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A RESULT CONCERNING ADDITIVE FUNCTIONS IN HERMITIAN BANACH *-ALGEBRAS AND AN APPLICATION

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ABSTRACT. Let $\mathcal{A}$ be a complex hermitian Banach *-algebra with an identity element e. Suppose there exists an additive function $f\colon \mathcal{A} \to \mathcal{A}$ such that $f(a) = -a^*af(a^{-1})$ holds for all normal invertible elements $a \in \mathcal{A}$. We prove that in this case f is of the form $f(a) = f(ie)k$, where $a = h + ik$. Using this result we generalize S. Kurepa's extension of Jordan-Neumann characterization of pre-Hilbert space.

This research has been inspired by the work of S. Kurepa [**2**, **3**] and P. Vrbová [**6**]. All algebras and vector spaces in this paper will be over the complex field. Algebras are assumed to have an identity element, which will be denoted by e. An algebra $\mathcal{A}$ is called a *-algebra if there exists an involution (conjugate-linear anti-isomorphism of period two) $a \mapsto a^*$ on $\mathcal{A}$. An element $h \in \mathcal{A}$ is said to be hermitian if $h^* = h$, and $u \in \mathcal{A}$ is said to be unitary if $u^*u = uu^* = e$. An element $a \in \mathcal{A}$ will be called normal if $a^*a = aa^*$. It is easy to see that each element $a \in \mathcal{A}$ has a unique decomposition $a = h + ik$ with hermitian h and k. An element $a \in \mathcal{A}$ is normal if and only if h and k commute.

A *-algebra which is also a Banach algebra is called a Banach *-algebra. A Banach *-algebra is called hermitian if each hermitian element has real spectrum. Let $\mathcal{A}$ be a hermitian Banach *-algebra and let $h \in \mathcal{A}$ be a hermitian element. It is convenient to write $h > 0$ ($h \geq 0$) if the spectrum of h is positive (nonnegative). The notation $h > k$ ($h \geq k$) means $h-k > 0$ ($h-k \geq 0$). The most important hermitian Banach *-algebras are B^*-algebras (i.e. Banach *-algebras in which $\|a^*a\| = \|a\|^2$ is fulfilled for all a). For basic facts concerning hermitian Banach *-algebras, we refer the reader to V. Pták's paper [**5**].

Let X and $\mathcal{A}$ be a vector space and an algebra, respectively. Suppose that X is a left $\mathcal{A}$-module. A left $\mathcal{A}$-module X will be called unitary if $ex = x$ holds for all $x \in X$, and will be called irreducible if for each pair $x, y \in X$, $x \neq 0$, there exists $a \in \mathcal{A}$ such that $ax = y$.

First we shall consider the following result.

THEOREM 1. *Let $\mathcal{A}$ be a hermitian Banach *-algebra. Suppose there exists an additive function $f\colon \mathcal{A} \to \mathcal{A}$ such that $f(a) = -a^*af(a^{-1})$ holds for all normal invertible elements $a \in \mathcal{A}$. In this case $f(a) = f(ie)k$ is fulfilled for all $a = h + ik$.*

REMARK. If $\mathcal{A}$ is the complex number field, then the theorem above reduces to a result due to P. Vrbová [**6**].

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For the proof of Theorem 1 we need the lemma below. We omit the proof since it is an easy consequence of Ford's square root lemma [**1** or **5**, (1.5)].

