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A REPRESENTATION THEOREM FOR "PREFERENCE FOR FLEXIBILITY"

BY DAVID M. KREPS[1]

This paper concerns individual choice among "opportunity sets," from which the individual will later choose a single object. In particular, it concerns preference relations on opportunity sets which satisfy "preference for flexibility," a set is at least as good as all of its subsets, but which may not satisfy "revealed preference," the union of two sets may be strictly preferred to each one taken separately. A representation theorem is given which "rationalizes" such choice behavior as being as if the individual is "uncertain about future tastes."

1. INTRODUCTION

IN MANY PROBLEMS of individual choice, the choice is made in more than one stage. At early stages, the individual makes decisions which will constrain the choices that are feasible later. In effect, these early choices amount to choice of a subset of items from which subsequent choice will be made. This paper concerns choice among such opportunity sets, where the individual has a "desire for flexibility" which is "irrational" if the individual knows what his subsequent preferences will be.

A simple example is that of making reservations at a restaurant. Imagine that the only way that restaurants vary is in the menu of meals which they will serve. The individual is assumed to know the menus at all restaurants that he might select. Eventually, the individual will choose a meal, but his initial choice is of a restaurant/menu from which he will later choose his meal. Let Z be the set of possible meals, with generic element z. Let X be the set of all conceivable menus, with generic element x. That is, X is the set of nonempty subsets of Z. Initial choice is the selection of one x from some subset of X (the set of available menus), and subsequent choice is the selection of one z from the x chosen. For simplicity, assume that Z is finite.

The standard model of this situation posits a complete and transitive binary relation $\gtrsim$ on Z, which represents the individual's (weak) preferences over meals. Choice of a menu is induced from $\gtrsim$ in the obvious fashion: Define

$$x \mathrel{\dot{\gtrsim}} x' \text{ iff for all } z' \in x' \text{ there exists } z \in x \text{ such that } z \gtrsim z'. \tag{1.1}$$

Choice of a menu then amounts to selecting a $\dot{\gtrsim}$-maximal element of the set of available menus. This relation $\dot{\gtrsim}$ is complete and transitive and satisfies

$$x \mathrel{\dot{\gtrsim}} x' \text{ implies } x \mathrel{\dot{\sim}} x \cup x', \tag{1.2}$$

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where $\dot{\sim}$ is the indifference relation induced from $\dot{\succsim}$. In fact, (1.2) characterizes all preference relations on X which arise from some $\succsim$ on Z in the following sense: A binary relation $\dot{\succsim}$ on X is complete and transitive and satisfies (1.2) if and only if there exists some complete and transitive $\succsim$ on Z such that (1.1) holds. (The proof of this is left to the reader.)

This paper considers preferences on X which do not satisfy (1.2), hence which cannot be "rationalized" as above by some $\succsim$ on Z. Of interest are preferences which exhibit a "desire for flexibility":

(1.3) $\quad x \supseteq x'$ implies $x \dot{\succsim} x'$.

But, as opposed to preferences which satisfy (1.2), $x \cup x'$ may be *strictly* preferred to both x and x'. Consider the menus/meals example. Suppose that the individual prefers a menu containing only *steak* to one containing only *chicken*. But he strictly prefers a menu with both *steak* and *chicken* to either of the first two, because it gives him greater flexibility. This sort of preference for flexibility is discussed by Koopmans [**8**]. (Note that Koopmans has a multiperiod setting, with consumption in each period. Thus the terminology in [**8**] is most directly comparable with Section 4 of this paper.) Koopmans' Axiom 1 is roughly equivalent to the supposition here that choice today can be represented by a preference relation $\dot{\succsim}$, and his Axiom 2 is (1.3). (Again, comparison with (4.2) instead of (1.3) is most natural.)

Why should anyone desire such flexibility? The individual choosing a menu would likely make some statement about being uncertain about something or other. For some reason, he is unsure about what will be his mood on the evening in question. Perhaps he is unsure about what he will have had for dinner on the previous evening. Perhaps he cannot explain why he is uncertain about his future mood, but he claims to be unsure of it nonetheless. This rationalization for (1.3) is naturally termed "uncertainty about future tastes." (Cf. Koopmans [**8**, pp. 246, 254].) It *could* be modeled as follows:

