he visited Leningrad (Ladyzhenskaya [13], Ladyzhenskaya-Solonnikov [14]). The solvability of Leray's problem was shown firstly by Amick [3] and also by Ladyzhenskaya-Solonnikov [14]. However in their case, the domain was simply connected. We consider 2-dimensional multiply-connected unbounded domain. Namely, let Ω be a 2-dimensional domain as follows.

$$\Omega = T \setminus \bigcup_{i=1}^{N} \overline{O_i}$$

where T is a straight channel

$$T = \mathbf{R} \times (-1, 1) = \{\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2 \mid -\infty < x_1 < +\infty, -1 < x_2 < 1\}\,,$$

and O_i's are simply connected bounded domains mutually disjoint, closure of which are contained in T. We denote the boundary by

$$\partial\Omega = \bigcup_{i=0}^{N} \Gamma_i\,,$$

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where Γ_0 is the boundary of T and Γ_i is that of O_i $(1 \le i \le N, N \ge 2)$. Let Γ_0^+ (resp. Γ_0^-) be the upper part (resp. the lower part) of Γ_0. We consider the boundary value problem of the Navier-Stokes equations

$$\text{(NS)} \qquad \begin{cases} (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} - \nabla p & \text{in } \Omega\,, \\ \operatorname{div} \mathbf{u} \quad = 0 & \text{in } \Omega\,, \end{cases}$$

with the boundary condition

$$\text{(BC)} \qquad \begin{cases} \mathbf{u} = \boldsymbol{\beta} & \text{on } \partial\Omega\,, \\ \mathbf{u} \to \mu \mathbf{U} & \text{as } |x_1| \to \infty \text{ in } \Omega \end{cases}$$

where $\mathbf{u}$ is the velocity, p is the pressure, ν is the kinematic viscosity (positive constant), $\boldsymbol{\beta}$ is a given function on $\partial\Omega = \bigcup_{i=0}^{N} \Gamma_i$ compactly supported, $\mathbf{U}$ the Poiseuille flow in T with flux 1:

$$\text{(U)} \qquad \mathbf{U} = \frac{3}{4}(1 - x_2^2, 0)$$

and μ a constant. For the boundary value $\boldsymbol{\beta}$, we suppose only the general outflow condition

$$\text{(GOC)} \qquad \int_{\partial\Omega} \boldsymbol{\beta} \cdot \mathbf{n} d\sigma = \sum_{i=0}^{N} \int_{\Gamma_i} \boldsymbol{\beta} \cdot \mathbf{n} d\sigma = 0\,,$$

$\mathbf{n}$ being the unit outward normal vector to $\partial\Omega$. We note that under a more stringent outflow condition

$$\text{(SOC)} \qquad \int_{\Gamma_i} \boldsymbol{\beta} \cdot \mathbf{n} d\sigma = 0 \quad (1 \le i \le N)\,, \quad \int_{\Gamma_0^+} \boldsymbol{\beta} \cdot \mathbf{n} d\sigma = \int_{\Gamma_0^-} \boldsymbol{\beta} \cdot \mathbf{n} d\sigma = 0\,,$$

the existence of solution is known for small $|\mu|$.

Suppose that the domains O_i's are symmetric with respect to the x_1-axis, every $\Gamma_i (1 \le i \le N)$ intersects the x_1-axis and the boundary value $\boldsymbol{\beta}$ is also symmetric. Then we can show the existence of solution to (NS) (BC) for small $|\mu|$ without the stringent outflow condition nor smallness assumption on $\boldsymbol{\beta}$. Here the vector field $\boldsymbol{\varphi}(\mathbf{x}) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2))$ is called symmetric with respect to the x_1-axis if

$$\varphi_1(x_1, x_2) = \varphi_1(x_1, -x_2)\,, \quad \varphi_2(x_1, x_2) = -\varphi_2(x_1, -x_2)$$

holds.

As for the regularity of the solution, Amick [4] proved it together with the exponential decay of the solution at the infinity under his setting. We also study these properties of the solution. Lemma 5 below plays a key roll in our proof of the exponential decay of the solution.

In Section 2, we state the notation and results concerning the extension of the boundary value (Lemma 1), the existence of solution (Theorem 1), the regularity of the solution obtained (Theorem 2) and the asymptotic behavior of the solution (Theorem 3). The proof of Theorem 1 and Theorem 2 is found in Section 4, the proof of Theorem 3 in Section 5.

For the bounded symmetric domain in $\mathbf{R}^2$ with symmetric data, Amick [2] obtained the existence result by contradiction argument. The second author showed the similar result constructively by the virtual drain method, *i.e.*, by constructing explicitly an appropriate solenoidal extension of the boundary value [9]. We make use of this result.

2. Notation and results.

Let $\mathbf{C}_0^\infty(\Omega)$ be the set of all smooth vector valued functions with compact support in Ω. Let $L^2(\Omega)$ be the set of all vector valued square integrable functions in Ω with the inner product $(\cdot,\cdot)_\Omega$ and the norm $\|\cdot\|_\Omega$. If there is no confusion, we denote the inner product and the norm by $(\cdot,\cdot)$ and $\|\cdot\|$. $L^p(\Omega)$ is the set of all vector valued functions $\mathbf{u}$ such that $|\mathbf{u}|^p$ is integrable in Ω. The norm is denoted by $\|\mathbf{u}\|_{L^p(\Omega)}$ or simply by $\|\mathbf{u}\|_p$. $W^{m,p}(\Omega)$ is the standard Sobolev spaces; $W_0^{m,p}(\Omega)$ is the closure of $\mathbf{C}_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. $H^m(\Omega) = W^{m,2}(\Omega)$. $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

Let $\mathbf{C}_{0,\sigma}^\infty(\Omega)$ be the set of all smooth solenoidal (*i.e.* divergence free) vector valued functions with compact support in Ω. $H_\sigma = H_\sigma(\Omega)$ is the closure of $\mathbf{C}_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)$. $V = V(\Omega)$ is the completion of $\mathbf{C}_{0,\sigma}^\infty(\Omega)$ in the Dirichlet norm $\|\nabla\cdot\|$, which is equivalent to the $H^1(\Omega)$-norm by virtue of the Poincaré inequality (Lemma 3). Let $\mathbf{C}_{0,\sigma}^{\infty,S}(\Omega)$ be the set of functions in $\mathbf{C}_{0,\sigma}^\infty(\Omega)$ symmetric with respect to the x_1-axis. $V^S = V^S(\Omega)$ is the completion of $\mathbf{C}_{0,\sigma}^{\infty,S}(\Omega)$ in the Dirichlet norm $\|\nabla\cdot\|$.

