successfully achieves exponential stability under the controllability assumption of the bilinear system.

The future work is to extend the control design in this note to division controllers for more general nonlinear systems when the targeted equilibrium point is a singular point of the division controller. Such cases can be found in the feedback linearization control when the nonlinear system does not have a well-defined relative degree.

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A Result on Common Quadratic Lyapunov Functions

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***Abstract*—In this note, we define strong and weak common quadratic Lyapunov functions (CQLFs) for sets of linear time-invariant (LTI) systems. We show that the simultaneous existence of a weak CQLF of a special form, and the nonexistence of a strong CQLF, for a pair of LTI systems, is characterized by easily verifiable algebraic conditions. These conditions are found to play an important role in proving the existence of strong CQLFs for general LTI systems.**

***Index Terms*—Quadratic stability, stability theory, switched linear systems.**

I. INTRODUCTION

The existence or nonexistence of common quadratic Lyapunov functions (CQLFs) for two or more stable linear time-invariant (LTI) systems is closely connected to recent work on the design and stability of switching systems [1], [2]. In this context, numerous papers have appeared in the literature [2]–[6] in which sufficient conditions have been derived under which two stable dynamical systems

$$\Sigma_{A_i}: \dot{x} = A_i x, \qquad A_i \in \mathbb{R}^{n\times n}, \qquad i \in \{1, 2\}$$

have a CQLF. If the matrix $P = P^T > 0$, $P \in \mathbb{R}^{n\times n}$, simultaneously satisfies the Lyapunov equations $A_i^T P + PA_i = -Q_i$, $i \in \{1, 2\}$, where $Q_i > 0$, then $V(x) = x^T Px$ is said to be a strong CQLF for Σ_{A_1} and Σ_{A_2}. If $Q_i \geq 0$ for $i \in \{1, 2\}$ then $V(x)$ is said to be a weak CQLF. This technical note considers pairs of stable LTI systems for which a strong CQLF does not exist, but for which a weak CQLF exists where $-Q_1$ and $-Q_2$ are both negative semidefinite and of rank $n-1$. We derive a result that can be used to determine necessary and sufficient conditions for the existence of a strong CQLF for certain classes of stable LTI systems.

II. MATHEMATICAL PRELIMINARIES

In this section, we present some results and definitions that are useful in proving the principal result of this note. Throughout, the following notation is adopted

$\mathbb{R}$ and $\mathbb{C}$	fields of real and complex numbers, respectively;
$\mathbb{R}^n$	n-dimensional real Euclidean space;
$\mathbb{R}^{n\times n}$	space of $n \times n$ matrices with real entries;
x_i	ith component of the vector x in $\mathbb{R}^n$;
a_{ij}	entry in the (i, j) position of the matrix A in $\mathbb{R}^{n\times n}$.

Where appropriate, the proofs of individual lemmas are presented in the Appendix.

i) ***Strong and weak common quadratic Lyapunov functions:*** Consider the set of LTI systems

$$\Sigma_{A_i}: \dot{x} = A_i x, \qquad i \in \{1, 2, \ldots, M\} \tag{1}$$

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where M is finite and the $A_i, i \in \{1, 2, \ldots, M\}$ are constant Hurwitz matrices in $\mathbb{R}^{n\times n}$ (i.e., the eigenvalues of A_i lie in the open left half of the complex plane and hence the Σ_{A_i} are stable LTI systems). Let the matrix $P = P^T > 0$, $P \in \mathbb{R}^{n\times n}$, be a simultaneous solution to the Lyapunov equations

$$A_i^T P + PA_i = -Q_i, \qquad i \in \{1, 2, \ldots, M\}. \tag{2}$$

Then, $V(x) = x^T Px$ is a strong quadratic Lyapunov function for the LTI system Σ_{A_i} if $Q_i > 0$, and is said to be a *strong CQLF* for the set of LTI systems Σ_{A_i}, $i \in \{1, \ldots, M\}$, if $Q_i > 0$ for all i. Similarly, $V(x)$ is a weak quadratic Lyapunov function for the LTI system Σ_{A_i} if $Q_i \geq 0$, and is said to be a *weak CQLF* for the set of LTI systems Σ_{A_i}, $i \in \{1, \ldots, M\}$, if $Q_i \geq 0$ for all i.