Posit a random variable s (the state of the individual's tastes or mood), a subjective probability measure π on the possible states, and a "state dependent cardinal utility function" $U: Z \times S \to R$, such that if $v: X \to R$ is defined by

$$(1.4) \qquad v(x) := \sum_s \pi(s) \cdot [\max_{z \in x} U(z, s)],$$

then v represents preference on X. This sort of representation gives rise to preferences which satisfy (1.3), and allows models in which $\{$*steak*, *chicken*$\}$ is strictly preferred to $\{$*steak*$\}$ and to $\{$*chicken*$\}$: Take $S = \{s_1, s_2\}$. If tastes s_1 are realized, the individual likes steak better than chicken—say, $U(\textit{steak}, s_1) = 1$ and $U(\textit{chicken}, s_1) = 0$. If s_2 occurs, he likes chicken better than steak: $U(\textit{steak}, s_2) = 0$ and $U(\textit{chicken}, s_2) = 1$. He thinks that s_1 is more likely than s_2, say, $\pi(s_1) = .9$. Then the menu $\{$*steak*$\}$ has "expected utility" .9, and $\{$*chicken*$\}$, .1. But $\{$*steak*, *chicken*$\}$ is best of all, having "expected utility" 1.

The representation given in (1.4) has appeared in the literature, notably in the work of Goldman [**2**, **3**]. (See also Henry [**6**] and Jones and Ostroy [**7**].) These papers show that preference for flexibility as represented in (1.4) can, in certain circumstances, lead to interesting economic phenomena, such as "liquidity demand for money."

The purpose of this paper is to provide the connection between preferences for opportunity sets that satisfy the qualitative axiom (1.3) and those that can be represented as in (1.4). If preferences can be represented as in (1.4), then they satisfy (1.3), but the converse fails. In particular, preferences represented as in (1.4) satisfy

(1.5) $x \dot{\sim} x \cup x'$ implies that for all x'', $x \cup x'' \dot{\sim} x \cup x' \cup x''$.

This does not follow from (1.3). It is contended, however, that (1.5) is a reasonable axiom. If $x \dot{\sim} x \cup x'$, then the flexibility gained by adjoining x' to x is of no value. When then should adding x' to the larger $x \cup x''$ be of any value? In any case, the conjunction of (1.3) and (1.5) is necessary and sufficient for a "rationalization" of $\dot{\succsim}$ of the type in (1.4).

THEOREM 1: *If Z is finite, then a binary relation $\dot{\succsim}$ on X is complete and transitive and satisfies* (1.3) *and* (1.5) *if and only if there exists a finite set S and a function $U: Z \times S \to R$ such that*

(1.6) $$v(x) := \sum_s [\max_{z \in x} U(z, s)]$$

represents $\dot{\succsim}$.

Note that the probability measure π is suppressed here, as it obviously has no real significance. It will become apparent in the proof of this theorem that the "additive representation" in (1.6) also has limited significance—the representation is basically ordinal in character, and an ordinal restatement will be given in the sequel.

The approach taken in this paper is exclusively descriptive. Preferences over opportunity sets are taken as given, and the representation "explains" these preferences as being as if the individual were maximizing a "state dependent utility function of subsequent consumption." But there is no claim that the individual would acknowledge that the representation has any validity. He *might* be calculating exactly in the fashion of the representation, using "states" that to him are clearly identified sources of subjective uncertainty. However, a given preference relation can be represented in many ways, involving many different collections of "states." (Theorem 2 gives a characterization of collections of "states" for which the representation is possible; see Section 3.) There is no way to guarantee that the representation that is constructed in the proof of the theorem is the one that the individual is using. A normative approach to this problem would look quite different: If an individual expressed a desire for flexibility and approached me for advice, I would advise him (i) to try to assess what is the uncertainty that he perceives and (ii) to consider, if only to sharpen his assessments, how he would rank "opportunity sets contingent on the states" that he perceives. (Moreover, part (ii) of this normative program would add substance to the additive nature of the representation.) The normative content of this paper is only that an individual who wishes to obey (1.3) and (1.5) might find it helpful to seek a representation of his preferences as in (1.4), using (i) and (ii) above.

Two connections with the literature should be noted. The phenomenon "preference for flexibility" is natural when there is explicit uncertainty and sequential decision making—individuals will want to be able to adapt their actions to circumstances as the uncertainty concerning those circumstances resolves. Among the many papers which make this point are Hart [**5**], Marshak and Nelson [**10**], and Merkhofer [**11**]. The difference between those papers and the analysis here is that in those papers, the uncertainty is explicit. Here there is no explicit uncertainty.

A different treatment of preference for opportunity sets appears in the literature under the rubric "changing tastes." Representative papers are Strotz [**13**], Hammond [**4**], and Yaari [**14**]. (Stigler and Becker [**12**] give an alternative view of this phenomenon.) In these papers, changes in tastes are anticipated, and the individual attempts to constrain later choices so that eventual choice is as good as possible from the point of view of current tastes. In such situations flexibility is not desirable. Rather, a desire for *inflexibility* or *precommitment* characterizes the choice of opportunity sets.