By definition, $\mathbf{u}$ is called a weak solution to (NS), (BC) if $\mathbf{u}$ is expressible of the form

$$\mathbf{u} = \mathbf{w} + \mathbf{b} + \mu\mathbf{U},$$

where $\mathbf{w} \in V$, $\mathbf{b} \in H^1(\Omega)$ is a solenoidal extension of the boundary value

$$\mathbf{b} = \boldsymbol{\beta} - \mu\mathbf{U} \quad \text{on } \partial\Omega,$$

and satisfies the weak form of the Navier-Stokes equations:

$$\nu(\nabla\mathbf{u},\nabla\mathbf{v}) + ((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}) = 0 \quad (\forall\mathbf{v}\in\mathbf{C}_{0,\sigma}^\infty(\Omega)). \tag{1}$$

Next lemma is crucial for the existence proof.

LEMMA 1. *Suppose that $\partial\Omega$ is smooth and symmetric with respect to the x_1-axis, that every Γ_i $(1 \le i \le N)$ intersects the x_1-axis and that the boundary value $\boldsymbol{\beta}_0 \in H^{1/2}(\partial\Omega)$ is symmetric with respect to the x_1-axis, vanishes on Γ_0 and satisfies (GOC). Then for every $\varepsilon > 0$ there exists a symmetric solenoidal extension $\mathbf{b}_\varepsilon$ of $\boldsymbol{\beta}_0$ such that*

$$|((\mathbf{v}\cdot\nabla)\mathbf{v},\mathbf{b}_\varepsilon)| \le \varepsilon\|\nabla\mathbf{v}\|^2 \quad (\forall\mathbf{v}\in V^S(\Omega)).$$

The proof of Lemma 1 is similar to [9] and is omitted.

REMARK 1. As is noted in [9], the support of $\mathbf{b}_\varepsilon$ is contained in the union of narrow collar neighbourhood of the boundary $\Gamma_1,\cdots,\Gamma_N$ and a narrow neighbourhood of segments on the x_1-axis joining Γ_1 and $\Gamma_2,\cdots,\Gamma_{N-1}$ and Γ_N.

THEOREM 1. *Suppose that the boundary $\partial\Omega$ is smooth and symmetric with respect to the x_1-axis, that every Γ_i $(1 \leq i \leq N)$ intersects the x_1-axis and that the boundary value $\boldsymbol{\beta}$ is a smooth function on $\partial\Omega$, symmetric with respect to the x_1-axis, vanishes on Γ_0 and satisfies (GOC). If $|\mu|$ is sufficiently small, then there exists a symmetric weak solution to (NS), (BC).*

REMARK 2. The boundary value $\boldsymbol{\beta}$ is not necessarily small.

REMARK 3. It is well known (e.g. Ladyzhenskaya [12], Galdi [10]) that for the weak solution $\mathbf{u}$ to (NS) (BC), there exists a scalar function $p \in L^2_{loc}(\Omega)$ such that

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{u}\cdot\nabla\mathbf{u}), \mathbf{v}) = (p, \operatorname{div}\mathbf{v}) \quad (\forall\mathbf{v} \in \mathbf{C}_0^\infty(\Omega)) .$$

p is called an associated pressure of $\mathbf{u}$. We call $\{\mathbf{u}, p\}$ solution pair to (NS) (BC).

THEOREM 2. *The solution pair $\{\mathbf{u}, p\}$ to (NS), (BC) obtained in Theorem 1 is smooth in $\bar{\Omega}$.*

Before stating a slight generalization of Theorems 1 and 2, we introduce the following notation. For $t < s$, we put

$$\Omega^{t,s} = \{(x_1, x_2) \in \Omega \mid t < x_1 < s\} .$$

Let R be a positive number such that

$$\Omega^{R,\infty} \cap T = \{x \in T \mid x_1 > R\} , \quad \Omega^{-\infty,-R} \cap T = \{x \in T \mid x_1 < -R\} . \tag{2}$$

REMARK 4. Theorems 1 and 2 hold true if the set $\Omega \setminus \{\Omega^{-\infty,-R} \cup \Omega^{R,\infty}\}$ is not contained in the channel T, and if the boundary value $\boldsymbol{\beta}$ does not vanish identically on Γ_0, but is of a bounded support and satisfies the following condition.

$$\int_{\Gamma_0^+} \boldsymbol{\beta}\cdot\mathbf{n}dx_1 = \int_{\Gamma_0^-} \boldsymbol{\beta}\cdot\mathbf{n}dx_1 = 0 .$$

Let $P = -\frac{3}{2}\nu x_1$ be an associated pressure of $\mathbf{U}$. Let $\alpha = (\alpha_1, \alpha_2)$ be a multi-index and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}$, $D_i = \partial/\partial x_i$ $(i = 1, 2)$.

THEOREM 3. *Let R be a sufficiently large positive number mentioned in (2). The solution pair $\{\mathbf{u}, p\}$ to (NS), (BC) obtained in Theorem 1 tends to the Poiseuille flow $\{\mu\mathbf{U}, \mu P\}$ exponentially, as $|x_1| \to \infty$, that is, there exist positive constants σ and C_α such that*

$$|D^\alpha\{\mathbf{u}(x) - \mu\mathbf{U}(x)\}| \leq C_\alpha e^{-\sigma|x_1|} \quad (|x_1| > R) \tag{3}$$

$$|D^\alpha\{\nabla p(x) - \mu\nabla P(x)\}| \leq C_\alpha e^{-\sigma|x_1|} \quad (|x_1| > R) \tag{4}$$

for every multi-index α.

3. Preliminaries.

We begin with some lemmas which are necessary later. The straightforward calculation yields

LEMMA 2. *Let* $\mathbf{U}$ *be the Poiseuille flow* (U). *Then*

$$|((\mathbf{w}\cdot\nabla)\mathbf{U},\mathbf{w})| \le \|\mathbf{w}\|^2 \quad (\forall \mathbf{w}\in V(\Omega))$$

holds.

Since the Poincaré inequality holds for a domain bounded in one direction, we obtain

LEMMA 3. *The following inequalities hold with some domain constants* κ_0, κ_1.

$$\|w\|_{L^2(T)} \le \kappa_0 \|\nabla w\|_{L^2(T)} \quad (\forall w \in H_0^1(T)) \tag{5}$$

$$\|w\|_{L^4(T)} \le \kappa_1 \|\nabla w\|_{L^2(T)} \quad (\forall w \in H_0^1(T))\,. \tag{6}$$

We need the following result for divergence operator which is due to Bogovskii [6]. See also Galdi [10], Babuska-Aziz [5].

LEMMA 4. *Let* Q *be a bounded domain, star-like with respect to every point of a ball* $B(x_0, a) \subset Q$. *Suppose* $f(x) \in L^p(Q)$ *satisfying*

$$\int_Q f(x)dx = 0\,.$$

Then there exists $\mathbf{v} \in W_0^{1,p}(Q)$ *such that*

$$\begin{cases} \operatorname{div} \mathbf{v} = f & \text{in } Q \\ \mathbf{v} \quad\ = 0 & \text{on } \partial Q \end{cases} \tag{7}$$

satisfying

$$\|\mathbf{v}\|_{W^{1,p}(Q)} \le c\|f\|_{L^p(Q)}$$

where c is a constant dependent only on the diameter of Q and a.

