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A REMARK ON UNIPOTENT GROUPS OF CHARACTERISTIC $p>0$

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Borel and Springer [1] deal with a unipotent group U defined over a field of prime characteristic p and investigate the conditions of the existence of the one dimensional subgroup to which a given element of the Lie algebra $L(U)$ of U is tangent. They use the lemma (9.15) (ii) (p. 493) in the proof of the last theorem (9.16) (ii) (p. 495). In this report we shall show that this lemma is not correct (cf. § 2). But a modification of it does not disturb the truth of the theorem. Moreover we want to show that the theorem (9.16) (iii) in [1] is still true under a weaker assumption (cf. § 1). The notations in this report are the same as in [1].

§ 1. In this section we give a modification of the lemma (9.15) (ii) in [1]. It is given as a corollary of the following lemma. The form of the weight in the assumption is changed from $(p^i+p^j)a$ to p^ia+p^jb $(a, b\in\Psi)$. The proof proceeds similarly to that given in [1].

LEMMA. *Let U be a unipotent k-group and T a k-torus which acts k-morphically on U. Let N be a connected central k-subgroup of U, stable under T, such that U/N is commutative. Let U_1, W and W_1 be subgroups of U containing N, stable under T, such that U_1 and W_1 are normal subgroups of U and W, respectively, and such that U/U_1 and W/W_1 are isomorphic to $\boldsymbol{G}_a$. Put $\Phi(T, U/U_1)=\{a\}$ and $\Phi(T, W/W_1)=\{b\}$. If $\Phi(T, N)$ does not contain any element of the form p^ia+p^jb $(i, j\geqq 0)$ and the commutator groups (U, W_1) and (U_1, W) are trivial, then (U, W) is also trivial.*

Proof. Let

$$\alpha:\quad U\times W\rightarrow N$$

be the commutator map, sending (x, y) to $x\cdot y\cdot x^{-1}\cdot y^{-1}$. Using that (U, W_1) and (U_1, W) are trivial, we see that this induces a T-equivariant morphism

$$\alpha':\quad U/U_1\times W/W_1\rightarrow N.$$

By Chevalley ([2] Exp. 9, lemme 2, p. 1), we may find a composition series

$$N=N_0\supset N_1\supset\cdots\supset N_q=\{e\}$$

of connected subgroups, stable under T, such that successive quotients are isomorphic

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to $\boldsymbol{G}_a$. Now we want to prove that $\operatorname{Im} \alpha' \subset N_{j+1}$ assuming $\operatorname{Im} \alpha' \subset N_j$. The map α' defines naturally a T-equivariant morphism of varieties,

$$\beta\colon\ U/U_1 \times W/W_1 \to N_j/N_{j+1}.$$

We see immediately that the restriction of β to $U/U_1 \times \{y\}$ or to $\{x\} \times W/W_1$ is a group homomorphism (where x and y are fixed elements of U/U_1 and W/W_1, respectively). Identifying U/U_1, W/W_1 and N_j/N_{j+1} to $\boldsymbol{G}_a$, β is given by a polynomial $F(X, Y) \in \bar{k}[X, Y]$ with the following conditions

$$F(X+X', Y) = F(X, Y) + F(X', Y), \tag{1}$$

$$F(X, Y+Y') = F(X, Y) + F(X, Y') \tag{2}$$

and

$$F(t^a X, t^b Y) = t^c F(X, Y) \qquad \text{for some } c \in \Phi(T, N) \tag{3}$$

$$(X, X'; Y, Y' \in \boldsymbol{G}_a, t \in T).$$

If $F \neq 0$, then the above conditions imply that $c = p^i a + p^j b$ $(i, j \geqq 0)$, which contradicts the assumption of the lemma. Hence $F = 0$. By induction on j, this shows that α is trivial. Therefore $(U, W) = \{e\}$. (Q.E.D.)

COROLLARY. *Let U be a unipotent k-group and T a k-torus which acts k-morphically on U. Let N be a connected central k-subgroup of U, stable under T, such that T has no fixed point $\neq e$ in $U' = U/N$. Let Φ and Ψ be the sets of weights of T in N and U', respectively. If U/N is commutative and Φ does not contain any element of the form $p^i a + p^j b$ for $a, b \in \Psi$ $(i, j \geqq 0)$, then U is commutative.*

Proof. By Chevalley ([2] loc. cit.), we may find composition series

$$U = U_m \supset U_{m-1} \supset \cdots \supset U_0 = N$$

of connected subgroups, stable under T, such that successive quotients are isomorphic to $\boldsymbol{G}_a$. The set of weights of T in U_i/N is a subset of Ψ. We prove by induction on i that U_i is central in U. Since $N = U_0$ is cetral, we may assume that U_i is central in U. It suffices to show that $(U_{i+1}, U_{k+1}) = \{e\}$ under the assumption that $(U_{i+1}, U_k) = \{e\}$ $(i \leqq k < m)$. This follows immediately by the lemma. Hence U is commutative. (Q.E.D.)