ii) ***The matrix pencil*** $\sigma_{\gamma[0,\infty)}[A_1, A_2]$***:*** The matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$, for $A_1, A_2 \in \mathbb{R}^{n\times n}$, is the parameterized family of matrices $\sigma_{\gamma[0,\infty)}[A_1, A_2] = A_1 + \gamma A_2$, $\gamma \in [0, \infty)$. We say that the pencil is *nonsingular* if $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is nonsingular for all $\gamma \geq 0$. Otherwise, the pencil is said to be *singular*. Further, a pencil is said to be *Hurwitz* if its eigenvalues are in the open left half of the complex plane for all $\gamma \geq 0$.

iii) The following result provides a useful test for the singularity of a matrix pencil.

Lemma 2.1 [7]: Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ with A_1 nonsingular. A necessary and sufficient condition for singularity of the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is that the matrix product $A_1^{-1}A_2$ has a negative (real) eigenvalue. (If A_2 is also nonsingular, then this is equivalent to $A_1 A_2^{-1}$ having a negative (real) eigenvalue).

iv) **The stability of** Σ_A **and** $\Sigma_{A^{-1}}$***:*** The relationship between a matrix, its inverse, and a quadratic Lyapunov function will arise in our discussion. In this context we note the following fundamental result that appeared in [8]. Consider the LTI systems Σ_A and $\Sigma_{A^{-1}}$ where $A \in \mathbb{R}^{n\times n}$ is Hurwitz. Then, any quadratic Lyapunov function for Σ_A is also a quadratic Lyapunov function for $\Sigma_{A^{-1}}$.

Comment: Suppose that $V(x)$ is a strong CQLF for the stable LTI systems Σ_{A_1}, Σ_{A_2}. It is easily verified that the same function $V(x)$ will be a strong quadratic Lyapunov function for the systems $\Sigma_{\sigma_{\gamma[0,\infty)}[A_1,A_2]}$ and $\Sigma_{\sigma_{\gamma[0,\infty)}[A_1,A_2^{-1}]}$ for all $\gamma \in [0, \infty)$. Hence, $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are both necessarily Hurwitz for all $\gamma \in [0, \infty)$. Thus, the nonsingularity of these two pencils is a necessary condition for the existence of a strong CQLF for the systems Σ_{A_1}, Σ_{A_2}.

v) **Lemma 2.2:** Let $u, v, x, y \in \mathbb{R}^n$ be any four nonzero vectors. There exists a nonsingular $T \in \mathbb{R}^{n\times n}$ such that each component of the vectors Tu, Tv, Tx, Ty is nonzero.

vi) **Lemma 2.3:** Let x, y, u, vbe four nonzero vectors in $\mathbb{R}^n$ such that for all Hermitian matrices $P \in \mathbb{R}^{n\times n}$, $x^T Py = -ku^T Pv$ with $k > 0$. Then, either

$$x = \alpha u \text{ for some real scalar } \alpha, \text{ and } y = -\left(\frac{k}{\alpha}\right)v, \text{ or}$$
$$x = \beta v \text{ for some real scalar } \beta \text{ and } y = -\left(\frac{k}{\beta}\right)u.$$

III. Main Results

We consider pairs of stable LTI systems for which no strong CQLF exists, but for which a weak CQLF exists with Q_i, $i \in \{1, 2\}$, of rank $n - 1$. Our principal result, Theorem 3.1, establishes a set of easily verifiable algebraic conditions, that are satisfied when such a weak CQLF exists.

Theorem 3.1: Let A_1, A_2 be two Hurwitz matrices in $\mathbb{R}^{n\times n}$ such that a solution $P = P^T \geq 0$ exists to the nonstrict Lyapunov equations

$$A_i^T P + PA_i = -Q_i \leq 0, \qquad i \in \{1, 2\} \tag{3}$$

for some positive–semidefinite matrices Q_1, Q_2 both of rank $n - 1$. Furthermore suppose that no strong CQLF exists for Σ_{A_1} and Σ_{A_2}. Under these conditions, at least one of the pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$, $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is singular. Equivalently, by Lemma 2.1, at least one of the matrix products $A_1 A_2$ and $A_1 A_2^{-1}$ has a real negative eigenvalue.

