A REMARK ON THE INVARIANTS OF W^*-ALGEBRAS[1]

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The purpose of this paper is to supplement the theory of invariants of W^*-algebras accomplished by [1], [5], [6], [9] and [10] etc., with some considerations and to show some spatial isomorphism theorems.

Through the following discussion $M^\natural$ denotes the center of a W^*-algebra M on a Hilbert space H. We denote by $z(e)$ the central envelope of a projection e. By an isomorphism, we mean a *-preserving isomorphism. Ω denotes always the spectrum of $M^\natural$ and Ω_{eH} the spectrum of Me.

1. Definition of the Invariant. We employ the invariant of M in the same sense as in Pallu de la Barrière [1] if M is finite with finite commutator and denote it by $\overline{C}(t)$.

Next, let M be a W^*-algebra and α an infinite cardinal. A central projection e is called a homogeneous projection of type F_α if it is a "projection of type S_α" in the sense of Griffin [5] and "projection de uniforme d'ordre α" in [1] and e is called a homogeneous projection of type C_α if it is an "α-dimensional projection" in the sense of Griffin [6] and "homogeneous projection of order α" in Suzuki [13].

The following multiplicity theorem of a W^*-algebra is well known (cf. [1], [5], [6], [13]).

THEOREM 1. *Let M be a W^*-algebra on a Hilbert space H and π_1 (resp. π_2) the set of infinite cardinal α for which there exists a homogeneous projection of type F_α (resp. C_α). Then there exists a family of orthogonal central projections $\{e_\alpha\}_{\alpha\in\pi_1}$ (resp. $\{e_\beta\}_{\beta\in\pi_2}$) such that*

$$1 = e_1 + \sum_{\alpha\in\pi_1} e_\alpha + \sum_{\beta\in\pi_2} e_\beta$$

where $e_\alpha(\alpha \in \pi_1$ or $\pi_2)$ is a maximal homogeneous projection of type F_α or of type C_α and e_1 a maximal central finite projection. This decomposition is unique.

We denote by K_α an open and closed set in Ω corresponding to e_α in Theorem 1. A function $p(t)$ defined on a dense open set $\bigcup_{\alpha\in\{1\}\cup\pi_1\cup\pi_2} K_\alpha$ in Ω is called *the algebraic invariant of M*, if

$$p(t) = \alpha \qquad \text{for } t \in K_\alpha.$$

Now let $\{p_1, p_2\}$ be an orthogonal central decomposition where M_{p_1} is finite with finite commutator and M_{p_2} does not contain such coupled component. Denote by $\overline{C}(t)$, $p(t)$, and $p'(t)$ the invariant of M_{p_1}, the algebraic invariants

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of M_{p_2} and M'_{p_2} respectively. $p(t)$ and $p'(t)$ are defined on dense open sets Ω_1, Ω_2 in Ω_{p_2H}.

We define *the invariant of M* as follows;

$$C(t) = \overline{C}(t) \qquad \text{for } t \in \Omega_{p_1H}$$
$$= (p(t), p'(t)) \text{ for } t \in \Omega_1 \cap \Omega_2,$$

where $(p(t),\ p'(t))$ means a formal coupled function on $\Omega_1 \cap \Omega_2$. $C(t)$ is defined on a dense open set of Ω.

2. Spatial Isomorphism Theorem. The following lemma is due to Pallu de la Barrière [1].

LEMMA 1. *Let M be a finite W^*-algebra with finite commutator and $\overline{C}(t)$ the invariant of M. For any projection $e' \in M'$, let $\overline{C}_{e'H}(t)$ be the invariant of Me'. Then we have*

$$\overline{C}_{e'H}(t) = \overline{C}(t)\ e'^{\natural}(t) \text{ for all } t \in \Omega_{e'H} \text{ and } \overline{C}(t) \neq \infty.$$