In order to prove the exponential decay of the solution to the Poiseuille flow, we need next lemma proved originally by Horgan-Wheeler [11]. For the completeness, we give a proof following Galdi [10, Lemma VI. 2.2].

LEMMA 5. *Suppose* $y(t) \in C^1[R, \infty)$, $y(t) \ge 0$ $(t \ge R)$, *and for some* $\xi > 0$, $\eta \in \mathbf{R}$, $y(t)$ *satisfies the differential inequality*

$$y'(t) + \xi \int_t^\infty y(s)ds \le \eta y(t) \quad (\forall t \ge R)\,. \tag{8}$$

Then there exist positive constants λ *and* σ *such that*

$$y(t) \le \lambda e^{-\sigma(t-R)} \quad (t \ge R)$$

holds true.

PROOF. Let α be the positive root of the equation $\alpha^2 - \eta\alpha - \xi = 0$, *i.e.*,

$$\alpha = (\sqrt{\eta^2 + 4\xi} + \eta)/2\,.$$

Note that $\alpha > \eta$. Put

$$F(t) = e^{-\eta(t-R)}\left\{y(t) + \alpha \int_t^\infty y(s)ds\right\}.$$

Differentiating with respect to t, we have

$$F'(t) + \alpha F(t) = \left\{y'(t) - \eta y(t) + (\alpha^2 - \eta\alpha)\int_t^\infty y(s)ds\right\} e^{-\eta(t-R)}.$$

By the differential inequality (8),

$$F'(t) + \alpha F(t) \le (\alpha^2 - \eta\alpha - \xi)\int_t^\infty y(s)ds e^{-\eta(t-R)}$$

holds true. Since $\alpha^2 - \eta\alpha - \xi = 0$, $F'(t) + \alpha F(t) \le 0$. Integrating the differential inequality $(e^{\alpha t}F(t))' \le 0$ from R to t, and using the definiton of $F(t)$, we get

$$y(t) + \alpha \int_t^\infty y(s)ds \le F(R)e^{-(\alpha-\eta)(t-R)}.$$

Therefore, we have

$$-\frac{d}{dt}\left\{e^{-\alpha(t-R)}\int_t^\infty y(s)ds\right\} \le F(R)e^{-(2\alpha-\eta)(t-R)}.$$

Since $2\alpha - \eta > 0$ and $e^{-\alpha(t-R)}\int_t^\infty y(s)ds \to 0$ $(t \to \infty)$, we integrate the both sides and obtain

$$\int_R^\infty y(s)ds \le \frac{F(R)}{2\alpha - \eta}.$$

Using the definition of F and this estimate, we have

$$F(R) \le \frac{2\alpha - \eta}{\alpha - \eta} y(R).$$

Therefore

$$y(t) \le F(R)e^{-(\alpha-\eta)(t-R)} \le \frac{2\alpha - \eta}{\alpha - \eta} y(R)e^{-(\alpha-\eta)(t-R)}.$$

We can choose σ and λ as follows.

$$\sigma \equiv \alpha - \eta = (\sqrt{\eta^2 + 4\xi} - \eta)/2, \quad \lambda = \frac{\sqrt{\eta^2 + 4\xi}}{\sigma} y(R)$$

and the lemma is proved. Q.E.D.

4. Existence and regularity of solution.

First, we give a proof of Theorem 1. The pressure corresponding to the Poiseuille flow $\mathbf{U} = \frac{3}{4}((1 - x_2^2), 0)$ is given by $P = -\frac{3}{2}\nu x_1$. That is, $\mathbf{U}$ and P satisfy the Navier-Stokes equations.

$$\begin{cases} -\nu\Delta\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + \nabla P = \mathbf{0} & \text{in } T \\ \operatorname{div}\mathbf{U} = 0 & \text{in } T \\ \mathbf{U} = \mathbf{0} & \text{on } \Gamma_0. \end{cases}$$

Remark that $\mu\mathbf{U}$ and μP also satisfy the above equations. Assume that $|\mu|$ is so small that the inequality $\nu-\kappa_0^2|\mu|>0$ holds, where κ_0 is the constant introduced in Lemma 3. Let us choose $\varepsilon>0$ such that $\nu-\kappa_0^2|\mu|-\varepsilon>0$ holds. Since $\boldsymbol{\beta}-\mu\mathbf{U}$ satisfies the hypothesis of Lemma 1, there exists its solenoidal symmetric extension $\mathbf{b}$ satisfying

$$|((\mathbf{w}\cdot\nabla)\mathbf{b},\mathbf{w})|\le\varepsilon\|\nabla\mathbf{w}\|^2\quad(\forall\mathbf{w}\in V^S(\Omega))\,.$$

Note that $\mathbf{b}$ is of compact support. Furthermore, since $\partial\Omega$ and $\boldsymbol{\beta}$ are smooth, $\mathbf{b}$ is also smooth.

We look for the solution $\{\mathbf{u},p\}$ to (NS) (BC) in the following form.

$$\begin{cases}\mathbf{u}=\mathbf{w}+\mu\mathbf{U}+\mathbf{b}\\ p=q+\mu P\,.\end{cases}\tag{9}$$

The function $\mathbf{w}\in V(\Omega)$ satisfies the following equation.

$$\begin{aligned}\nu(\nabla\mathbf{w},\nabla\varphi)&+((\mathbf{w}\cdot\nabla)\mathbf{w},\varphi)+\mu((\mathbf{w}\cdot\nabla)\mathbf{U},\varphi)+\mu((\mathbf{U}\cdot\nabla)\mathbf{w},\varphi)\\&+((\mathbf{w}\cdot\nabla)\mathbf{b},\varphi)+((\mathbf{b}\cdot\nabla)\mathbf{w},\varphi)\\&=(\mathbf{F},\varphi)-\nu(\nabla\mathbf{b},\nabla\varphi)\quad(\forall\varphi\in\mathbf{C}_{0,\sigma}^{\infty}(\Omega))\end{aligned}\tag{10}$$

where

$$\mathbf{F}=-(\mathbf{b}\cdot\nabla)\mathbf{b}-\mu(\mathbf{U}\cdot\nabla)\mathbf{b}-\mu(\mathbf{b}\cdot\nabla)\mathbf{U}\,.$$

Let Ω^n, $n=1,2,\cdots$, be an expanding sequence of bounded symmetric domain with smooth boundary such that

$$\Omega^n\subset\Omega^{n+1}\to\Omega\,,\quad\partial\Omega^n\cap\partial\Omega\to\partial\Omega\quad\text{as }n\to\infty\,.$$

We suppose that $\Gamma_1,\Gamma_2,\cdots,\Gamma_N$ are the inner boundary of $\partial\Omega^1$, and the support of $\mathbf{b}$ is contained in $\overline{\Omega^1}$. We consider the stationary Navier-Stokes equations in Ω^n.