This paper is organized as follows. The key to the analysis and the representation is a "domination relation" $\dot{\geqslant}$ defined by $x \dot{\geqslant} x'$ if $x \dot{\sim} x \cup x'$. This relation is analyzed in Section 2. Section 3 contains the ordinal restatement of Theorem 1, the proof of Theorem 1, discussion concerning the "ordinal nature" of the representation, and a characterization of the possible "states" for a given preference relation $\dot{\gtrsim}$. In Section 4 the results of Section 3 are adapted to the context of a two period consumption-investment budgeting problem. Section 5 gives extensions of the results for infinite Z.

2. THE DOMINATION RELATION

Given is a complete and transitive binary relation $\dot{\gtrsim}$ on X which satisfies (1.3) and (1.5). Define a binary relation on X by

(2.1) $\quad x \dot{\geqslant} x'$ if $x \dot{\sim} x \cup x'$.

This is a "domination" relation in the following sense: If $x \dot{\geqslant} x'$, then the flexibility gained by adjoining x' to x is of no value. Looking ahead to the eventual representation, no matter what state ensues, there is something in x as good as everything in x'. (Of course, what this something is may depend on the state.) So x as a set dominates x'.

Note that (1.5) can be rephrased in terms of $\dot{\geqslant}$: If $x \dot{\geqslant} x'$ and $x'' \supseteq x$, then $x'' \dot{\geqslant} x'$. Further properties of $\dot{\geqslant}$ are compiled in the following lemma.

LEMMA 1: *If* $\dot{\gtrsim}$ *is complete and transitive and satisfies* (1.3) *and* (1.5), *then* $\dot{\geqslant}$ *defined by* (2.1) *satisfies*:

(a) $\dot{\geqslant}$ *is reflexive and transitive.*
(b) $x \supseteq x'$ *implies* $x \dot{\geqslant} x'$.
(c) *If* $x \dot{\geqslant} x' \supseteq x''$, *then* $x \dot{\geqslant} x''$.
(d) *If* $x_1 \dot{\geqslant} x_2$ *and* $x_3 \dot{\geqslant} x_4$, *then* $x_1 \cup x_3 \dot{\geqslant} x_2 \cup x_4$.
(e) *For every* x *there exists some set* $x' \supseteq x$ *such that* $x \dot{\geqslant} x''$ *if and only if* $x'' \subseteq x'$.

PROOF: (a) Reflexivity is obvious. If $x \dot{\geqslant} x' \dot{\geqslant} x''$, then $x \dot{\sim} x \cup x' \dot{\gtrsim} x' \dot{\sim} x' \cup x''$. Applying (1.5) to $x' \dot{\sim} x' \cup x''$ yields $x \cup x' \dot{\sim} x \cup x' \cup x''$, thus $x \dot{\sim} x \cup x' \dot{\sim} x \cup x' \cup x'' \dot{\gtrsim} x \cup x'' \dot{\gtrsim} x$ (the last two by (1.3)), and so $x \dot{\sim} x \cup x''$.

(b) Obvious by the definition of $\dot{\geqslant}$ and by (1.3).

(c) Follows from (a) and (b).

(d) $x_1 \dot{\geqslant} x_2$ implies $x_1 \cup x_3 \dot{\sim} x_1 \cup x_2 \cup x_3$, and $x_3 \dot{\geqslant} x_4$ implies $x_1 \cup x_2 \cup x_3 \dot{\sim} x_1 \cup x_2 \cup x_3 \cup x_4$. Thus $x_1 \cup x_3 \dot{\sim} x_1 \cup x_2 \cup x_3 \cup x_4$.

(e) By (d), if $x \dot{\geqslant} x_1$ and $x \dot{\geqslant} x_2$, then $x \dot{\geqslant} x_1 \cup x_2$. So for x', take the union for all x'' such that $x \dot{\geqslant} x''$. Since Z is finite, this is a finite union. Thus by induction $x \dot{\geqslant} x'$: Apply (c) *Q.E.D.*

Part (e) of the lemma is the key to the subsequent analysis. It says that for every set x there is some largest set x' which x dominates. In terms of the eventual representation, this is the set of all z such that no matter what state occurs, there is something in x as good as z. Let $f: X \to X$ be the map which carries each x into "its" x'. Formally, let

$$f(x) := \bigcup_{\{x' \in X : x \dot{\geqslant} x'\}} x'. \tag{2.2}$$

Part (e) of the lemma establishes that

$$x'' \subseteq f(x) \text{ if and only if } x \dot{\geqslant} x''. \tag{2.3}$$

Note that $x \subseteq f(x)$ and $x \dot{\geqslant} f(x)$. Moreover:

LEMMA 2: (a) *For all* x, $f(f(x)) = f(x)$. (b) $x \dot{\geqslant} x'$ *if and only if* $f(x) \supseteq f(x')$.