LEMMA 2. *Let $\mathcal{A}$ be a hermitian Banach *-algebra. For each $h > 0$ there corresponds $u > 0$, such that $u^2 = h$. Moreover, u commutes with each element which commutes with h.*

PROOF OF THEOREM 1. Let us first prove that

$$f(h) = 0 \tag{1}$$

holds for all $h \in \mathcal{A}$, $0 < h < e$. Since in this case $e - h^2 > 0$, there exists, by Lemma 2, a hermitian element k, such that k commutes with h, and $e - h^2 = k^2$, whence it follows that $u = h + ik$ is a unitary element. Therefore, according to the requirements of the theorem, we have

$$f(h) + f(ik) = f(u) = -u^*uf(u^{-1}) = -f(h - ik) = -f(h) + f(ik),$$

whence it follows $f(h) = 0$. Let us prove that

$$f(te) = 0 \tag{2}$$

for each real number t. If $0 \leq t < 1$, then (2) follows from (1). It is easy to see that $f(e) = 0$. Therefore (2) holds for all $t \in [0, 1]$. If $t > 1$, we have $0 < t^{-1} < 1$, whence $f(te) = -t^2 f(t^{-1}e) = 0$, which proves that (2) holds for all nonnegative real numbers and therefore also for all real numbers. Let us prove that

$$f(h) = 0 \tag{3}$$

for all hermitian $h \in \mathcal{A}$. Therefore, let h be an arbitrary hermitian element, and let us choose a real number t such that $te + h > e$. Then $0 < (te + h)^{-1} < e$. According to (1) we have $f(te + h) = -(te + h)^2 f((te + h)^{-1}) = 0$. Hence $f(h) = f(-te)$ and, according to (2), $f(h) = 0$. Now we intend to prove that

$$f(ih) = hf(ie) \tag{4}$$

holds for all $h \in \mathcal{A}$, $0 < h < e$. From $0 < h < e$ it follows that $h - h^2 > 0$. By Lemma 2 there exists a hermitian element k, such that k commutes with h, and that $h - h^2 = k^2$. The element $a = k + ih$ is normal, since h and k commute. Since a can be expressed in the form $a = h(h^{-1}k + ie)$, it is obvious that a is invertible (recall that $\mathcal{A}$ is by assumption hermitian). Therefore using the requirements of the theorem and (3) we obtain

$$\begin{aligned} f(ih) = f(k) + f(ih) = f(a) &= -a^*af(a^{-1}) = -a^*af((a^*a)^{-1}a^*) \\ &= -(h^2 + k^2)f((h^2 + k^2)^{-1}(k - ih)) = -hf(h^{-1}(k - ih)) \\ &= -hf(h^{-1}k) + hf(ie) = hf(ie). \end{aligned}$$

Let us prove that

$$f(ite) = tf(ie) \tag{5}$$

holds for each real number t. If $0 \leq t \leq 1$, then (5) follows from (4). If $t > 1$, then $0 < t^{-1} < 1$, and we have

$$f(ite) = -t^2 f((ite)^{-1}) = -t^2 f(-it^{-1}e) = t^2 t^{-1} f(ie) = tf(ie),$$

which proves that (5) holds for all nonnegative real numbers and therefore also for all real numbers. We shall prove that

$$f(ih) = hf(ie) \tag{6}$$

is fulfilled for each hermitian element $h \in \mathcal{A}$. Therefore, let h be an arbitrary hermitian element, and let us choose a real number t such that $te + h > e$. Then $0 < (te+h)^{-1} < e$. According to (4) we have

$$\begin{aligned} f(i(te+h)) &= -(te+h)^2 f((i(te+h))^{-1}) \\ &= (te+h)^2(te+h)^{-1}f(ie) = (te+h)f(ie). \end{aligned}$$

Using the additivity of the function f and (5), we obtain $f(ih) = hf(ie)$. From (3), (6) and the fact that each $a \in \mathcal{A}$ can be expressed in the form $a = h + ik$, where h and k are hermitian, it follows $f(a) = kf(ie)$, which completes the proof of the theorem.

Let X and $\mathcal{A}$ be a complex vector space and a complex *-algebra, respectively. Suppose that X is a left $\mathcal{A}$-module. A mapping $B(\cdot,\cdot)\colon X \times X \to \mathcal{A}$ is called an $\mathcal{A}$-bilinear form if

1° $B(a_1x_1 + a_2x_2, y) = a_1B(x_1,y) + a_2B(x_2,y),\ x_1, x_2, y \in X,\ a_1, a_2 \in \mathcal{A}$,

2° $B(x, a_1y_1 + a_2y_2) = B(x,y_1)a_1^* + B(x,y_2)a_2^*,\ x, y_1, y_2 \in X,\ a_1, a_2 \in \mathcal{A}$.