$$(\mathrm{NS})_n\qquad\begin{cases}(\mathbf{u}\cdot\nabla)\mathbf{u}=\nu\Delta\mathbf{u}-\nabla p & \text{in }\Omega^n\,,\\ \operatorname{div}\mathbf{u}\quad=0 & \text{in }\Omega^n\,,\end{cases}$$

with the boundary condition

$$(\mathrm{BC})_n\qquad\begin{cases}\mathbf{u}=\boldsymbol{\beta} & \text{on }\partial\Omega\cap\partial\Omega^n\,,\\ \mathbf{u}=\mu\mathbf{U} & \text{on }\partial\Omega^n\setminus\partial\Omega\,.\end{cases}$$

A function $\mathbf{u}$ is called a weak solution to $(\mathrm{NS})_n$ $(\mathrm{BC})_n$, if $\mathbf{u}\in H^1(\Omega^n)$, $\operatorname{div}\mathbf{u}=0$,

$$\nu(\nabla\mathbf{u},\nabla\mathbf{v})+((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v})=0\quad(\forall\mathbf{v}\in V(\Omega^n))\,,$$

and $\mathbf{u}$ satisfies the boundary condition $(\mathrm{BC})_n$ in the trace sense.

It was established by Fujita [9] that there exists $\mathbf{u}_n=\mathbf{w}_n+\mathbf{b}+\mu\mathbf{U}$, satisfying

$$\nu(\nabla\mathbf{u}_n,\nabla\mathbf{v})+((\mathbf{u}_n\cdot\nabla)\mathbf{u}_n,\mathbf{v})=0\quad(\forall\mathbf{v}\in V(\Omega^n))\,,\tag{11}$$

where $\mathbf{w}_n\in V^S(\Omega^n)$. Substituting $\mathbf{v}=\mathbf{w}_n$ in (11), we have

$$\nu\|\nabla\mathbf{w}_n\|^2=-\mu((\mathbf{w}_n\cdot\nabla)\mathbf{U},\mathbf{w}_n)-((\mathbf{w}_n\cdot\nabla)\mathbf{b},\mathbf{w}_n)-\nu(\nabla\mathbf{b},\nabla\mathbf{w}_n)+(\mathbf{F},\mathbf{w}_n)\,.$$

Since $\mathbf{w}_n\in V^S(\Omega^n)\subset V^S(\Omega)$,

$$|((\mathbf{w}_n\cdot\nabla)\mathbf{b},\mathbf{w}_n)|\le\varepsilon\|\nabla\mathbf{w}_n\|^2\quad(\forall n)\,.$$

According to Lemma 2 and Lemma 3,

$$|((\mathbf{w}_n \cdot \nabla)\mathbf{U}, \mathbf{w}_n)| \leq (\kappa_0 \|\nabla \mathbf{w}_n\|)^2 \quad (\forall n)\,.$$

Therefore,

$$\nu \|\nabla \mathbf{w}_n\|^2 \leq (|\mu|\kappa_0^2 + \varepsilon)\|\nabla \mathbf{w}_n\|^2 + \nu|(\nabla \mathbf{b}, \nabla \mathbf{w}_n)| + |(\mathbf{F}, \mathbf{w}_n)|\,.$$

Since $\nu - \kappa_0^2|\mu| - \varepsilon > 0$ holds and $\mathbf{F}$ and $\mathbf{b}$ are independent of n, we obtain an estimate for $\mathbf{w}_n$, *i.e.*, there exists a constant $M > 0$ independent of n such that $\|\nabla \mathbf{w}_n\| \leq M$ holds true. We remark that the norm $\|\mathbf{u}\|_{H^1(\Omega)}$ is equivalent to $\|\nabla \mathbf{u}\|_{L^2(\Omega)}$. By the standard argument, we can show the existence of the weak solution $\mathbf{u}$ to (NS) (BC). Since $\mathbf{w}_n, \mathbf{b}, \mathbf{U}$ are symmetric, $\mathbf{u}$ is also symmetric. We obtain the solution pair of the form (9), with $\mathbf{w} \in V^S(\Omega)$ and $q \in L^2_{loc}(\Omega)$ and Theorem 1 has been proved. Q.E.D.

Now we study the regularity of the solution to (NS) (BC) obtained in Theorem 1. Let $\{\mathbf{u}, p\}$ be the solution pair having the form (9). Then $\mathbf{w} \in V^S(\Omega)$ and $q \in L^2_{loc}(\Omega)$ satisfy the following equation.

$$\begin{aligned}(12)\qquad & \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) - (q, \operatorname{div} \mathbf{v}) + ((\mathbf{w} \cdot \nabla)\mathbf{w}, \mathbf{v}) + \mu((\mathbf{w} \cdot \nabla)\mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{w}, \mathbf{v}) \\ & \quad + ((\mathbf{w} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{w}, \mathbf{v}) = (\mathbf{F} + \nu\Delta\mathbf{b}, \mathbf{v}) \quad (\forall \mathbf{v} \in \mathbf{C}_0^\infty(\Omega))\,.\end{aligned}$$

Since $\mathbf{b}$ is smooth in Ω and of compact support, it is easy to see that $\mathbf{w}$ and ∇q are smooth in the closure of Ω. See, e.g., Galdi [10]. However, for the sake of completeness, we give a proof in Appendix, which is based on the regularity theorem for the Stokes equations.

5. Asymptotic behavior of the solution.

5.1. Uniform convergence of the solution. Let us study the asymptotic behavior of the solution to (NS) (BC) obtained in Theorem 1. First, we prove the uniform convergence of the solution to the Poiseuille flow.

LEMMA 6. *The solution pair* $\{\mathbf{u}, p\}$ *to* (NS), (BC) *obtained in Theorem* 1 *behaves asymptotically as follows*:

$$(13)\qquad \sup_{\Omega^{t,\infty}} |D^\alpha\{\mathbf{u}(x) - \mu\mathbf{U}(x)\}| \to 0 \quad (t \to \infty)\,,$$

$$(14)\qquad \sup_{\Omega^{t,\infty}} |D^\alpha\{\nabla p(x) - \mu\nabla P(x)\}| \to 0 \quad (t \to \infty)$$

for every multi-index α. *Similar result holds for* $t \to -\infty$.