LEMMA 2. *Let M_1 and M_2 be properly infinite W^*-algebras on each Hilbert space H_1 and H_2 with finite commutators and θ an isomorphism between them. There exists a continuous function $\gamma(t)$ defined on the common spectrum Ω of $M^i_t, i = 1,2$, and ranging over $[0, \infty]$ such that for any finite projection $e \in M_1$*

$$\overline{C}_{\theta(e)H_2}(t) = \gamma(t)\overline{C}_{eH_1}(t) \text{ for all } t \in \Omega_{eH_1} = \Omega_{\theta(e)H_2},$$

where $\overline{C}_{eH_1}(t)$ and $\overline{C}_{\theta(e)H_2}(t)$ are the invariants of M_{1e} and $M_{2\theta(e)}$.

PROOF. By J. Dixmier [4: Proposition 2] one can easily verify that there exists a W^*-algebra N on a Hilbert space K and two projections f'_1, f'_2 in N' whose central envelopes are the identity such that θ may be identified with the isomorphism

$$\theta : a'_{f_1} \to a'_{f_2} \qquad \text{for all } a \in N.$$

Thus, we can assume that $M_1 = N'_{f'_1}$, $M_2 = N'_{f'_2}$. Since f'_i $(i = 1,2)$ are finite, $f'_0 = f'_1 \vee f'_2$ is finite, too. Take a finite projection $e \in N'_{f'_1}$. Without loss of generality we may assume that $z(e) = 1$ in $N'_{f'_1}$ (i. e. $= f'_1$). We find a projection $e_0 \in N$ with $e = e_0 f'_1$; e_0 is finite in N with $z(e_0) = 1$. As $N'_{f'_0}$ and $N'_{f_0 e_0}$ are finite, we can consider canonical applications $\natural_1$ and $\natural_2$ in $N''_{f'_0}$ and $N'_{f'_0 e_0}$, respectively. Then, applying lemma 1 to $N'_{f'_0 e_0}$, we have

$$\overline{C}_{eH_1}(t) = \overline{C}_{e_0 f'_1 K}(t) = \overline{C}_{e_0 f'_0 K}(t)(e_0 f'_1)^{\natural_2}(t)$$
$$\overline{C}_{\theta(e)H_2}(t) = \overline{C}_{e_0 f'_2 K}(t) = \overline{C}_{e_0 f'_0 K}(t)\,(e_0 f'_2)^{\natural_2}(t)$$

for $t \in \Omega_{e_0 f'_1 K} = \Omega_{e_0 f'_2 K}$ such as $\overline{C}_{e_0 f'_0 K}(t) \neq \infty$. Since $N'_{f'_0}$ is isomorphic to $N'_{e_0 f'_0}$ we get the identification $(e_0 f'_i)^{\natural_2} = f'^{\natural_1}_i$ $(i = 1, 2)$. Moreover, by an isomorphism between $N'_{f'_1}$ and $N'_{f'_0}$, we get the further identification, that is, $\{f'^{\natural_1}_i(t)\}_{i = 1, 2}$ can be considered as continuous functions on $\Omega_{f'_1 K}$. Therefore putting $\gamma(t) = f'^{\natural_1}_2(t)/f'^{\natural_1}_1(t)$, we have a continuous function on $\Omega_{f'_1 K}$. Q. E. D.

We are now going to prove our main

THEOREM 2. *Let M_1 and M_2 be W^*-algebras and θ an isomorphism between them. By the isomorphism θ we may identify two spectra Ω_1, Ω_2 of $M_1^\natural$, $M_2^\natural$ respectively. Suppose M_1 and M_2 have the same invariant $C(t)$. Then*

$1°$ *if $C(t) \neq (\alpha, 1)$ for all infinite cardinal α, θ is spatial,*

$2°$ *if $C(t) = (\alpha, 1)$, and $\gamma(t) = 1$, then θ is spatial.*

PROOF. Following the same notations as in the proof of Lemma 2 we may assume

$$\theta : a_{f_1}' \to a_{f_2}' \qquad \text{for all } a \in N$$

where $z(f_1') = z(f_2') = 1$. Hence, to prove θ being spatial is reduced to prove that $f_1' \sim f_2'$ mod N', so that it suffices to prove this fact in the following cases, separately.