Comment: The following facts are established in Theorem 3.1.

a) Vectors $x_1, x_2 \in \mathbb{R}^n$ exist such that $Q_1 x_1 = 0$ and $Q_2 x_2 = 0$.

b) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hyperplanes in the space of symmetric matrices defined by the following equations (in the free parameter H):

$$\mathcal{H}_1\colon x_1^T HA_1 x_1 = 0, \ \mathcal{H}_2\colon x_2^T HA_2 x_2 = 0. \tag{4}$$

Then, $\mathcal{H}_1$ and $\mathcal{H}_2$ define the same plane.

c) There is some real $\alpha_0 > 0$ with $x_1^T HA_1 x_1 = -\alpha_0 x_2^T HA_2 x_2$, for all $H = H^T$.

Proof of Theorem 3.1: As Q_1 and Q_2 are of rank $n - 1$, there are nonzero vectors x_1, x_2 such that $x_1^T Q_1 x_1 = 0$, $x_2^T Q_2 x_2 = 0$. The proof of Theorem 3.1 is split into two main stages.

Stage 1: The first stage in the proof is to show that if there exists a Hermitian matrix $\overline{P}$ satisfying

$$x_1^T \overline{P} A_1 x_1 < 0 \quad x_2^T \overline{P} A_2 x_2 < 0 \tag{5}$$

then a strong CQLF exists for Σ_{A_1} and Σ_{A_2}.

Note that as $x^T PA_1 x$ is a scalar for any x, we can write $x^T Q_1 x = 2x^T PA_1 x$. The same obviously holds for $x^T Q_2 x$.

Now, assume that there is some $\overline{P}$ satisfying (5). We shall show that by choosing $\delta_1 > 0$ sufficiently small, it is possible to guarantee that $A_1^T(P + \delta_1 \overline{P}) + (P + \delta_1 \overline{P})A_1$ is negative definite. First, consider the set

$$\Omega_1 = \{x \in \mathbb{R}^n \colon \|x\| = 1 \text{ and } x^T \overline{P} A_1 x \geq 0\}.$$

Note that if the set Ω_1 was empty, then any positive constant $\delta_1 > 0$ would make $A_1^T(P + \delta_1 \overline{P}) + (P + \delta_1 \overline{P})A_1$ negative definite. Hence, we assume that Ω_1 is nonempty.

The function that takes x to $x^T \overline{P} A_1 x$ is continuous. Thus Ω_1 is closed and bounded, hence, compact. Furthermore x_1 (or any nonzero multiple of x_1) is not in Ω_1 and, thus, $x^T PA_1 x$ is strictly negative on Ω_1.

Let M_1 be the maximum value of $x^T \overline{P} A_1 x$ on Ω_1, and let M_2 be the maximum value of $x^T PA_1 x$ on Ω_1. Then by the final remark in the previous paragraph, $M_2 < 0$. Choose any constant $\delta_1 > 0$ such that

$$\delta_1 < \frac{|M_2|}{M_1 + 1} = C_1$$

and consider the Hermitian matrix

$$P + \delta_1 \overline{P}.$$

By separately considering the cases $x \in \Omega_1$ and $x \notin \Omega_1$, $\|x\| = 1$, it follows that for all nonzero vectors x of norm 1

$$x^T \left(A_1^T \left(P + \delta_1 \overline{P}\right) + \left(P + \delta_1 \overline{P}\right) A_1\right) x < 0$$

provided $0 < \delta_1 < (|M_2|/M_1 + 1)$. Since the above inequality is unchanged if we scale x by any nonzero real number, it follows that

$A_1^T(P+\delta_1\overline{P})+(P+\delta_1\overline{P})A_1$ is negative definite. By a standard result of systems theory, this implies that the matrix $P+\delta_1\overline{P}$ is positive definite.