Let us prove the lemma when $t \to +\infty$. The case $t \to -\infty$ can be proved similarly. Since $\mathbf{w}$ and ∇q are smooth in $\Omega^{R,\infty}$ (Theorem 2), and $\mathbf{b} \equiv 0$ in $\Omega^{R,\infty}$, we see that

$$(15)\qquad \begin{cases} -\nu\Delta\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \mu(\mathbf{w} \cdot \nabla)\mathbf{U} + \mu(\mathbf{U} \cdot \nabla)\mathbf{w} + \nabla q = \mathbf{0} & \text{in } \Omega^{R,\infty} \\ \operatorname{div} \mathbf{w} = 0 & \text{in } \Omega^{R,\infty} \end{cases}$$

hold. Let

$$Q_0 = \{(x_1, x_2) \mid -5/8 < x_1 < 5/8, -1 < x_2 < 1\}\,,$$

$$Q_1 = \{(x_1, x_2) \mid -1 < x_1 < 1, -1 < x_2 < 1\}.$$

Let ω_0 and Ω_0 be bounded symmetric domains with smooth boundary satisfying

$$Q_0 \subset \omega_0 \subset \Omega_0 \subset Q_1,$$

such that the right (resp. left) component of $\partial\omega_0 \setminus \partial\Omega$ is congruent with the right (resp. left) component of $\partial\Omega_0 \setminus \partial\Omega$. Let $t_0 \geq R$ and put

$$\Omega^{(k)} = \{(x_1, x_2) \mid (x_1 - k, x_2) \in \Omega_0\}, \quad k = t_0 + 1, t_0 + 2, \cdots,$$

$$\omega^{(k)} = \{(x_1, x_2) \mid (x_1 - k, x_2) \in \omega_0\}, \quad k = t_0 + 1, t_0 + 2, \cdots.$$

$\Omega^{(k)}$'s (resp. $\omega^{(k)}$'s) are congruent figures.

Let $\psi(x)$ be a scalar function in $C^\infty(\bar{T})$, $\psi(x_1, -x_2) = \psi(x_1, x_2)$, $0 \leq \psi(x) \leq 1$, $\psi(x) \equiv 1$ ($x \in \overline{\omega_0}$); $\equiv 0$ ($x \in \bar{T} \setminus \overline{\Omega_0}$), and put

$$\psi_k(x_1, x_2) \equiv \psi(x_1 - k, x_2).$$

Let m_k be the mean value of q over $\Omega^{(k)}$, *i.e.*, $m_k = (1/|\Omega^{(k)}|) \iint_{\Omega^{(k)}} q(x)dx$, where $|\Omega^{(k)}|$ is the measure of $\Omega^{(k)}$. Since the pressure q is free by additive constant, we take $q - m_k$ instead of q in (15). Put

$$\begin{cases} \mathbf{W}^{(k)} = \psi_k \mathbf{w} \\ \pi^{(k)} = \psi_k(q - m_k). \end{cases}$$

Using the equation (15), we can write

$$\nu\Delta\mathbf{W}^{(k)} - \nabla\pi^{(k)} = \psi_k \mathbf{G}_0 + \mathbf{G}_k \tag{16}$$

$$\operatorname{div} \mathbf{W}^{(k)} = \mathbf{w} \cdot \nabla\psi_k \quad \text{in } \Omega^{(k)}, \quad \mathbf{W}^{(k)} = \mathbf{0} \quad \text{on } \partial\Omega^{(k)}, \tag{17}$$

with

$$\mathbf{G}_0 = \mathbf{G}_0(\mathbf{w}, \mathbf{U}) = (\mathbf{w} \cdot \nabla)\mathbf{w} + \mu(\mathbf{U} \cdot \nabla)\mathbf{w} + \mu(\mathbf{w} \cdot \nabla)\mathbf{U}$$
$$\mathbf{G}_k = \mathbf{G}_k(\mathbf{w}, q) = 2\nu\nabla\mathbf{w}\nabla\psi_k + \nu(\Delta\psi_k)\mathbf{w} - (\nabla\psi_k)(q - m_k).$$

Since $\operatorname{div} \mathbf{w} = 0$ in $\Omega^{(k)}$, and $\mathbf{w} = 0$ on $\partial\Omega \cap \partial\Omega^{(k)}$, we have

$$\operatorname{div}(\psi_k \mathbf{w}) = (\nabla\psi_k) \cdot \mathbf{w} \in H_0^1(\Omega^{(k)}), \quad \int_{\Omega^{(k)}} \operatorname{div}(\psi_k \mathbf{w})dx = \int_{\partial\Omega^{(k)}} (\psi_k \mathbf{w}) \cdot \mathbf{n}d\sigma = 0.$$

Let us show $\mathbf{G}_0, \mathbf{G}_k \in L^{4/3}(\Omega^{(k)})$. By the inequality (6), we obtain

$$\|(\mathbf{w} \cdot \nabla)\mathbf{w}\|_{L^{4/3}(\Omega^{(k)})} \leq \|\nabla\mathbf{w}\|\|\mathbf{w}\|_{L^4(\Omega^{(k)})} \leq \kappa_1\|\nabla\mathbf{w}\|^2. \tag{18}$$

On the other hand,

$$\|(\mathbf{U} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{U}\|_{L^{4/3}(\Omega^{(k)})} \leq a\|(\mathbf{U} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{U}\|_{\Omega^{(k)}} \leq a(1 + 2\kappa_0)\|\nabla\mathbf{w}\|_{\Omega^{(k)}},$$

where κ_0 is the constant in (5) and $a = |\Omega^{(k)}|^{1/4}$. Note that κ_0 and a are constants independent of k. Therefore

$$\begin{aligned} \|\psi_k \mathbf{G}_0\|_{L^{4/3}(\Omega^{(k)})} &\leq \|\mathbf{G}_0\|_{L^{4/3}(\Omega^{(k)})} \\ &\leq \kappa_1\|\nabla\mathbf{w}\|^2_{\Omega^{(k)}} + a(1 + 2\kappa_0)|\mu|\|\nabla\mathbf{w}\|_{\Omega^{(k)}} \leq C_1\|\nabla\mathbf{w}\|_{\Omega^{(k)}} \end{aligned} \tag{19}$$

where $C_1 = \kappa_1 \beta + a(1 + 2\kappa_0)|\mu|$ and $\beta = \|\nabla \mathbf{w}\|_\Omega$. Since $q \in L^2_{loc}(\Omega)$, we have

$$\|\mathbf{G}_k\|_{L^{4/3}(\Omega^{(k)})} \leq a\|\mathbf{G}_k\|_{\Omega^{(k)}} \leq C_2\{\nu\|\nabla \mathbf{w}\|_{\Omega^{(k)}} + \|q - m_k\|_{\Omega^{(k)}}\} \tag{20}$$

where $C_2 = a(2 \sup|\nabla\psi| + \kappa_0 \sup|\Delta\psi|)$. Therefore, $\psi_k \mathbf{G}_0 + \mathbf{G}_k$ is in $L^{4/3}(\Omega^{(k)})$. According to the well known estimate for solutions to the Stokes inhomogeneous boundary value problem (16), (17) (Cattabriga [7]), we have

$$\mathbf{W}^{(k)} \in W^{2,4/3}(\Omega^{(k)}), \quad \nabla\pi^{(k)} \in L^{4/3}(\Omega^{(k)}).$$

Furthermore

$$\begin{aligned} \nu\|\mathbf{W}^{(k)}\|_{W^{2,4/3}(\Omega^{(k)})} &+ \|\nabla\pi^{(k)}\|_{L^{4/3}(\Omega^{(k)})} \\ &\leq C_3\{\|\psi_k \mathbf{G}_0 + \mathbf{G}_k\|_{L^{4/3}(\Omega^{(k)})} + \|\mathbf{w} \cdot \nabla\psi_k\|_{W^{1,4/3}(\Omega^{(k)})}\} \\ &\leq C_4\{\|\nabla \mathbf{w}\|_{\Omega^{(k)}} + \|q - m_k\|_{\Omega^{(k)}}\} \end{aligned} \tag{21}$$

where the constants C_3 and C_4 do not depend on k. Now we proceed to the estimation for the pressure q.