Case $(1°)$ $C(t) \neq (\alpha, 1)$.

a. f_1', f_2' are finite. In this case N_{f_1}' is finite, otherwise this yields an excluded case. $f_0' = f_1' \vee f_2'$ is finite, too. Since N_{f_0}' becomes finite we have, by lemma 1 applying to N_{f_0}',

$$\overline{C}_{f_1 K}'(t) = \overline{C}_{f_0 K}'(t) f_1^\natural(t) \text{ and } \overline{C}_{f_2 K}'(t) = \overline{C}_{f_0 K}'(t) f_2^\natural(t)$$

for $t \in \Omega_{f_1 K}'$ and $\overline{C}_{f_0 K}'(t) \neq \infty$. Therefore $f_1^\natural(t) = f_2^\natural(t)$ for such t. But as N_{f_0}' is isomorphic to N_{f_i}', we get $f_1^\natural(t) = f_2^\natural(t)$ over a dense open set in $\Omega_{f_0 K}'$, whence $f_1^\natural = f_2^\natural$. We get $f_1' \sim f_2'$ mod N_{f_0}', $f_1' \sim f_2'$ mod N'.

b. f_1', f_2' are homogeneous projections of type $C_{\aleph_0}$ in N_{f_1}' aud N_{f_2}'. We may assume that N_{f_i}' $(i = 1, 2)$ are countably decomposable, so that f_1' and f_2' are countably decomposable in N'. Then we have

$$f_1' = \sum_{n=1}^{\infty} e_n' \qquad \text{with } e_n' \prec f_2' \text{ mod } N'.$$

On the other hand we may write $f_2' = \sum_{n=1}^{\infty} f_{2n}'$ where $f_2' \sim f_{2n}'$ for all n. Therefore $f_1' \prec f_2'$; by symmetry it follows $f_1' \sim f_2'$ mod N'.

c. f_1, f_2 are homogeneous projections of type F_α (or C_α) in N_{f_i}' $(i = 1, 2)$. (In case of type C_α we assume $\alpha > \aleph_0$). We may consider as in case *b*, that $N_{f_i}^\natural$ are countably decomposable for $i = 1, 2$. Then we can find an infinite family $\{e_i'\}_{i \in I}$ of orthogonal, equivalent, finite (resp. cyclic) projections in N_{f_1}' such that $f_1' = \sum_{i \in I} e_i'$, where the cardinal of I is α. Now take a fixed projection e_{i_0}' and a central projection g such that $g e_{i_0}' \succ g f_2'$ and $(1 - g) e_{i_0}' \prec (1 - g) f_2'$. In both cases $g e_{i_0}' \succ g f_2'$ is impossible except for $g = 0$. Hence $e_{i_0}' \prec f_2'$. Take a maximal family $\{\overline{e_j'}\}_{j \in J}$ of orthogonal, equivalent, finite (resp. cyclic) projections such that $\overline{e_j'}$ is contained in f_2' and equivalent to e_{i_0}'. Notice that we may consider J as an infinite index set. Then we can choose a central projection h such that $\sum_{j \in J} h \overline{e_j'} \sim h f_2' \neq 0$. Therefore the cardinal of J is α, which implies that $f_1' = \sum_{i \in I} e_i' \sim \sum_{j \in J} \overline{e_j'} \preceq f_2'$. By symmetry, this shows $f_1' \sim f_2'$

mod N'.

Case (2°), $C(t) = (\alpha, 1)$.

Put $f_0' = f_1' \vee f_2'$, then f_0' is finite. If $\gamma(t) = 1$, $f_1' \sim f_2'$ mod $N'_{f_0'}$ from the definition of $\gamma(t)$. Hence $f_1' \sim f_2'$ mod N'. Q. E. D.