PROOF: For (a), note that $x \dot{\geqslant} f(x) \dot{\geqslant} f(f(x))$, so $x \dot{\geqslant} f(f(x))$. Apply (2.3). For (b), $x \dot{\geqslant} x'$ if and only if $x \dot{\geqslant} f(x')$ (by transitivity of $\dot{\geqslant}$) if and only if $f(x) \supseteq f(x')$.
Q.E.D.

3. THE REPRESENTATION THEOREM

Theorem 1 can be "restated" in ordinal form as follows.

THEOREM 1′: *If* Z *is finite, a binary relation* $\dot{\gtrsim}$ *on* X *is complete and transitive and satisfies* (1.3) *and* (1.5) *if and only if there exist a finite set* S, *a function* $U: Z \times S \to R$, *and a strictly increasing function* $u: R^S \to R$ *such that if* $w: X \to R^S$

is defined by

$$(3.1)\qquad (w(x))(s) := \max_{z\in x} U(z, s),$$

then $u \circ w$ $(: X \to R)$ *represents* $\dot{\gtrsim}$.

"Strictly increasing" means that if w and w' from R^S are such that $w(s) \geqslant w'(s)$ for every s, with strict inequality for some s, then $u(w) > u(w')$. In Theorem 1, u is component addition. Otherwise the two statements are identical. Of course, it is certainly not evident that they are equivalent.

PROOF: The "if" part of the theorem is left to the reader. Suppose $\dot{\gtrsim}$ is complete and transitive and satisfies (1.3) and (1.5). Let $v: X \to R$ be a numerical representation of $\dot{\gtrsim}$. Define $S := \{x \in X : x = f(x)\}$ and

$$U(z, s) := \begin{cases} 1 & \text{if } \quad z \notin s, \\ 0 & \text{if } \quad z \in s. \end{cases}$$

Then w defined by (3.1) is

$$(w(x))(s) = \begin{cases} 1 & \text{if } \quad x \backslash s \neq \varnothing, \\ 0 & \text{if } \quad x \subseteq s. \end{cases}$$

The key observation is that for this w,

$$(3.2)\qquad w(x) \geqslant w(x') \text{ if and only if } x \dot{\geqslant} x'.$$

For if $x \dot{\ngeqslant} x'$, then $x' \nsubseteq f(x)$, so that $(w(x))(f(x)) = 0 < 1 = (w(x'))(f(x))$. While if $x \dot{\geqslant} x'$, then $(w(x))(s) = 0$ implies $x \subseteq s$; thus $f(x) \subseteq f(s) = s$ (Lemma 2(b)); thus $x' \subseteq s$ and $(w(x'))(s) = 0$. So $w(x) \geqslant w(x')$.

Now define u on $w(X)$ by

$$u(w(x)) := v(x).$$

This is well defined: If $w(x) = w(x')$, then $x \dot{\geqslant} x' \dot{\geqslant} x$ so $x \dot{\sim} x'$ and $v(x) = v(x')$. Moreover, u is strictly increasing on $w(X)$: If $w(x) \geqslant w(x')$ with strict inequality for some s, then $x \dot{\geqslant} x' \dot{\ngeqslant} x$ (by (3.2)) and $v(x) > v(x')$.

Since $w(X)$ is finite and S is finite, u can be extended to all of R^S in a strictly increasing fashion. (Alternatively, this can be deduced from the proof of Theorem 1.) *Q.E.D.*