A mapping $Q\colon X \to \mathcal{A}$ is called an $\mathcal{A}$-quadratic form if

3° $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y),\ x, y \in X$,

4° $Q(ax) = aQ(x)a^*,\ x \in X,\ a \in \mathcal{A}$.

Let us consider two examples of $\mathcal{A}$-bilinear forms.

EXAMPLE 1. Let $\mathcal{A}$ be a *-algebra and $\mathcal{L} \subset \mathcal{A}$ a left ideal. Considering $\mathcal{L}$ as a left $\mathcal{A}$-module, one can introduce an $\mathcal{A}$-bilinear form $B(\cdot,\cdot)$ as follows $B(x,y) = xy^*,\ x, y \in \mathcal{L}$.

EXAMPLE 2. Let X be a Hilbert space and let us denote by $L(X)$ the algebra of all bounded linear operators of X into itself. Let the involution on $L(X)$ be the adjoint operation. X can be considered as a unitary irreducible left $L(X)$-module (multiplication by $A \in L(X)$ is operator action on X). A simple calculation shows that the mapping $B(\cdot,\cdot)\colon X \times X \to L(X)$ defined by the relation $B(x,y)z = (z,y)x$, where $(\cdot,\cdot)$ denotes the inner product in X, is an $\mathcal{A}$-bilinear form.

It is easy to see that each $\mathcal{A}$-bilinear form gives rise to the $\mathcal{A}$-quadratic form by the relation $Q(x) = B(x,x)$. It seems natural to ask whether the converse is also true. More precisely, we consider the following

PROBLEM. Let X and $\mathcal{A}$ be a vector space and a *-algebra, respectively. Suppose that X is a left $\mathcal{A}$-module, and that there exists an $\mathcal{A}$-quadratic form $Q\colon X \to \mathcal{A}$. Does there exist an $\mathcal{A}$-bilinear form $B(\cdot,\cdot)\colon X \times X \to \mathcal{A}$ such that $Q(x) = B(x,x)$ holds for all $x \in X$?

It follows from a result of S. Kurepa [**3**] that the answer to the question above is affirmative if $\mathcal{A}$ is the complex number field. His result can be formulated as follows.

THEOREM 3 (S. KUREPA [**3**]). *Let X be a vector space over the complex field C. Suppose there exists a mapping $Q\colon X \to C$ such that $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$, $Q(\lambda x) = |\lambda|^2 Q(x)$ holds for all pairs $x, y \in X$ and all $\lambda \in C$. Under these conditions the mapping $B(\cdot,\cdot)\colon X \times X \to C$ defined by*

$$B(x,y) = \frac{1}{4}(Q(x+y) - Q(x-y)) + \frac{i}{4}(Q(x+iy) - Q(x-iy))$$

is additive in both arguments, and $B(\lambda x, y) = \lambda B(x,y), B(x, \lambda y) = \overline{\lambda} B(x,y)$ *hold for all pairs* $x, y \in X$ *and all* $\lambda \in C$. *For each* $x \in X$ *the relation* $Q(x) = B(x,x)$ *is fulfilled.*

REMARKS. The theorem above can be considered as an extension of the well-known result due to P. Jordan and J. von Neumann which characterizes pre-Hilbert space among all normed spaces. It should be mentioned that P. Vrbová [**6**] has obtained a simple proof of S. Kurepa's theorem. Using Theorem 1 and an approach from [**6**] we prove the result below which can be considered as a generalization of Theorem 3.

