

# **A Review and Comparison of Methods for Detecting Outliers in Univariate Data Sets**

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# **A Review and Comparison of Methods for Detecting Outliers in Univariate Data Sets**

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Most real-world data sets contain outliers that have unusually large or small values when compared with others in the data set. Outliers may cause a negative effect on data analyses, such as ANOVA and regression, based on distribution assumptions, or may provide useful information about data when we look into an unusual response to a given study. Thus, outlier detection is an important part of data analysis in the above two cases. Several outlier labeling methods have been developed. Some methods are sensitive to extreme values, like the SD method, and others are resistant to extreme values, like Tukey's method. Although these methods are quite powerful with large normal data, it may be problematic to apply them to non-normal data or small sample sizes without knowledge of their characteristics in these circumstances. This is because each labeling method has different measures to detect outliers, and expected outlier percentages change differently according to the sample size or distribution type of the data.

Many kinds of data regarding public health are often skewed, usually to the right, and lognormal distributions can often be applied to such skewed data, for instance, surgical procedure times, blood pressure, and assessment of toxic compounds in environmental analysis. This paper reviews and compares several common and less common outlier labeling methods and presents information that shows how the percent of outliers changes in each method according to the skewness and sample size of lognormal distributions through simulations and application to real data sets. These results may help establish guidelines for the choice of outlier detection methods in skewed data, which are often seen in the public health field.

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## 1.0 INTRODUCTION

This chapter consists of two sections: the Background and Outlier Detection Method. In the Background, basic ideas of an outlier are discussed such as definitions, features, and reasons to detect outliers. In the Outlier Detection Method section, characteristics of the two kinds of outlier detection methods are described briefly: formal and informal tests.

### 1.1 BACKGROUND

Observed variables often contain outliers that have unusually large or small values when compared with others in a data set. Some data sets may come from homogeneous groups; others from heterogeneous groups that have different characteristics regarding a specific variable, such as height data not stratified by gender. Outliers can be caused by incorrect measurements, including data entry errors, or by coming from a different population than the rest of the data. If the measurement is correct, it represents a rare event. Two aspects of an outlier can be considered.

The first aspect to note is that outliers cause a negative effect on data analysis. Osborne and Overbay (2004) briefly categorized the deleterious effects of outliers on statistical analyses:

- 1) Outliers generally serve to increase error variance and reduce the power of statistical tests.
- 2) If non-randomly distributed, they can decrease normality (and in multivariate analyses, violate assumptions of sphericity and multivariate normality), altering the odds of making both Type I and Type II errors.
- 3) They can seriously bias or influence estimates that may be of substantive interest.

The following example simply shows how one outlier can highly distort the mean, variance, and 95% confidence interval for the mean. Let's suppose there is a simple data set composed of data points 1, 2, 3, 4, 5, 6, 7 and its basic statistics are as shown in Table 1. Now,



let's replace data point 7 with 77. As shown in Table 2, the mean and variance of the data are much larger than that of the original data set due to one unusual data value, 77. The 95% confidence interval for the mean is also much broader because of the large variance. It may cause potential problems when data analysis that is sensitive to a mean or variance is conducted.

**Table 1: Basic Statistic of a Simple Data Set**

Mean	Median	Variance	95 % Confidence Interval for the mean
4	4	4.67	[2.00 to 6.00]

**Table 2: Basic Statistic After Changing 7 into 77 in the Simple Data Set**

Mean	Median	Variance	95 % Confidence Interval for the mean
14	4	774.67	[-11.74 to 39.74]

The second aspect of outliers is that they can provide useful information about data when we look into an unusual response to a given study. They could be the extreme values sitting apart from the majority of the data regardless of distribution assumptions. The following two cases are good examples of outlier analysis in terms of the second aspect of an outlier: 1) to identify medical practitioners who under- or over-utilize specific procedures or medical equipment, such as an x-ray instrument; 2) to identify Primary Care Physicians (PCPs) with inordinately high Member Dissatisfaction Rates (MDRs) (MDRs = the number of member complaints / PCP practice size) compared to other PCPs.<sup>23</sup>

In summary, there are two reasons for detecting outliers. The first reason is to find outliers which influence assumptions of a statistical test, for example, outliers violating the normal distribution assumption in an ANOVA test, and deal with them properly in order to improve statistical analysis. This could be considered as a preliminary step for data analysis. The second reason is to use the outliers themselves for the purpose of obtaining certain critical information about the data as was shown in the above examples.

## 1.2 OUTLIER DETECTION METHOD

There are two kinds of outlier detection methods: formal tests and informal tests. Formal and informal tests are usually called tests of discordancy and outlier labeling methods, respectively.

Most formal tests need test statistics for hypothesis testing. They are usually based on assuming some well-behaving distribution, and test if the target extreme value is an outlier of the distribution, i.e., whether or not it deviates from the assumed distribution. Some tests are for a single outlier and others for multiple outliers. Selection of these tests mainly depends on numbers and type of target outliers, and type of data distribution.<sup>1</sup> Many various tests according to the choice of distributions are discussed in Barnett and Lewis (1994) and Iglewicz and Hoaglin (1993). Iglewicz and Hoaglin (1993) reviewed and compared five selected formal tests which are applicable to the normal distribution, such as the Generalized ESD, Kurtosis statistics, Shapiro-Wilk, the Boxplot rule, and the Dixon test, through simulations.