The same argument can be used to show that there is some $C_2>0$ such that

$$x^T\left(A_2^T\left(P+\delta_1\overline{P}\right)+\left(P+\delta_1\overline{P}\right)A_2\right)x<0$$

for all nonzero x, for $0<\delta_1<C_2$. So, if we choose $\delta>0$ less than the minimum of C_1, C_2, we would have a positive–definite matrix

$$P_1=P+\delta\overline{P}$$

which defined a strong CQLF for Σ_{A_1} and Σ_{A_2}.

Stage 2: So, under our assumptions, no Hermitian solution $\overline{P}$ exists satisfying (5). We now show that such a solution $\overline{P}$ would exist unless one of the two pencils $\sigma_{\gamma[0,\infty)}[A_1,A_2]$, $\sigma_{\gamma[0,\infty)}[A_1,A_2^{-1}]$ was singular.

As there is no Hermitian solution to (5), any Hermitian H that makes the expression $x_1^THA_1x_1$ negative will make the expression $x_2^THA_2x_2$ positive. More formally

$$x_1^THA_1x_1<0\iff x_2^THA_2x_2>0 \tag{6}$$

for Hermitian H. It follows from this that

$$x_1^THA_1x_1=0\iff x_2^THA_2x_2=0.$$

The expressions $x_1^THA_1x_1$, $x_2^THA_2x_2$, viewed as functions of H, define linear functionals on the space of Hermitian matrices. Moreover, we have seen that the null sets of these functionals are identical. So they must be scalar multiples of each other. Furthermore, (6) implies that they are negative multiples of each other. That is

$$x_1^THA_1x_1=-kx_2^THA_2x_2 \tag{7}$$

with $k>0$, for all Hermitian matrices H.

Now, Lemma 2.3 implies that either $x_1=\alpha x_2$ and $A_1x_1=-(k/\alpha)A_2x_2$ or $x_1=\beta A_2x_2$ and $A_1x_1=-(k/\beta)x_2$. To begin with, consider the former situation. Then, we have

$$A_1(\alpha x_2)=-\left(\frac{k}{\alpha}\right)A_2x_2$$
$$\Rightarrow\left(A_1+\left(\frac{k}{\alpha^2}\right)A_2\right)x_2=0$$

and, thus, the pencil $\sigma_{\gamma[0,\infty)}[A_1,A_2]$ is singular. It follows from Lemma 2.1 that the matrix $A_1A_2^{-1}$ has a negative eigenvalue.

On the other hand, in the latter situation, we have that

$$x_2=\frac{1}{\beta}A_2^{-1}x_1.$$

Thus

$$A_1x_1=-\left(\frac{k}{\beta^2}\right)A_2^{-1}x_1$$
$$\Rightarrow\left(A_1+\left(\frac{k}{\beta^2}\right)A_2^{-1}\right)x_1=0.$$

Thus, in this case the pencil $\sigma_{\gamma[0,\infty)}[A_1,A_2^{-1}]$ is singular. It follows from Lemma 2.1 that the matrix A_1A_2 has a negative eigenvalue. This completes the proof of Theorem 3.1.

Comment: A crucial point in the proof of Theorem 3.1 is that there is a unique hyperplane containing the matrix P which separates the sets $\{\overline{P}>0: A_1^T\overline{P}+\overline{P}A_1<0\}$ and $\{\overline{P}>0: A_2^T\overline{P}+\overline{P}A_2<0\}$. For the question of CQLF existence for three or more LTI systems, such a hyperplane need not exist and alternative methods would need to be considered.

IV. Application of Main Result

In this section, we present an example to illustrate the use of Theorem 3.1.