THEOREM 4. *Let* X *be a vector space and* $\mathcal{A}$ *a commutative hermitian Banach* $*$*-algebra. Let* X *be a unitary* $\mathcal{A}$*-module, and suppose that there exists an* $\mathcal{A}$*-quadratic form* $Q: X \to A$. *In this case the mapping* $B(\cdot,\cdot): X \times X \to \mathcal{A}$ *defined by*

$$B(x,y) = \frac{1}{4}(Q(x+y) - Q(x-y)) + \frac{i}{4}(Q(x+iy) - Q(x-iy))$$

is an $\mathcal{A}$*-bilinear form. For all* $x \in X$ *the relation* $Q(x) = B(x,x)$ *holds.*

PROOF. Let us first prove that the function $S(\cdot,\cdot)$ defined by the relation $S(x,y) = Q(x+y) - Q(x-y)$ is additive in both variables. This part of the proof goes through as in the proof of Theorem 3 (see [**3**] and also [**4**] for some generalizations), but we shall write it down for the sake of completeness. It is easy to see that $Q(0) = 0$ and $Q(-x) = Q(x),\ x \in X$. For arbitrary elements $x, y, u \in X$ we have

$$\begin{aligned} S(x+y, 2u) &= Q(x+y+2u) - Q(x+y-2u) \\ &= Q((x+u) + (y+u)) + Q((x+u) - (y+u)) \\ &\quad - Q((x-u) + (y-u)) - Q((x-u) - (y-u)). \end{aligned}$$

Using the relation $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$ we obtain

$$\begin{aligned} S(x+y, 2u) &= 2Q(x+u) + 2Q(y+u) - (2Q(x-u) + 2Q(y-u)) \\ &= 2S(x,u) + 2S(y,u). \end{aligned}$$

Hence

$$S(x+y, 2u) = 2S(x,u) + 2S(y,u). \tag{7}$$

Putting $y = 0,\ x = z$ we obtain $S(z, 2u) = 2S(z,u)$. Substituting z by $x+y$ and using (7) we finally obtain

$$2S(x+y, u) = S(x+y, 2u) = 2S(x,u) + 2S(y,u)$$

which proves that the function $S(\cdot,\cdot)$ is additive in the first variable. Since $S(x,y) = S(y,x)$ for all pairs $x, y \in X$ (this follows from the relation $Q(-x) = Q(x)$) it follows that the function $S(\cdot,\cdot)$ is additive also in the second variable. From the fact that $S(\cdot,\cdot)$ is additive in both variables, it follows that the same is true for the function $B(\cdot,\cdot)$ defined by the relation $B(x,y) = \frac{1}{4}S(x,y) + \frac{i}{4}S(x,iy)$. Therefore, since it is easy to see that $Q(x) = B(x,x)$ holds for all $x \in X$, it remains to prove that

$$B(ax, y) = aB(x,y), \quad B(x, ay) = a^* B(x,y) \tag{8}$$

is fulfilled for all pairs $x, y \in X$ and all $a \in \mathcal{A}$. Now we are going to use the condition $Q(ax) = a^*aQ(x)$. First of all it follows from the condition above that

$$S(ax, y) = a^*aS(x, a^{-1}y) \tag{9}$$

holds for all pairs $x, y \in X$ and all invertible $a \in \mathcal{A}$. Let us prove that $B(\cdot,\cdot)$ satisfies the relations

$$B(ix, y) = iB(x, y), \tag{10}$$

$$B(x, iy) = -iB(x, y) \tag{11}$$

Indeed,

$$\begin{aligned} 4B(ix, y) &= S(ix, y) + iS(ix, iy) = S(x, -iy) + iS(x, y) \\ &= i(S(x, y) - iS(x, -iy)) = i(S(x, y) + iS(x, iy)) = 4iB(x, y) \end{aligned}$$

which proves (10). Furthermore,

$$\begin{aligned} 4B(x, iy) &= S(x, iy) + iS(x, -y) = S(x, iy) - iS(x, y) \\ &= -i(S(x, y) + iS(x, iy)) = -4iB(x, y). \end{aligned}$$