LEMMA 7. *There is a constant C_5 independent of k such that the estimate*

$$\|q - m_k\|_{\Omega^{(k)}} \leq C_5\|\nabla \mathbf{w}\|_{\Omega^{(k)}} \tag{22}$$

holds.

PROOF. From (15) we have

$$-\nu\Delta \mathbf{w} + \nabla q = -(\mathbf{w} \cdot \nabla)\mathbf{w} - \mu(\mathbf{w} \cdot \nabla)\mathbf{U} - \mu(\mathbf{U} \cdot \nabla)\mathbf{w} = -\mathbf{G}_0(\mathbf{w}, \mathbf{U}).$$

Multiplying this equation by $\mathbf{v} \in H^1_0(\Omega^{(k)})$, integrating over $\Omega^{(k)}$, and also noting

$$(\text{const.}, \operatorname{div} \mathbf{v})_{\Omega^{(k)}} = 0,$$

we obtain

$$\nu(\nabla \mathbf{w}, \nabla \mathbf{v})_{\Omega^{(k)}} - (q - m_k, \operatorname{div} \mathbf{v})_{\Omega^{(k)}} = -(\mathbf{G}_0(\mathbf{w}, \mathbf{U}), \mathbf{v})_{\Omega^{(k)}}. \tag{23}$$

We look for a function $\mathbf{v}$ satisfying the following equation.

$$\begin{cases} \operatorname{div} \mathbf{v} = q - m_k & \text{in } \Omega^{(k)} \\ \mathbf{v} \quad = \mathbf{0} & \text{on } \partial\Omega^{(k)}. \end{cases} \tag{24}$$

Since $q - m_k \in L^2(\Omega^{(k)})$ and the integral $\iint_{\Omega^{(k)}}(q - m_k)dx$ vanishes, we can apply Lemma 4 and find the solution $\mathbf{v} \in H^1_0(\Omega^{(k)})$ to (24) such that the estimate

$$\|\mathbf{v}\|_{H^1_0(\Omega^{(k)})} \leq C_0\|q - m_k\|_{\Omega^{(k)}}$$

holds true where C_0 is a constant independent of k. Substituting this $\mathbf{v}$ into (23), we obtain

$$\nu(\nabla \mathbf{w}, \nabla \mathbf{v})_{\Omega^{(k)}} - \|q - m_k\|^2_{\Omega^{(k)}} = -(\mathbf{G}_0(\mathbf{w}, \mathbf{U}), \mathbf{v})_{\Omega^{(k)}}.$$

Therefore

$$\begin{aligned}\|q-m_k\|^2_{\Omega^{(k)}} &= \nu(\nabla\mathbf{w},\nabla\mathbf{v})_{\Omega^{(k)}} + (\mathbf{G}_0(\mathbf{w},\mathbf{U}),\mathbf{v})_{\Omega^{(k)}}\\ &\le \nu\|\nabla\mathbf{w}\|_{\Omega^{(k)}}\|\nabla\mathbf{v}\|_{\Omega^{(k)}} + \|\mathbf{G}_0\|_{L^{4/3}(\Omega^{(k)})}\|\mathbf{v}\|_{L^4(\Omega^{(k)})}\\ &\le \nu\|\nabla\mathbf{w}\|_{\Omega^{(k)}}\|\nabla\mathbf{v}\|_{\Omega^{(k)}} + \kappa_1\|\mathbf{G}_0\|_{L^{4/3}(\Omega^{(k)})}\|\nabla\mathbf{v}\|_{\Omega^{(k)}}\\ &\le C_0\|q-m_k\|_{\Omega^{(k)}}(\nu\|\nabla\mathbf{w}\|_{\Omega^{(k)}} + \kappa_1\|\mathbf{G}_0\|_{L^{4/3}(\Omega^{(k)})})\,.\end{aligned}$$

According to the estimate (19) for $\mathbf{G}_0$, we obtain

$$\|q-m_k\|_{L^2(\Omega^{(k)})} \le C_5\|\nabla\mathbf{w}\|_{\Omega^{(k)}}$$

where $C_5 = C_0(\nu+\kappa_1 C_1)$. The constant C_5 does not depend on k. Q.E.D.

We continue the proof of Lemma 6. Substituting (22) into (21), we obtain

$$\nu\|\mathbf{W}^{(k)}\|_{W^{2,4/3}(\Omega^{(k)})} + \|\nabla\pi^{(k)}\|_{L^{4/3}(\Omega^{(k)})} \le C_6\|\nabla\mathbf{w}\|_{\Omega^{(k)}} \tag{25}$$

where $C_6 = C_4(1+C_5)$. Since $\psi_k \equiv 1$ on the set $\omega^{(k)}$,

$$\mathbf{w} = \mathbf{W}^{(k)} \quad \text{and} \quad q = \pi^{(k)} + m_k \quad \text{in } \omega^{(k)}\,.$$

Therefore

$$\mathbf{w} \in W^{2,4/3}(\omega^{(k)})\,, \qquad \nabla q \in L^{4/3}(\omega^{(k)})$$

and

$$\nu\|\mathbf{w}\|_{W^{2,4/3}(\omega^{(k)})} + \|\nabla q\|_{L^{4/3}(\omega^{(k)})} \le C_6\|\nabla\mathbf{w}\|_{\Omega^{(k)}}\,. \tag{26}$$

According to the Sobolev imbedding theorem, the inclusion

$$W^{2,4/3}(\omega^{(k)}) \subset C(\overline{\omega^{(k)}})$$

holds. Therefore $\mathbf{w}$ is bounded and continuous in $\omega^{(k)}$ and

$$\|\mathbf{w}\|_{C(\overline{\omega^{(k)}})} \le C\|\mathbf{w}\|_{W^{2,4/3}(\omega^{(k)})} \le C_7\|\nabla\mathbf{w}\|_{\Omega^{(k)}}$$

where the constants C and $C_7 = CC_6$ do not depend on k. Since

$$\Omega^{(k)} \subset \Omega^{k-1,k+1} \subset \omega^{(k-1)} \cup \omega^{(k)} \cup \omega^{(k+1)}\,, \tag{27}$$

it holds that

$$\begin{aligned}\|\mathbf{w}\|_{C(\Omega^{(k)})} &\le \sup_{k-1\le j\le k+1}\|\mathbf{w}\|_{C(\omega^{(j)})} \le \sup_{k-1\le j\le k+1} C_7\|\nabla\mathbf{w}\|_{\Omega^{(j)}}\\ &\le C_7\beta \quad (\forall k \ge R+1)\end{aligned} \tag{28}$$