Example (Second-Order Systems): Let Σ_{A_1} and Σ_{A_2} be stable LTI systems with $A_1, A_2\in\mathbb{R}^{2\times 2}$. We note the following easily verifiable facts.

a) If a strong CQLF exists for Σ_{A_1} and Σ_{A_2}, then the pencils $\sigma_{\gamma[0,\infty)}[A_1,A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1,A_2^{-1}]$ are necessarily Hurwitz.

b) If A_1 and A_2 satisfy the nonstrict Lyapunov (3), then the matrices Q_1 and Q_2 are both rank 1 (rank $n-1$).

c) If a strong CQLF does not exist for Σ_{A_1} and Σ_{A_2} then a positive constant d exists such that a strong CQLF exists for Σ_{A_1-dI} and Σ_{A_2}. By continuity a nonnegative $d_1<d$ exists such that A_1-d_1I and A_2 satisfy Theorem 3.1 and one of the pencils $\sigma_{\gamma[0,\infty)}[A_1-d_1I,A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1-d_1I,A_2^{-1}]$ is necessarily singular. Hence, it follows that one of the pencils $\sigma_{\gamma[0,\infty)}[A_1,A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1,A_2^{-1}]$ is not Hurwitz.

Items a)–c) establish the following facts. Given two stable second order LTI systems Σ_{A_1} and Σ_{A_2}, a necessary condition for the existence of a strong CQLF is that the pencils $\sigma_{\gamma[0,\infty)}[A_1,A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1,A_2^{-1}]$ are Hurwitz. Conversely, a necessary condition for the nonexistence of a strong CQLF is that one of the pencils $\sigma_{\gamma[0,\infty)}[A_1,A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1,A_2^{-1}]$ is not Hurwitz. Together, these conditions yield the following known result [4], [6], [9].

> A necessary and sufficient condition for the LTI systems Σ_{A_1} and Σ_{A_2}, $A_1, A_2\in\mathbb{R}^{2\times 2}$ both Hurwitz, to have a strong CQLF is that the pencils $\sigma_{\gamma[0,\infty)}[A_1,A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1,A_2^{-1}]$ are Hurwitz.

V. Concluding Remarks

In this note, a result related to strong and weak CQLFs has been derived. It is shown that if a strong CQLF does not exist for a pair of stable LTI systems, but a weak CQLF of a specific form exists, then at least one of the matrix pencils $A_1+\gamma A_2$, $A_1+\lambda A_2^{-1}$ is singular for some positive γ (*or* λ) (and at least one of the matrix products A_1A_2 or $A_1A_2^{-1}$ has a negative eigenvalue). It is possible to adapt the method of proof of Theorem 3.1 to obtain corresponding results for discrete-time systems involving the *bilinear* or *Cayley* transform $C(A)=(A-I)(A+I)^{-1}$ ([10]).

Appendix

A. Proof of Lemma 2.2

Consider the norm $\|A\|_\infty=\sup\{|a_{ij}|: 1\le i,j\le n\}$ on $\mathbb{R}^{n\times n}$, and let z be any nonzero vector in $\mathbb{R}^n$. Then, it is easy to see that the set $\{T\in\mathbb{R}^{n\times n}: \det(T)\neq 0, (Tz)_i\neq 0, 1\le i\le n\}$ is open. On the other hand, if $T\in\mathbb{R}^{n\times n}$ is such that $(Tz)_i=0$ for some i, an arbitrarily small change in an appropriate element of the ith row of T will result in a matrix T' such that $(T'z)_i\neq 0$. From this, it follows that arbitrarily close to the original matrix T, there is some $T_1\in\mathbb{R}^{n\times n}$ such that T_1z is nonzero component-wise.

Now, to prove the Lemma, simply select a nonsingular T_0 such that T_0x is nonzero component-wise. Suppose that some component of T_0y is zero. By the arguments in the previous paragraph, it is clear that we can select a nonsingular $T_1\in\mathbb{R}^{n\times n}$ such that each component of T_1x and T_1y is nonzero. Now, it is simply a matter of repeating this step for the remaining vectors u and v to complete the proof of the Lemma.