A spatial isomorphism between two W^*-algebras also induces an isomorphism between their commutators and these two isomorphisms coincide each other on the center. For the inverse statement we get

THEOREM 3. *Let M_1 and M_2 be W^*-algebras which do not contain such coupled components as (II_∞, II_1) and (II_1, II_1). Suppose that θ and θ' are isomorphisms between M_1' and M_2, M_1' and M_2' respectively and coincides each other on the center of M_1. Then θ is spatial.*

For the proof we need the following

LEMMA 3. *Let M_1 and M_2 be W^*-algebras with commutative commutators. Then any isomorphism between M_1 and M_2 is spatial.*

PROOF OF THEOREM 3. Without loss of generality we may restrict our proof to each separated coupled components which (M_1, M_1') contain. But since, except for the cases of $(I_\infty,$ finite) and (finite and type I, finite), the invariant of M_1 is constructed by couplings of algebraic invariants and theorem 2 may be applicable it is sufficient to deal with the above excluded cases. Now take an abelian projection e of M_1 with $z(e) = 1$. Then θ induces an isomorphism θ_1 between M_{1e} and $M_{2\theta(e)}$. For the commutator of M_{1e} we define θ_1' as $\theta_1'(a_e')$ $= \theta'(a')\theta(e)$. Then one verifies easily that this is an isomorphism between M_{1e}' and $M_{2\theta(e)}'$. By lemma 3, θ_1' is spatial. Moreover θ_1 is also spatial because $M_{1e} = M_{1e}''$. Hence $\overline{C}_{\theta(e)H_2}(t) = \overline{C}_{eH_1}(t)$ for $t \in \Omega_{eH_1} = \Omega_{\theta(e)H_2}$. Therefore, if it is the case of $(I_\infty,$ finite) $\gamma(t) = 1$ on Ω_{eH_1} by Lemma 2. But, as $z(e) = 1$ we have $\gamma(t) = 1$ on Ω. That is, θ is spatial. If M_1 is finite and type I with finite commutator we have $\overline{C}_{eH_1}'(t) = \overline{C}_{\theta(e)H_2}'(t)$, where $\overline{C}_{eH_1}'(t)$, $\overline{C}_{\theta(e)H_2}'(t)$ denote the invariants of M_{1e}' and $M_{2\theta(e)}'$. By lemma 1 applying M_1' and M_2' we get

$$\overline{C}_{eH_1}'(t) = \overline{C}_1'(t)e^1(t) \text{ for } t \in \Omega_{eH_1} \text{ and } \overline{C}_1'(t) \neq \infty,$$

$$\overline{C}_{\theta(e)H_2}'(t) = \overline{C}_2'(t,\theta,e)^1(t) \text{ for } t \in \Omega_{\theta(e)H_2} \text{ and } C_2'(t) \neq \infty.$$

Therefore we get $\overline{C}_1'(t) = \overline{C}_2'(t)$ on Ω considering $z(e) = 1$, which implies $\overline{C}_1(t)$ $= C_2(t)$. Hence θ is spatial.

REMARK. Theorem 3 fails in the excluded cases. In the case of (II_1, II_1), take a standard approximately finite factor M on a separable Hilbert space H. Let R^2 be a two-dimensional Euclidean space and construct $H \otimes R^2$, tensor product of H and R^2. Consider an ampliation from M to $M \otimes 1$ over $H \otimes R^2$. Then $(M \otimes 1)' = M' \otimes B(R^2)$ is also an approximately finite factor by Misonou [10], and so $(M \otimes 1)'$ is isomorphic to M'. Thus M is coupled isomorphic to $M \otimes 1$ in the sense of theorem 3. But $\overline{C}(t) = 2\overline{C}_1(t) \neq \overline{C}_1(t)$ where $\overline{C}(t)$ and $\overline{C}_1(t)$ denote the invariants of $M \otimes 1$ and M respectively, whence M is not

spatially isomorphic to $M \otimes 1$. In the case (II_∞, II_1), we shall only refer to the paper [14], [7] (an example of the W^*-algebra with a non-unitarily induced center-elementwise-invariant automorphism).

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