Now we intend to prove that for the function $f\colon \mathcal{A} \to \mathcal{A}$, defined by the relation

$$f(a) = B(ax, y) - B(x, ay), \tag{12}$$

where x and y are fixed vectors, the requirements of Theorem 1 are fulfilled. Since the additivity of the function above follows from the fact that $B(\cdot,\cdot)$ is additive in both variables, it remains to show that $f(a) = -a^*af(a^{-1})$ holds for all invertible $a \in \mathcal{A}$. We have

$$\begin{aligned} 4f(a) &= S(ax, y) + iS(ax, iy) - (S(x, ay) + iS(x, iay)) \\ &= S(ax, y) + iS(ax, iy) - (S(ay, x) + iS(iay, x)). \end{aligned}$$

Using (9) we obtain

$$\begin{aligned} 4f(a) &= a^*a(S(x, a^{-1}y) + iS(x, ia^{-1}y)) - a^*a(S(y, a^{-1}x) + iS(iy, a^{-1}x)) \\ &= 4a^*a(B(x, a^{-1}y) - B(a^{-1}x, y)) = -4a^*af(a^{-1}). \end{aligned}$$

According to Theorem 1 we have $f(h + ik) = f(ie)k$ for all hermitian h and k. In particular, $f(h) = 0$ which implies

$$B(hx, y) = B(x, hy) \tag{13}$$

for all hermitian $h \in \mathcal{A}$ and all pairs $x, y \in X$. If we put $a = ih$, h hermitian, we obtain

$$B(ihx, y) - B(x, ihy) = f(ih) = hf(ie) = h(B(ix, y) - B(x, iy)).$$

Using (10), (11) and (13) we obtain

$$B(hx, y) = hB(x, y). \tag{14}$$

Therefore according to (10), (11), (13) and (14) it follows that (8) holds. The proof of the theorem is complete.

REMARK. It would be interesting to know whether Theorem 4 holds also in the noncommutative case.

We conclude with the following purely algebraic result.

THEOREM 5. *Let X be a vector space and $\mathcal{A}$ a commutative *-algebra. Let X be a unitary irreducible $\mathcal{A}$-module, and suppose that there exists an $\mathcal{A}$-quadratic form $Q\colon X \to \mathcal{A}$. In this case the mapping $B(\cdot,\cdot)\colon X \times X \to \mathcal{A}$ defined by*

$$B(x,y) = \frac{1}{4}(Q(x+y) - Q(x-y)) + \frac{i}{4}(Q(x+iy) - Q(x-iy))$$

is an $\mathcal{A}$-bilinear form. For all $x \in X$ the relation $Q(x) = B(x,x)$ holds.

PROOF. It remains to prove that

$$S(hx,y) = hS(x,y), \quad S(x,hy) = hS(x,y), \tag{15}$$

where $S(x,y)$ stands for $Q(x+y) - Q(x-y)$, holds for all pairs $x,y \in X$ and each hermitian $h \in \mathcal{A}$, since the rest of the proof goes through as in the proof of Theorem 4. Therefore, let $x,y \in X$, $h \in \mathcal{A}, h^* = h$ be arbitrary, and let us prove (15). We may assume that $x \neq 0$, since there is nothing to prove if $x = y = 0$. By the requirements of the theorem there exists $a \in \mathcal{A}$ such that $y = ax$. We have

$$\begin{aligned} S(hx,y) &= S(hx,ax) = Q((h+a)x) - Q((h-a)x) \\ &= (h+a)^*(h+a)Q(x) - (h-a)^*(h-a)Q(x) \\ &= h((e+a)^*(e+a) - (e-a)^*(e-a))Q(x) \\ &= h(Q((e+a)x) - Q((e-a)x)) = hS(x,y). \end{aligned}$$

Similarly, we obtain that the relation $S(x,hy) = hS(x,y)$ holds. The proof of the theorem is complete.

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