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A REMARK ON TRANSITIVITY OF OPERATOR ALGEBRAS

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Let H be a Hilbert space, $B(H)$ the algebra of all bounded linear operators in H, and $\mathfrak{A}$ a C*-subalgebra strongly dense in $B(H)$. If an operator $B \in B(H)$, a finite set of vectors $x_1, \ldots, x_n$ in H, and a number $\varepsilon > 0$ are arbitrarily given then by the definition of the strong operator topology there is an element $A \in \mathfrak{A}$ such that $|Ax_i - Bx_i| < \varepsilon$ for $i = 1, \ldots, n$. The Kaplansky density theorem (see [2], p. 43) asserts that A can be chosen with $|A| \leqq |B|$. On the other hand, it follows from a result proved here that there exists an operator $C \in \mathfrak{A}$ such that $|C| \leqq |B| + \varepsilon$ only, but $Cx_i = Bx_i$ for $i = 1, \ldots, n$. Clearly for this purpose it will suffice to suppose $x_1, \ldots, x_n$ orthonormal, and we shall do so henceforth.

Given another set of vectors $y_1, \ldots, y_n$ there are, of course, operators in $B(H)$ transforming x_i into y_i for $i = 1, \ldots, n$. The norm of any such operator must be clearly $\geqq \beta$ where

$$\beta = \sup |\lambda_1 y_1 + \ldots + \lambda_n y_n|$$

is taken over all complex λ_j with $|\lambda_1|^2 + \ldots + |\lambda_n|^2 = 1$. It is obvious that the operator V defined by the formula

$$Vz = (z, x_1)\, y_1 + \ldots + (z, x_n)\, y_n\,, \quad z \in H$$

has norm $|V| = \beta$ and satisfies $Vx_i = y_i$ for $i = 1, \ldots, n$. However, V need not lie in $\mathfrak{A}$. We shall show in Theorem 2 that, for each $\varepsilon > 0$, there exists an operator T in $\mathfrak{A}$ such that $Tx_i = y_i$ for $i = 1, \ldots, n$, and $|T| \leqq \beta + \varepsilon$. Clearly this estimate is the best possible. Transitivity of strongly dense C*-algebras has been proved first by R. V. KADISON [3]. In the present remark, we use a method suggested for that purpose by V. PTÁK [5], obtaining thereby a significant simplification of the proof as well as an improvement of the estimate in Dixmier's book [1], p. 43–44.

The case $n = 1$ has been solved by V. Pták [5] and the general case goes similarly. It is based on the Pták Induction Theorem recently obtained in [4]; see also [5], [6] where further important applications to various problems of analysis are described.

For the present remark a somewhat special version of the induction theorem will be quite sufficient. It is formulated as Theorem 1 below after some necessary definitions.

If (E, d) is a metric space and $x \in E$, we denote by $U(x, r)$ the set $U(x, r) = \{y \in E; d(y, x) \leqq r\}$, r being a positive number. Let $R = \{r; 0 < r < t\}$ be an interval with $t > 0$ fixed. Assume that for each $r \in R$ a set $W(r) \subset E$ is given and put

$$W(0) = \bigcap_{s>0} (\bigcup_{r \leqq s} W(r))^{-} .$$

It can be easily seen that $W(0)$ is in fact the set of those $x \in E$ for which there are a sequence $r_n \to 0$ and points $x_n \in W(r_n)$ with $x_n \to x$. In this situation we can state

Theorem 1. *Let (E, d) be complete. Let $0 < k < 1$ be fixed. Suppose the implication*

$$x \in W(r) \Rightarrow U(x, r) \cap W(kr) \neq \emptyset$$

to be true for any $r \in R$. If at least one of the sets $W(r)$, $r \in R$ is non-void, then so is $W(0)$.

The proof is straightforward and can be found in any of [4], [5], [6]. Now we can state

Theorem 2. *Let $\mathfrak{A}$ be a strongly dense C^*-subalgebra of $B(H)$ Let $x_1, \ldots, x_n$ be orthonormal vectors and let $y_1, \ldots, y_n$ be given vectors in H; denote by β the lowest possible norm of an operator in $B(H)$ taking x_i into y_i for $i = 1, \ldots, n$. Then, for each $\varepsilon > 0$, there exists an operator C in $\mathfrak{A}$ such that $Cx_i = y_i$ for $i = 1, \ldots, n$, and $|C| \leqq \beta + \varepsilon$.*

Proof. Clearly we may assume $\beta = 1$. Let $\varepsilon > 0$ be given. Put $k = \varepsilon/(1 + \varepsilon)$, and for each $0 < r < 1$ construct a set $W(r)$ in $\mathfrak{A}$ as follows

$$W(r) = \{T \in \mathfrak{A};\ |T| \leqq (1 + \varepsilon)(1 - r),\ |Tx_i - y_i| < r/n \text{ for } i = 1, \ldots n\} .$$

We have to verify the implication assumed in Theorem 1. Hence take a $T \in W(r)$. Define an operator S by the formula

$$Sz = (z, x_1)(y_1 - Tx_1) + \ldots + (z, x_n)(y_n - Tx_n), \quad z \in H .$$

Then $S \in B(H)$, $|S| \leqq r$, and $Sx_i = y_i - Tx_i$. By the Kaplansky density theorem there is a $Q \in \mathfrak{A}$ such that $|Q| \leqq r$ and $|Qx_i - Sx_i| < kr/n$ for $i = 1, \ldots, n$. Then the sum $T + Q$ lies in $\mathfrak{A} \cap U(T, r)$; moreover it belongs to $W(kr)$ since

$$|T + Q| \leqq |T| + |Q| \leqq (1 + \varepsilon)(1 - r) + r = (1 + \varepsilon)(1 - kr)$$

and

$$|(T + Q)x_i - y_i| \leqq |Tx_i - y_i + Sx_i| + |Qx_i - Sx_i| < 0 + kr/n = kr/n$$

for $i = 1, \ldots, n$.

Also $W(k)$ is non-void since the operator V can be approximated, in virtue of the Kaplansky density theorem again, by an element $W \in \mathfrak{A}$ of norm not exceeding 1 in such a way that $|Wx_i - Vx_i| < k/n$, $i = 1, \ldots, n$. In view of $(1 + \varepsilon)(1 - k) = 1$ and $Vx_i = y_i$, this W belongs to $W(k)$.

By Theorem 1 the set $W(0)$ is non-void, and any element $C \in W(0)$ is clearly a solution. Thus the theorem is proved.

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