The theorem and its proof can be explained as follows. Each state s is characterized by $U(\cdot, s)$ and represents a possible "second period preference relation." Thus $(w(x))(s)$ represents the "maximal second period utility obtainable from the opportunity set x in state s." The theorem states that first period preference is weakly decomposable into the vector of these "maximal obtainable utilities." This, by itself, is without content. If all preference relations are admitted as states, then $x \to w(x)$ is one-to-one which is sufficient to represent *any* complete and transitive binary relation. The content of the theorem comes from the requirement that u is strictly increasing. That is, to be "consistent," preference for an opportunity set is required to be strictly increasing in the vector of "possible second period utilities" which make up the decomposition. This requirement rules out certain second period preference relations as "states." For example, if $\{chicken, steak\} \dot{\sim} \{steak\}$, then in no state can *chicken* be strictly preferred to *steak*. In general, if $x \dot{\geqslant} x'$, then $\max_{z \in x} U(z, s) \geqslant \max_{z \in x'} U(z, s)$ must hold for every s. Following the analysis of Section 2, a necessary and sufficient condition for this is that for every s and real number r, the set $x = \{z \in Z : U(z, s) \leqslant r\}$ satisfies $f(x) = x$. On the other hand, there must be "sufficiently many" states to achieve the representation: If $x \dot{>} x'$, then there must exist *some* state s with $\max_{z \in x} U(z, s) > \max_{z \in x'} U(z, s)$. For given S and U, these two conditions are necessary and sufficient for there to exist a strictly increasing u that represents $\dot{\gtrsim}$. (See Theorem 2. Note that these two conditions are necessary and sufficient for (1.3) and (1.5).) In the proof of the theorem, such an S and U pair is produced. The first condition is met by construction, and the second holds since if $x \dot{>} x'$, then $x \not\subseteq f(x')$ and $(w(x))(f(x')) = 1 > 0 = (w(x'))(f(x'))$.

While this S and U pair meet the conditions, there are other pairs which are adequate. This is of particular interest because the set S in the proof is quite large. For example, if condition (1.2) holds, so that a representation with singleton S is possible, the proof uses as many states as there are indifference classes of $\dot{\gtrsim}$. It may be desirable to economize on the number of states needed. One way to do this is the following. Let $F := \{x \in X : f(x) = x\}$. (This notation will be used throughout the sequel.) Let S be a collection of chains in F (ordered by set inclusion), with Z in each chain and with every $x \in F$ found in at least one chain in S. And (for v a numerical representation of $\dot{\gtrsim}$), let $U(z, s) := \min\{v(x) : z \in x \in s\}$. Such S and U admit a representation. Note that in cases where (1.2) holds, this can be used to obtain a representation with one state.

This can be generalized to give a "uniqueness" result for the representation. Suppose that U and u give a representation. It is clear that if U' is such that for each s, $U(\cdot, s)$ and $U'(\cdot, s)$ are ordinally equivalent, then there is some u' such that U' and u' give a representation. Otherwise, the "uniqueness" of the representation depends on the "possible second period preference relations" that make up the decomposition.

THEOREM 2: *Let $\dot{\gtrsim}$ be complete and transitive and satisfy* (1.3) *and* (1.5). *Then a representation of the form given in Theorem* 1′ *is possible for given S and U if and only if* (a) *for each $s \in S$, the set $\{x \in X : x = \{z \in Z : U(z, s) \leqslant r\}$ for $r \in R\}$ forms a chain in F, and* (b) *every $x \in F$ can be written as $\bigcap_{s \in S} x(s)$, where $x(s)$ is selected from the chain in F corresponding to s.*

The proof is quite simple and is left to the reader. Note that since this establishes necessary and sufficient conditions, it would be useful in a normative analysis: Suppose the individual has preferences which satisfy (1.3) and (1.5) and that he nominates a specific S and U. (That is, he identifies the set of "second period preference relations" that he deems possible.) Then the theorem can be used to check whether his professed preferences are consistent with this S and U, in the sense that a representation of the former is possible using the latter.

The additive representation is derived from the following lemma.

LEMMA 3: *Suppose Y is an arbitrary finite set endowed with two binary relations B and G such that*: (a) *B is complete and transitive.* (b) *G is reflexive.* (c) *If $y\ G\ y'$ and $y \neq y'$, then not $y'\ B\ y$. Then there exist negative numbers $a(y)$ such that $y'\ B\ y''$ if and only if $\Sigma_{\{y:yGy'\}}\, a(y) \geqslant \Sigma_{\{y:yGy''\}}\, a(y)$.*

PROOF: By (a), B is a (weak) preference relation. Let B° denote the induced indifference relation and B^* the induced strict preference relation. Also, let G^* denote the relation given by $y\ G^*\ y'$ if $y\ G\ y'$ and $y \neq y'$. Then (c) can be rewritten $y\,G^*\,y'$ implies $y\,B^*\,y'$. Writing $w(y')$ for $\Sigma_{\{y:yGy'\}}\, a(y)$ and $w^*(y)$ for $\Sigma_{\{y:yG^*y'\}}\, a(y)$, by definition $w(y') = a(y') + w^*(y')$.