where $\beta = \|\nabla\mathbf{w}\|_\Omega$. Furthermore, by (28), we see $\mathbf{G}_0 \in L^2(\Omega^{(k)})$ and

$$\|\mathbf{G}_0\|_{\Omega^{(k)}} \le \|\mathbf{w}\|_{C(\Omega^{(k)})}\|\nabla\mathbf{w}\|_{\Omega^{(k)}} + |\mu|(1+2\kappa_0)\|\nabla\mathbf{w}\|_{\Omega^{(k)}} \le C\|\nabla\mathbf{w}\|_{\Omega^{(k)}}$$

where $C = C_7\beta + |\mu|(1+2\kappa_0)$ is a constant independent of k. As for $\mathbf{G}_k$, using (20) and (22), we obtain

$$\|\mathbf{G}_k\|_{\Omega^{(k)}} \le C\|\nabla\mathbf{w}\|_{\Omega^{(k)}}$$

where $C = C_2(\nu+C_5)/a$ is a constant independent of k.

Here and after C denotes various positive constant independent of k. Repeating the preceding procedure, we conclude that $\mathbf{W}^{(k)} \in H^2(\Omega^{(k)})$ and $\nabla\pi^{(k)} \in L^2(\Omega^{(k)})$

$$\nu\|\mathbf{W}^{(k)}\|_{H^2(\Omega^{(k)})} + \|\nabla\pi^{(k)}\|_{L^2(\Omega^{(k)})} \le C\|\nabla\mathbf{w}\|_{\Omega^{(k)}}$$

therefore, $\mathbf{w} \in H^2(\omega^{(k)})$, $\nabla q \in L^2(\omega^{(k)})$. Thanks to the inclusion (27), we have

$$\nu\|\mathbf{w}\|_{H^2(\Omega^{(k)})} + \|\nabla q\|_{L^2(\Omega^{(k)})} \le C \sum_{j=k-1}^{k+1} \|\nabla\mathbf{w}\|_{\Omega^{(j)}}\,, \tag{29}$$

for a constant C independent of k. This estimate assures that $\mathbf{w} \in H^2(\Omega^{k,\infty})$ and $\nabla q \in L^2(\Omega^{k,\infty})$.

Now we can show easily $\mathrm{div}(\psi_k\mathbf{w}) \in H^2(\Omega^{(k)}) \cap H^1_0(\Omega^{(k)})$ and that $\nabla\mathbf{G}_0, \nabla\mathbf{G}_k \in L^2(\Omega^{(k)})$, that is, $\mathbf{G}_0, \mathbf{G}_k \in H^1(\Omega^{(k)})$. We apply the argument as before and obtain

$$\mathbf{W} \in H^3(\Omega^{(k)})\,, \quad \nabla\pi \in H^1(\Omega^{(k)})\,.$$

This means that

$$\mathbf{w} \in H^3(\omega^{(k)})\,, \quad \nabla q \in H^1(\omega^{(k)})\,.$$

As before we can show easily

$$\nu\|\mathbf{w}\|_{H^3(\Omega^{(k)})} + \|\nabla q\|_{H^1(\Omega^{(k)})} \le C \sum_{j=k-2}^{k+2} \|\nabla\mathbf{w}\|_{\Omega^{(j)}}\,. \tag{30}$$

Similarly,

$$\nu\|\mathbf{w}\|_{H^{2+\ell}(\Omega^{(k)})} + \|\nabla q\|_{H^{\ell}(\Omega^{(k)})} \le C \sum_{j=k-1-\ell}^{k+1+\ell} \|\nabla\mathbf{w}\|_{\Omega^{(j)}}\,, \tag{31}$$

for $\ell = 2, 3, \cdots$.

Now we estimate $\mathbf{w}$ in $\Omega^{k,\infty}$. Using (29) and the inclusion

$$\omega^{(j)} \subset \Omega^{(j)}\,, \quad \Omega^{k,\infty} \subset \bigcup_{j\ge k} \omega^{(j)} \subset \Omega^{k-1,\infty}\,,$$

we obtain

$$\begin{aligned} \|\mathbf{w}\|_{C(\overline{\Omega^{k,\infty}})} &\le \sup_{j\ge k} \|\mathbf{w}\|_{C(\overline{\omega^{(j)}})} \le \sup_{j\ge k} \|\mathbf{w}\|_{C(\overline{\Omega^{(j)}})} \\ &\le C \sup_{j\ge k} \|\mathbf{w}\|_{H^2(\Omega^{(j)})} \le C \sup_{j\ge k} \sum_{j-1}^{j+1} \|\nabla\mathbf{w}\|_{\Omega^{(i)}} \le 3C\|\nabla\mathbf{w}\|_{\Omega^{k-2,\infty}}\,. \end{aligned} \tag{32}$$

The right hand side tends to 0 as $k \to \infty$, because the constant C does not depend on k and $\nabla\mathbf{w} \in L^2(\Omega)$. Therefore (13) for $|\alpha| = 0$ is proved. Repeating the above argument, we have the estimate for $|\alpha| > 0$:

$$
(33) \quad \|D^\alpha \mathbf{w}\|_{C(\overline{\Omega^{k,\infty}})} \le \sup_{j\ge k} \|D^\alpha \mathbf{w}\|_{C(\overline{\omega^{(j)}})} \le \sup_{j\ge k} \|D^\alpha \mathbf{w}\|_{C(\overline{\Omega^{(j)}})} \le C \sup_{j\ge k} \|\mathbf{w}\|_{H^{|\alpha|+2}(\Omega^{(j)})}
$$

$$
\le C \sup_{j\ge k} \sum_{j-1-|\alpha|}^{j+1+|\alpha|} \|\nabla \mathbf{w}\|_{\Omega^{(i)}} \le (2|\alpha|+3)C\|\nabla \mathbf{w}\|_{\Omega^{k-2-|\alpha|,\infty}} .
$$

∇q can be similarly estimated by (30) and (31).

This completes the proof of Lemma 6. Q.E.D.

