MATH. SCAND. 67 (1990) 223–226.

A REMARK ON THE RETRACTING OF A BALL ONTO A SPHERE IN AN INFINITE DIMENSIONAL HILBERT SPACE

TOMASZ KOMOROWSKI and JACEK WOŚKO

1. Introduction.

Let $(H, \|\cdot\|)$ be an infinite dimensional Hilbert space. In this paper we consider the problem of the existence of a lipschitzian retraction of a unit ball onto an unit sphere in H. Let us formulate it more precisely. Let $B = (x \in H \colon \|x\| \leqq 1)$ ne the closed unit ball and $S = (x \in H \colon \|x\| = 1)$ be the unit sphere.

The mapping $R \colon B \to S$ is said to be a lipschitzian retraction B onto S if:

(1) R satisfies the Lipschitz condition i.e. $\|Rx - Ry\| \leqq k \|x - y\|$ for all $x, y \in B$,

(2) $Rx = x$ for $x \in S$.

The problem of the existence of such a retraction in any normed space was considered in papers [N], [B–S] and a construction of one was given there. However it is fairly complicated. Below we present the much simpler construction of a Lipschitz retraction in any Hilbert space.

2. Construction.

To make our consideration clear we divide it into two steps.

The first step. We will construct a certain regular retraction of the B onto the S in $L^2[0,1]$-space, however it will not be a lipschitzian one. Let $p \in (0,1)$. We define

$$t(f) = \sup\left\{t \in [0,1] \colon \int_0^t \frac{f^2}{\|f\|^2} = \frac{1 - \|f\|}{1 - p}\right\}$$

for every $f \in B$, $f \neq 0$.

Received July 14, 1988; in revised form December 6, 1989.

Let $R{:}B \to S$ be given in the following way:

$$\text{if} \qquad \|f\| \geqq p \quad \text{then} \quad Rf = \begin{cases} \dfrac{|f|}{\|f\|}, & t \leqq t(f) \\ \dfrac{f}{\|f\|}, & t(f) < t \leqq 1, \end{cases}$$

$$\text{if} \qquad \|f\| \leqq p \quad \text{then} \quad Rf = \frac{\dfrac{|f|}{p} + 1 - \dfrac{\|f\|}{p}}{\left\| \dfrac{|f|}{p} + 1 - \dfrac{\|f\|}{p} \right\|}.$$

It is easy to observe two facts: R is well defined and R is the retraction of B onto S.

For $\|f\|, \|g\| \leqq p$ we have $\| |f|/p + 1 - \|f\|/p\|^2 \geqq \|f^2/p^2 + (1 - \|f\|/p)^2 \geqq 1/2$ and elementary arguments prove that in this case $\|Rf - Rg\| \leqq (2\sqrt{2}/p)\|f - g\|$. If $\|f\|, \|g\| \geqq p$ and $t(g) \geqq t(f)$ then:

$$(1) \qquad \|Rf - Rg\|^2 = \int_0^{t(f)} \left(\frac{|f|}{\|f\|} - \frac{|g|}{\|g\|} \right)^2 + \int_{t(f)}^{t(g)} \left(\frac{f}{\|f\|} - \frac{|g|}{\|g\|} \right)^2 + $$

$$+ \int_{t(g)}^{1} \left(\frac{f}{\|f\|} - \frac{g}{\|g\|} \right)^2 \leqq$$

$$\leqq \left\| \frac{f}{\|f\|} - \frac{g}{\|g\|} \right\|^2 + 2 \int_{t(f)}^{t(g)} \left(\frac{f^2}{\|f\|^2} + \frac{g^2}{\|g\|^2} \right).$$

On the other hand from definitions of $t(f)$, $t(g)$ we obtain:

$$(2) \quad \int_{t(f)}^{t(g)} \left(\frac{f^2}{\|f\|^2} + \frac{g^2}{\|g\|^2} \right) = \int_{t(f)}^{1} \frac{f^2}{\|f\|^2} - \int_{t(g)}^{1} \frac{g^2}{\|g\|^2} + \int_{t(g)}^{1} \left(\frac{g^2}{\|g\|^2} - \frac{f^2}{\|f\|^2} \right) +$$

$$\int_0^{t(f)} \left(\frac{f^2}{\|f\|^2} - \frac{g^2}{\|g\|^2} \right) + \int_0^{t(g)} \frac{g^2}{\|g\|^2} - \int_0^{t(f)} \frac{f^2}{\|f\|^2} \leqq 1 - \frac{1 - \|f\|}{1 - p} -$$

$$\left(1 - \frac{1 - \|g\|}{1 - p} \right) + \int_0^1 \left| \frac{g^2}{\|g\|^2} - \frac{f^2}{\|f\|^2} \right| + \frac{1 - \|g\|}{1 - p} - \frac{1 - \|f\|}{1 - p} \leqq$$

$$\frac{2}{1-p}\|f-g\| + \int_0^1 \left|\frac{g}{\|g\|} - \frac{f}{\|f\|}\right|\left|\frac{g}{\|g\|} + \frac{f}{\|f\|}\right| \leqq \frac{2}{1-p}\|f-g\| +$$

$$\left\|\frac{g}{\|g\|} - \frac{f}{\|f\|}\right\|\left\|\frac{g}{\|g\|} + \frac{f}{\|f\|}\right\| \leqq \frac{2}{1-p}\|f-g\| + 2\left\|\frac{g}{\|g\|} - \frac{f}{\|f\|}\right\|.$$