The constants $a(y)$ are defined "inductively." Begin with the B°-equivalence class of B-most preferred elements of Y, and set $a(y')$ equal to any negative number for y' in this set. Now proceed "downward" through the B°-equivalence classes. Noting that $w^*(y')$ is fixed once $a(y)$ has been defined for all y such that $y\ B^*\ y'$ is the key. This permits the selection of $a(y')$ in the induction step such that $a(y') + w^*(y')$ is (i) equal across the B°-equivalence class and (ii) smaller than $w(y)$ for any y such that $y\ B^*\ y'$. Since there are finitely many elements of Y, there are finitely many B°-equivalence classes, and the induction procedure will give the representation. *Q.E.D.*

PROOF OF THEOREM 1: The "if" part follows from Theorem 1′. Suppose that $\dot{\succsim}$ is complete and transitive and satisfies (1.3) and (1.5). As in the proof of Theorem 1′, set $S = F$. Then $(S, \dot{\succsim}, \supseteq)$, where $\supseteq$ is set inclusion, satisfies (a), (b), and (c) of Lemma 3. (Here, $\dot{\succsim}$ plays the role of B, and $\supseteq$ the role of G.) That (a) and (b) hold is immediate. And (c) follows from Lemma 2(a). So there exist $a(s)$ as in Lemma 3. Define $U: Z \times S \to R$ by

$$U(z, s) = \begin{cases} 0 & \text{if} \quad z \notin s, \\ a(s) & \text{if} \quad z \in s. \end{cases} \tag{3.3}$$

For $x \in X$,

$$\sum_{s \in S} \max_{z \in x} U(z, s) = \sum_{\{s \in S: s \supseteq x\}} a(s) = \sum_{\{s \in S: s \supseteq f(x)\}} {}^{\circ}a(s). \tag{3.4}$$

Also, $x \dot{\succsim} x'$ if and only if $f(x) \dot{\succsim} f(x')$ if and only if

$$\sum_{\{s \in S: s \supseteq f(x)\}} a(s) \geqslant \sum_{\{s \in S: s \supseteq f(x')\}} a(s). \tag{3.5}$$

Combining (3.4) and (3.5) gives the representation. *Q.E.D.*

Note that the collection of states/second period preference relations used is identical with the collection used in the proof of Theorem 1′. Additive representations are possible for other collections of states. For example, let S be a set of chains in F such that every $x \in F$ is in at least one chain in S. For $x \in F$, let $n(x)$ denote the number of chains in S that contain x, and for $\{a(x); x \in F\}$ the constants in the proof of Theorem 1, let $U(z, s) := \Sigma_{\{x \in s : z \in x\}} a(x)/n(x)$. The reader can verify that S and U so defined compose an additive representation.

It is not the case, however, that every collection of "second period preferences" which admits an ordinal representation also admits an additive representation. Examples are easy to construct where the "independence" assumption which is necessary for an additive representation (cf. Krantz, et al. [**9**, Ch. 6]) is violated. It is possible to obtain necessary and sufficient conditions for a collection of second period preferences to admit an additive representation, parallelling Theorem 2. But results obtained thus far are little more than conjunctions of Theorem 2 and general results on additive representation (e.g., [**9**, Ch. 9, Theorem 1]).

Note that in the construction of the additive representation, $w(X)$ is a very sparse subset of R^S, having the cardinality of S. It is this sparseness which makes the construction possible. (Put negatively, in cases where an additive representation is not possible, it would be because $w(X)$ is insufficiently sparse in R^S for the S chosen.) The normative approach mentioned above might ameliorate this sparseness, thereby lending greater significance to the additive representation. For if the range of possible second period preferences has meaning to the individual, it might be possible to elicit preferences for "opportunity sets contingent on second period preferences," even if these do not represent real choice possibilities.

4. TWO PERIOD CONSUMPTION-INVESTMENT BUDGETING

In the previous sections, the object of study is preference over "opportunity sets," subsets of some primitive set. The story has been that the individual selects one such "opportunity set" today, from which he will choose tomorrow. But in many economic applications of two period choice, first period choice is over pairs of the form (*immediate payoff, opportunity set for second period choice*). The canonical example is two period, consumption-investment budgeting—today's choice determines not only the constraint on tommorrow's consumption decision, but also determines today's consumption.