5.2. Exponential decay of the solution. Now we state the outline of the proof of Theorem 3. For $R < t < s$, we set

$$
\Omega^{t,s} = \{(x_1, x_2) \in \Omega \mid t < x_1 < s\}, \quad \Sigma(t) = \{(t, x_2) \in \Omega \mid -1 < x_2 < 1\}.
$$

Put $\mathcal{H}(t) \equiv \iint_{\Omega^{t,\infty}} |\nabla \mathbf{w}|^2 dx$. Then, differentiating with respect to t, we obtain

$$
\mathcal{H}'(t) = -\int_{\Sigma(t)} |(\nabla \mathbf{w})(t, x_2)|^2 dx_2 .
$$

The similar argument to [10] yields

$$
(34) \qquad \nu\mathcal{H}'(t) + 2(\nu - |\mu|\kappa_0^2)\int_t^\infty \mathcal{H}(x_1)dx_1 \le C\mathcal{H}(t) \quad (t \ge R),
$$

where C is a constant. If $|\mu|$ is so small that $\nu - |\mu|\kappa_0^2 > 0$, we can apply Lemma 5 to (34) and obtain the exponential decay of the Dirichlet norm of $\mathbf{w}$, that is, there exist positive constants σ and λ such that

$$
\mathcal{H}(t) \le \lambda^2 e^{-2\sigma(t-R)} \quad (t \ge R).
$$

Consequently, using the estimate (33), we obtain

$$
(35) \quad \|D^\alpha \mathbf{w}\|_{C(\overline{\Omega^{t,\infty}})} \le (2|\alpha|+3)C\|\nabla \mathbf{w}\|_{L^2(\Omega^{t-2-|\alpha|,\infty})} \le (2|\alpha|+3)C\mathcal{H}(t-2-|\alpha|)^{1/2}
$$

$$
\le (2|\alpha|+3)C\lambda e^{-\sigma(t-2-|\alpha|-R)}
$$

and the estimate (3) is proved. The estimate for ∇q can be shown similarly and Theorem 3 is proved. Q.E.D.

REMARK 5. In the forthcoming paper [16], the first author studies several symmetric flows in symmetric channels involving general outflow condition and shows the existence and asymptotic behavior of the solution in the criterion of Amick [3], [4].

Appendix. PROOF OF THEOREM 2.

Let $\mathbf{w} \in V(\Omega)$ and $q \in L^2_{loc}(\Omega)$ satisfy the following equation.

$$
\nu(\nabla \mathbf{w}, \nabla \mathbf{v}) - (q, \operatorname{div} \mathbf{v}) + ((\mathbf{w}\cdot\nabla)\mathbf{w}, \mathbf{v}) + \mu((\mathbf{w}\cdot\nabla)\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{w}, \mathbf{v})
$$

$$
+ ((\mathbf{w}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{w}, \mathbf{v}) = (\mathbf{F}', \mathbf{v}) \quad (\forall \mathbf{v} \in \mathbf{C}_0^\infty(\Omega)),
$$

where $\mathbf{F}' \equiv \mathbf{F} + \nu\Delta\mathbf{b}$ is a smooth function of compact support. Let us show that $\mathbf{w}, q \in C^\infty(\bar{\omega})$ for any bounded domain ω containing $\Omega^{-R,R}$ and contained in Ω. Let $a'' < a' < -R < R < b' < b''$, $\omega' = \Omega \cap \{a' < x_1 < b'\}$, $\omega'' = \Omega \cap \{a'' < x_1 < b''\}$ and Q be a bounded domain with smooth boundary such that $\omega \subset \omega' \subset \omega'' \subset Q \subset \Omega$. Let $\psi(x)$ be a smooth function defined in the closure of Ω such that

$$\psi(x) \equiv 1 \quad (x \in \bar{\omega}), \quad \psi(x) \equiv 0 \quad (x \in \bar{\Omega} \setminus \bar{Q}).$$

Put

$$\mathbf{W} \equiv \psi \mathbf{w}, \quad \pi \equiv \psi q.$$

Then, it is easy to see that $\mathbf{W} \in H_0^1(Q)$, $\pi \in L^2(Q)$ and they satisfy

$$\begin{aligned} &\nu(\nabla \mathbf{W}, \nabla \mathbf{v}) - (\pi, \operatorname{div} \mathbf{v}) = -(\psi \mathbf{G}_0 + \mathbf{G}_1, \mathbf{v}) \quad (\forall \mathbf{v} \in C_0^\infty(\Omega)) \\ &\operatorname{div} \mathbf{W} = \operatorname{div}(\psi \mathbf{w}) = \nabla \psi \cdot \mathbf{w}, \quad \mathbf{W}|_{\partial Q} = 0 \end{aligned} \tag{36}$$

where

$$\begin{aligned} \mathbf{G}_0 &= (\mathbf{w} \cdot \nabla)\mathbf{w} + \mu(\mathbf{w} \cdot \nabla)\mathbf{U} + \mu(\mathbf{U} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{w} - \mathbf{F}', \\ \mathbf{G}_1 &= 2\nu \nabla \psi \nabla \mathbf{w} + \nu(\Delta \psi)\mathbf{w} - (\nabla \psi) q. \end{aligned}$$

Note $\operatorname{div} \mathbf{W} = \operatorname{div}(\psi \mathbf{w}) = \nabla \psi \cdot \mathbf{w} + \psi \operatorname{div} \mathbf{w} = \nabla \psi \cdot \mathbf{w} \in H_0^1(Q)$. According to Hölder's inequality and the Sobolev imbedding theorem, we have

$$\|(\mathbf{w} \cdot \nabla)\mathbf{w}\|_{L^{4/3}(Q)} \le \|\mathbf{w}\|_{L^4(Q)} \|\nabla \mathbf{w}\|_{L^2(Q)} \le C\|\mathbf{w}\|_{H^1(Q)} \|\nabla \mathbf{w}\|_{L^2(Q)}$$

and it is easy to show $\mathbf{G}_0 \in L^{4/3}(Q)$. On the other hand, we can easily check $\mathbf{G}_1 \in L^2(Q)$. Therefore $\psi \mathbf{G}_0 + \mathbf{G}_1 \in L^{4/3}(Q)$. Using the well known result of Cattabriga [7] for the Stokes boundary value problem, we have $\mathbf{W} \in W^{2,4/3}(Q)$, $\nabla \pi \in L^{4/3}(Q)$. Since

$$\mathbf{w} = \mathbf{W} \quad \text{in } \omega \quad \text{and} \quad q = \pi \quad \text{in } \omega,$$

it holds that

$$\mathbf{w} \in W^{2,4/3}(\omega) \subset C(\bar{\omega}), \quad \nabla q \in L^{4/3}(\omega).$$

According to the above estimate, it is easy to check $\mathbf{G}_0 \in L^2(\omega)$. Repeating the previous argument, we see that

$$\mathbf{w} \in H^2(\omega), \quad \nabla q \in L^2(\omega).$$

Now, let us show further regularity of the solution $\mathbf{w}$, q. Let ω and Q be as before. It is easy to check that $\nabla \mathbf{G}_0, \nabla \mathbf{G}_1 \in L^2(Q)$, that is, $\mathbf{G}_0, \mathbf{G}_1 \in H^1(Q)$. We apply the argument as before and obtain

$$\mathbf{W} \in H^3(Q), \quad \nabla \pi \in H^1(Q).$$

This means that

$$\mathbf{w} \in H^3(\omega), \quad \nabla q \in H^1(\omega).$$

We continue in this fashion to show that

$$\mathbf{w} \in H^m(\omega), \quad \nabla q \in H^{m-2}(\omega), \quad m = 2, 3, \cdots.$$

And we see

$$\mathbf{w} \in C^\infty(\bar{\omega}), \quad \nabla q \in C^\infty(\bar{\omega}).$$

This completes the proof of Theorem 2. Q.E.D.

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