B. Proof of Lemma 2.3

We can assume that all components of x, y, u, v are nonzero. To see why this is so, suppose that the result was proven for this case and we were given four arbitrary nonzero vectors x, y, u, and v. We could transform them via a single nonsingular transformation T such that each component of Tx, Ty, Tu, Tv was nonzero (Lemma 2.2). Then for all Hermitian matrices P we would have $(Tx)^T P(Ty) = x^T(T^T PT)y$, and hence, that $(Tx)^T P(Ty) = -k(Tu)^T P(Tv)$. Then, $Tx = \alpha Tu$ and, thus, $x = \alpha u$ or $Tx = \beta Tv$ and $x = \beta v$. So, we shall assume that all components of x, y, u, v are nonzero. Suppose that x is not a scalar multiple of u to begin with. Then, for any index i with $1 \leq i \leq n$, there is some other index j and two nonzero real numbers c_i, c_j such that

$$x_i = c_i u_i \quad x_j = c_j u_j, \qquad c_i \neq c_j. \tag{8}$$

Choose one such pair of indexes i, j. Equating the coefficients of p_{ii}, p_{jj} and p_{ij}, respectively, in the identity $x^T Py = -ku^T Pv$ yields the following equations:

$$x_i y_i = -ku_i v_i \tag{9}$$
$$x_j y_j = -ku_j v_j \tag{10}$$
$$(x_i y_j + x_j y_i) = -k(u_i v_j + u_j v_i). \tag{11}$$

If we combine (8) with (9) and (10), we find

$$y_i = -\frac{k}{c_i} v_i \tag{12}$$
$$y_j = -\frac{k}{c_j} v_j. \tag{13}$$

Using (9)–(13), we find $c_i u_i y_j + c_j u_j y_i = -k(u_i v_j + u_j v_i)$. Hence, $u_i v_j (c_j - c_i/c_j) = u_j v_i (c_j - c_i/c_i)$. Recall that $c_i \neq c_j$, so we can divide by $c_j - c_i$ and rearrange the terms to get

$$\frac{c_i}{c_j} = \left(\frac{v_i}{v_j}\right)\left(\frac{u_j}{u_i}\right). \tag{14}$$

However, using (8), we find

$$\frac{c_i}{c_j} = \left(\frac{x_i}{x_j}\right)\left(\frac{u_j}{u_i}\right). \tag{15}$$

Combining (14) and (15) yields

$$\frac{v_i}{v_j} = \frac{x_i}{x_j}. \tag{16}$$

Thus, $x_i = cv_i$, $x_j = cv_j$ for some constant c. Now, if we select any other index k with $1 \leq k \leq n$, and write $x_k = c_k u_k$ then c_k must be different to at least one of c_i, c_j. Without loss of generality, we may take it that $c_k \neq c_i$. Then, the aforementioned argument can be repeated with the indexes i and k in place of i and j to yield

$$x_i = cv_i \quad x_k = cv_k. \tag{17}$$

However, this can be done for any index k so we conclude that $x = cv$ for a scalar c. So, we have shown that if x is not a scalar multiple of u, then it is a scalar multiple of v. To complete the proof, note that if $x = \beta v$ for a scalar β then by (9), $\beta v_i y_i = -ku_i v_i$ for all i. Thus $y = -(k/\beta)u$ as claimed. The same argument will show that if $x = \alpha u$ for a scalar α, then $y = -(k/\alpha)v$. Q.E.D

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A Robust Solver Using a Continuation Method for Nevanlinna–Pick Interpolation With Degree Constraint

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***Abstract*—This note modifies a previous algorithm for solving a certain convex optimization problem, introduced by Byrnes, Georgiou, and Lindquist, to determine any Nevanlinna–Pick interpolant satisfying degree constraint. The modified algorithm is based on a continuation method with predictor-corrector steps and it turns out to be quite efficient and numerically robust.**

***Index Terms*—Continuation method, degree constraint, Nevanlinna–Pick interpolation, predictor–corrector step.**

I. INTRODUCTION

This note proposes a new solver for computing interpolants for the Nevanlinna–Pick interpolation problem with degree constraint (NPDC), formulated as follows.

NPDC: Suppose that a set $\mathcal{D} := \{(z_j, w_j) \in \mathbb{C}^2\}_{j=0}^n$, with distinct $\{z_j\}$ and $|z_j| > 1$, is given under the following assumptions.

A1) The Pick matrix P is positive definite, where

$$P := \left[\frac{w_i + \bar{w}_j}{1 - z_i^{-1}\bar{z}_j^{-1}}\right]_{i,j=0}^n. \tag{1}$$

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