This is modelled as follows. There are finite sets Z_1 and Z_2, representing possible first and second period consumption bundles. The set of opportunity sets for second period choice is the set of all nonempty subsets of Z_2, denoted X. And choice today is from a subset of $Z_1 \times X$. It is assumed that choice today can be represented by a complete and transitive binary relation $\dot{\gtrsim}$. Interesting properties of this preference relation include:

(4.1) For z_1 fixed, if $(z_1, x) \dot{\gtrsim} (z_1, x')$, then $(z_1, x) \dot{\sim} (z_1, x \cup x')$.

(4.2) If $x \supseteq x'$, then $(z_1, x) \dot{\gtrsim} (z_1, x')$ for every $z_1 \in Z_1$.

(4.3) For z_1 fixed, if $(z_1, x) \dot{\sim} (z_1, x \cup x')$, then for all $x'' \in X$,

$$(z_1, x \cup x'') \dot{\sim} (z_1, x \cup x' \cup x'').$$

These are analogous to (1.2), (1.3), and (1.5), respectively. Analogous to Theorem 1 is the following.

THEOREM 3: *A binary relation* $\dot{\gtrsim}$ *on* $Z_1 \times X$ *is complete and transitive and satisfies* (4.2) *and* (4.3) *if and only if there exist a finite set S and a function* $U: Z_1 \times Z_2 \times S \to R$ *such that*

$$v(z_1, x) := \sum_{s \in S} \max_{z_2 \in x} U(z_1, z_2, s)$$

represents $\dot{\gtrsim}$. *Moreover,* (4.1) *holds if and only if the representation can be achieved with singleton S.*

The methods of Sections 2 and 3 easily adapt to this reformulation, so the proof is omitted. An ordinal version of Theorem 3, analogous to Theorem 1′, and a result analogous to Theorem 2 are easily constructed.

This suggests one direction for generalization of the results given here. Consider a more than two period problem. In the consumption-investment budgeting framework, this would concern choice behavior over immediate consumption, opportunity set pairs. But these opportunity sets are in turn composed of pairs of second period consumption bundles and third period opportunity sets, and so forth. In the framework of Sections 2 and 3, the object of study would be choice behavior from sets of subsets of sets of subsets of . . . of sets of subsets of Z. (For example, in a three period restaurant problem, the objects of choice today are sets of menus from which a menu is selected tomorrow from which a meal is selected two days hence.) This turns out to be a nontrivial generalization, and it will be pursued in a subsequent paper.

5. INFINITE Z

When Z is infinite, a few complications arise. For an ordinal representation in the style of Theorem 1′, two additional "continuity" conditions must be met.

THEOREM 4: *A binary relkation* $\dot{\gtrsim}$ *on* X *(for arbitrary* Z*) satisfies*

(a) $\dot{\gtrsim}$ *is complete and transitive*;

(b) $\dot{\gtrsim}$ *satisfies* (1.3) *and* (1.5);

(c) *there is a countable order dense subset of* X; *and*

(d) *if* $x \dot{\geqslant} x_a$ *for all* $a \in A$, *A an arbitrary index set, then* $x \dot{\geqslant} \bigcup_{a \in A} x_a$;

if and only if there exist a set S, a function $U: Z \times S \to R$, *and a function* $u: R^S \to R$ *such that if* $w: X \to R^S$ *is defined by*

$$(w(x))(s) := \sup_{z \in x} U(z, s),$$

then

(e) *u is strictly increasing on* $w(X)$, *and*

(f) $u \circ w: X \to R$ *represents* $\dot{\gtrsim}$.

The additional conditions are (c) and (d). The necessity of (c) should be apparent, as the theorem produces a numerical representation of $\dot{\gtrsim}$ (cf. Fishburn

[**1**, Ch. 3, Theorem 3.1]). The necessity of (d) is illustrated by the following example.

EXAMPLE 1: Let $Z'=\{0, 1, \ldots\}$ and $Z=\{\omega\}\cup Z'$. Define $\dot{\succsim}$ to have three indifference classes: (i) all subsets of Z which are finite subsets of Z'; (ii) all subsets of Z which contain ω and finitely many (possibly zero) elements of Z'; (iii) all subsets of Z which contain infinitely many elements of Z'. These indifference classes are ordered (iii) $\dot{\succ}$ (ii) $\dot{\succ}$ (i). It is easy to verify that (1.3) and (1.5) hold and that a finite order dense subset exists. But (d) of Theorem 2 is violated: $\{\omega\}\dot{\geqslant}\{z\}$ for every $z\in Z$, yet $\{\omega\}\dot{\not\geqslant} Z$. And no representation of the type given in the theorem is possible: $\{\omega\}\dot{\geqslant}\{z\}$ for every z implies that $U(\omega, s)\geqslant U(z, s)$ for all z and s (if a representation were possible), thus $U(\omega, s)\geqslant \sup_{z\in Z} U(z, s)$ and $\{\omega\}\dot{\geqslant} Z$.