Now we give a proof of the theorem (9. 16) (ii) in [1]. As in the proof of the theorem (9. 8) ([1], p. 487) we may assume that S is a maximal, k-split and one dimensional k-torus. Furthermore we may assume that $b \in \Phi(S, G) \subset \boldsymbol{N} b = \{nb; n \in \boldsymbol{N}\}$ by the theorem (9. 16) (i) in [1]. We use induction on $\dim U$. Let N be a non-trivial connected central k-subgroup of U, stable under S. If X does not belong to the Lie algebra $L(N)$ of N, we apply the induction assumption to G/N and get a commutative k-subgroup W/N of G/N such that $X \in L(W/N)$ and $\Phi(S, W/N) \subset \{p^j b; j \in \boldsymbol{N}\}$

Then W is commutative by the above corollary and $X\in L(W)$. Hence in any case X is tangent to a commutative k-subgroup of U, stable under S. Then we can find a desired subgroup V using Prop. (9. 10) ([1], p. 489).

Next we shall prove the following theorem, which is an improvement of the theorem (9. 16) (iii) in [1], and which covers the case where $\Phi(S, G)$ contains some element of the form $(p^i+p^i)b$ $(i\geqq 0)$.

THEOREM. *Let G be a solvable k-subgroup, U the unipotent radical of G and S a k-torus in G. Let b be a non-trivial character of S, and X a non-zero nilpotent element of $L(G)$ such that* $\mathrm{Ad}\, s(X)=s^b X$ $(s\in S)$. *If $\Phi(T, G)$ does not contain any element of the form $(p^i+p^j)b$ $(i, j\geqq 0;\ i\neq j)$ or $p^i b$ $(i\geqq 1)$. Then there exists a connected k-subgroup of U of dimension one, stable under S, and to which X is tangent.*

Proof. We proceed by induction on dim U. Let N be a non-trivial connected central k-subgroup of U, stable under S. If $X\in L(N)$, our result follows immediately by the theorem (9. 8) in [1]. Let $X\notin L(N)$. Applying the induction assumption to U/N we have a connected k-subgroup W of U, stable under S, such that W/N is isomorphic to $\boldsymbol{G}_a$ and $L(W/N)=\bar{k}X$. Let

$$\alpha:\quad W\times W\to N$$

be the commutator map. As in the proof of the lemma we get the following polynomial from this map,

$$F(X, Y)=c_1X^{p^i}Y^{p^j}+c_2X^{p^j}Y^{p^i}. \qquad (c_1, c_2\in\bar{k})$$

Since $F(X, X)=0$, we have $c_1=-c_2$. This implies $F=0$ if $i=j$. So W must be commutative. Then the theorem follows from the theorem (9. 8) in [1]. (Q.E.D.)

§ **2.** Here we give a counterexample of the lemma (9. 15) (ii) in [1]. Let U be the unipotent group

$$\left\{\begin{bmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{bmatrix};\ x, y, z\in\bar{k}\right\}$$

and T the one dimensional torus $\boldsymbol{GL}_1$.

We define an operation of T on U by

$$\left(t, \begin{bmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{bmatrix}\right)\to\begin{bmatrix}1 & t^a x & t^{a+b}y\\ 0 & 1 & t^b z\\ 0 & 0 & 1\end{bmatrix}, \qquad \left(t\in T, \begin{bmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{bmatrix}\in U\right)$$

where a and b are any fixed non-trivial characters of T.

Let

$$N=\left\{\begin{bmatrix}1 & 0 & x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix};\ x\in\bar{k}\right\}.$$

N is the connected central subgroup, stable under T, and U/N is commutative. We have $\Phi(T, U/N)=\{a, b\}$ and $\Phi(T, N)=\{a+b\}$. We can choose a, b such that $a+b$ is not of the form $(p^i+p^j)a$ or $(p^i+p^j)b$ for any $i, j\geqq 0$. (For example we may take $a=1$, $b=2$ if $p=3$.) Since U is not commutative, this is a counterexample of the lemma (9. 15) (ii) in [1].

References

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[2] Chevalley, C., Séminaire sur la classification des groupes de Lie algébriques. Paris (1958).

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