It is known that in a Hilbert space $\left\|\frac{g}{\|g\|} - \frac{f}{\|f\|}\right\| \leqq \frac{\|f-g\|}{\min(\|f\|, \|g\|)}$. Thus from (1), (2) and the above remark we obtain:

$$\|Rf - Rg\|^2 \leqq \frac{1}{p^2}\|f-g\|^2 + \frac{4}{p(1-p)}\|f-g\|. \tag{3}$$

The second step. We construct the Lipschitz retraction basing on the mapping R. We fix any $\varepsilon \in \left(0, \frac{1-p}{2}\right)$ and denote the set $(f \in B: 1-\varepsilon \geqq \|f\| \geqq p+\varepsilon)$ by P. By the separability of the $L^2[0,1]$ there exists a sequence $(g_i)_{i \geqq 1}$ of elements of P satisfying the following conditions:

(i) for every $i \neq j$: $\|g_i - g_j\| \geqq \varepsilon$.

(ii) for every $x \in P$ there exists index i such that $\|x - g_i\| \leqq \varepsilon$.
Let us denote $D_1 = [g_1, g_2, \ldots]$, $D_2 = (f: \|f\| \leqq p)$, and $D = D_1 \cup D_2 \cup S$. The set D has the following property:

(iii) for every $f \in B$ there exists $g \in D$ such that $\|f - g\| \leqq \varepsilon$.

We define $R_0: D \to S$ by $R_0 = R_{|D}$. After applying (3) we get

$$\|R_0 f - R_0 g\| \leqq \left(\frac{1}{p^2} + \frac{4}{p(1-p)\varepsilon}\right)^{1/2}\|f-g\|$$

if either $f, g \in D_1$ and $f \neq g$ or $f \in D_1$ and $g \in D_2 \cup S$ or $f \in S$ and $g \in D_2$ (because $\|f-g\| \geqq \varepsilon$). If either $f, g \in D_2$ or $f, g \in S$ we obtain

$$\|R_0 f - R_0 g\| \leqq \frac{2\sqrt{2}}{p}\|f-g\|.$$

Hence the Lipschitz constant of R_0 is equal to

$$L = \max\left[\frac{2\sqrt{2}}{p}, \left(\frac{1}{p^2}\frac{4}{p(1-p)\varepsilon}\right)^{1/2}\right].$$

By the Kirszbraun-Valentine theorem [O] there exists a Lipschitz extension of R_0 (also designated by R_0) on the whole B with the same Lipschitz constant. We have $R_0: B \to L^2[0,1]$. For every $f \in B$ (by (iii)) it is possible to find $g \in 0$ such that $\|f - g\| \leqq \varepsilon$, so

$$\|R_0 f\| \geqq \|R_0 g\| - \|R_0 f - R_0 g\| \geqq 1 - L\|f - g\| \geqq 1 - L\varepsilon.$$

If we put $R_1 f = \dfrac{R_0 f}{\|R_0 f\|}$ then $R_1 \colon B \to S$ is the Lipschitz retraction B onto S wih a constant

$$L_1 = \frac{L}{1 - L\varepsilon}.$$

REMARK 1. If we accept for instance $\varepsilon = \frac{1}{60}$, $p = \frac{1}{2}$ then $L_1 \approx 64{,}44\ldots$

REMARK 2. $L^2[0, 1]$ is the separable Hilbert space, so it may be treated as a subspace K of infinite dimensional Hilbert space H. Letting P be the orthogonal projection of H onto K it is easy to see that if we define $T = -R \circ P$ then T is a Lipschitz map from the unit ball B of H to itself such that there exists $k > 0$ and $\|x - Tx\| > k$. By the standard construction (see [G], [S–L]) it is possible to obtain a traction of the unit ball onto the unit sphere in every Hilbert space H.

REFERENCES

[B–S] Y. Benyamini and Y. Sternfeld, *Spheres in infinite-dimensional normed spaces are Lipschitz contractibile*, Proc. Amer. Math. Soc. 88 (1983), 439–445.

[N] B. Nowak, *On the Lipschitzian retraction of the unit ball in infinite-dimensional Banach spaces onto its boundary*, Bull. Acad. Poln. Sci. Sér. Sci. Math. 27 (1979), 861–864.

[O] Z. Opial, *Nonexpansive and Monotone Mappings in Banach spaces*, Lecture Notes, Brown University, 1967.

[G] K. Goebel, *On minimal displacement of points under Lipschitzian mappings*, Pacific J. Math. 45 (1973), 151–163.

[S–L] Y. Sternfield andP. K. Lin, *Convex sets with the Lipschitz fixed point property are compact*, Proc. Amer. Math. Soc. 93 (1985), 633–639.

INSTYTUT MATEMATYKI UMCS
PL. MARII CURIE-SKŁODOWSKIEJ 1
20–031 LUBLIN
POLAND