Another point to be noted is that u need only be strictly increasing on $w(X)$ and not on all of R^S. This is natural. If S is uncountable, then there is *no* strictly increasing function from R^S to R. Of course, if u is strictly increasing on $w(X)$, then it can be extended to R^S to be increasing.

The proof of Theorem 4 is almost identical with that of Theorem 1′. The fact that Z (hence X) was finite was used in two places in the proof of Theorem 1′: In the proof of Theorem 1′ itself, it was used to produce a numerical representation of $\dot{\succsim}$. And it was used in deducing Lemma 1(e) from Lemma 1(d). For infinite Z, condition (c) is needed to produce the numerical representation. And condition (d) directly yields Lemma 1(e). Note also that the remarks following Theorem 1′ and Theorem 2, when modified in the obvious fashion, apply to the case of infinite Z.

One case of infinite Z which is of special interest is where Z is a compact subset of a Polish (complete, separable metric) space. Let $X^c \subseteq X$ denote the set of closed nonempty subsets of Z, and for $x\in X$, let x^c denote the closure of x. The space X^c can be metrized by the standard Hausdorff metric, with respect to which it is a compact Polish space. Thus it is meaningful to speak of $\dot{\succsim}$ as being continuous when restricted to X^c.

COROLLARY: *If Z is a compact subset of a Polish space, and $\dot{\succsim}$ is complete and transitive, satisfies* (1.3) *and* (1.5), *and is continuous on X^c, then* (c) *and* (d) *of Theorem* 4 *hold, so a representation of the type given in Theorem* 4 *is possible. Moreover, for every $x\in X$, $x\dot{\sim}x^c$.*

The proof is left to the reader. If Z is as in the Corollary and $\dot{\succsim}$ is represented as in Theorem 4 with $U(\cdot, s)$ continuous for each s and u continuous (under pointwise convergence in $w(X)$), then $\dot{\succsim}$ is continuous on X^c. The converse seems reasonable: If $\dot{\succsim}$ is continuous on X^c, then a representation with continuous $U(\cdot, s)$ and u is possible. But I am unable to supply a proof of the converse—U as constructed in the proof of the theorem will be lower semi-continuous only.

For infinite Z, additive representations are not always possible when ordinal representations are. An example illustrates this.

EXAMPLE 2: Let Z be as in Example 1, and define $\gtrsim$ as having three indifference classes: (i) the set $\{\omega\}$; (ii) any set containing exactly one element of Z' (plus, possibly, ω); (iii) any set containing two or more elements of Z'. Of course, (iii) $\dot{\succ}$ (ii) $\dot{\succ}$ (i). This relation satisfies the conditions of Theorem 4, so an ordinal representation is possible. But there is no representation of $\dot{\gtrsim}$ having the form

$$v(x):=\int_S \sup_{z \in x} U(z, s) \cdot \pi(ds),$$

where S is a measure space, π is a probability measure on S, and, for each x, $\sup_{z\in x} U(z, s)$ is an integrable function of s. (The probability measure has been introduced to avoid the ambiguous $\sup_{z\in x} U(z, ds)$.) Suppose such a representation did exist. Then without loss of generality, assume that in this representation $\sup_{z\in Z} U(z, s) = 0$ for every s (thus $v(Z) = 0$) and $v(\{z\}) = -1$ for $z \in Z'$. Since for $z, z' \in Z'$, $z \neq z'$, it is true that $v(\{z, z'\})$ must equal 0, it follows that $\pi(S_z \cap S_{z'}) = 0$, where $S_z := \{s \in S : u(z, s) < 0\}$. Moreover, $U(\omega, s) \leq U(z, s)$ for all $z \in Z'$, thus

$$v(\{\omega\}) = \int_S U(\omega, s)\pi(ds) \leq \sum_{z=0}^{\infty} \int_{S_z} U(z, s)\pi(ds) = -\infty,$$

contradicting the integrability assumption. It will not help to replace the integrability assumption with a quasi-integrability assumption—adjoin two more states to Z, one which is dominated by $\{\omega\}$, the other which dominates all of Z as presently constituted. Then no additive representation is possible, even allowing quasi-integrability.

I am unable to provide positive results on when an additive representation is possible (short of inelegant restatements of [**9**, Ch. 9, Theorem 1]). But it is conjectured that an additive representation is always possible when Z is a compact Polish space and $\dot{\gtrsim}$ is continuous on X^c.

Stanford University

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