Even though formal tests are quite powerful under well-behaving statistical assumptions such as a distribution assumption, most distributions of real-world data may be unknown or may not follow specific distributions such as the normal, gamma, or exponential. Another limitation is that they are susceptible to masking or swamping problems. Acuna and Rodriguez (2004) define these problems as follows:

Masking effect: It is said that one outlier masks a second outlier if the second outlier can be considered as an outlier only by itself, but not in the presence of the first outlier. Thus, after the deletion of the first outlier the second instance is emerged as an outlier.

Swamping effect: It is said that one outlier swamps a second observation if the latter can be considered as an outlier only under the presence of the first one. In other words, after the deletion of the first outlier the second observation becomes a non-outlying observation.

Many studies regarding these problems have been conducted by Barnett and Lewis (1994), Iglewicz and Hoaglin (1993), Davies and Gather (1993), and Bendre and Kale (1987).

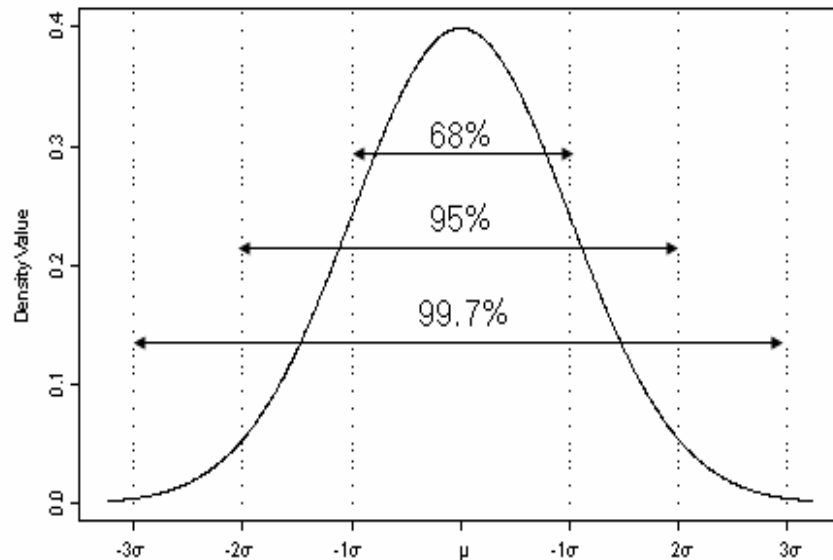
On the other hand, most outlier labeling methods, informal tests, generate an interval or criterion for outlier detection instead of hypothesis testing, and any observations beyond the interval or criterion is considered as an outlier. Various location and scale parameters are mostly employed in each labeling method to define a reasonable interval or criterion for outlier detection. There are two reasons for using an outlier labeling method. One is to find possible outliers as a screening device before conducting a formal test. The other is to find the extreme values away

from the majority of the data regardless of the distribution. While the formal tests usually require test statistics based on the distribution assumptions and a hypothesis to determine if the target extreme value is a true outlier of the distribution, most outlier labeling methods present the interval using the location and scale parameters of the data. Although the labeling method is usually simple to use, some observations outside the interval may turn out to be falsely identified outliers after a formal test when the outliers are defined as only observations that deviate from the assuming distribution. However, if the purpose of the outlier detection is not a preliminary step to find the extreme values violating the distribution assumptions of the main statistical analyses such as the t-test, ANOVA, and regression, but mainly to find the extreme values away from the majority of the data regardless of the distribution, the outlier labeling methods may be applicable. In addition, for a large data set that is statistically problematic, e.g., when it is difficult to identify the distribution of the data or transform it into a proper distribution such as the normal distribution, labeling methods can be used to detect outliers.

This paper focuses on outlier labeling methods. Chapter 2 presents the possible problems when labeling methods are applied to skewed data. In Chapter 3, seven outlier labeling methods are outlined. In Chapter 4, the average percentages of outliers in the standard normal and log normal distributions with the same mean and different variances is computed to compare the outlier percentage of the selected five outlier labeling methods according to the degree of the skewness and different sample sizes. In Chapter 5, the five selected methods are applied to real data sets.

## 2.0 STATEMENT OF PROBLEM

Outlier-labeling methods such as the Standard Deviation (SD) and the boxplot are commonly used and are easy to use. These methods are quite reasonable when the data distribution is symmetric and mound-shaped such as the normal distribution. Figure 1 shows that about 68%, 95%, and 99.7% of the data from a normal distribution are within 1, 2, and 3 standard deviations of the mean, respectively. If data follows a normal distribution, this helps to estimate the likelihood of having extreme values in the data<sup>3</sup>, so that the observation two or three standard deviations away from the mean may be considered as an outlier in the data.

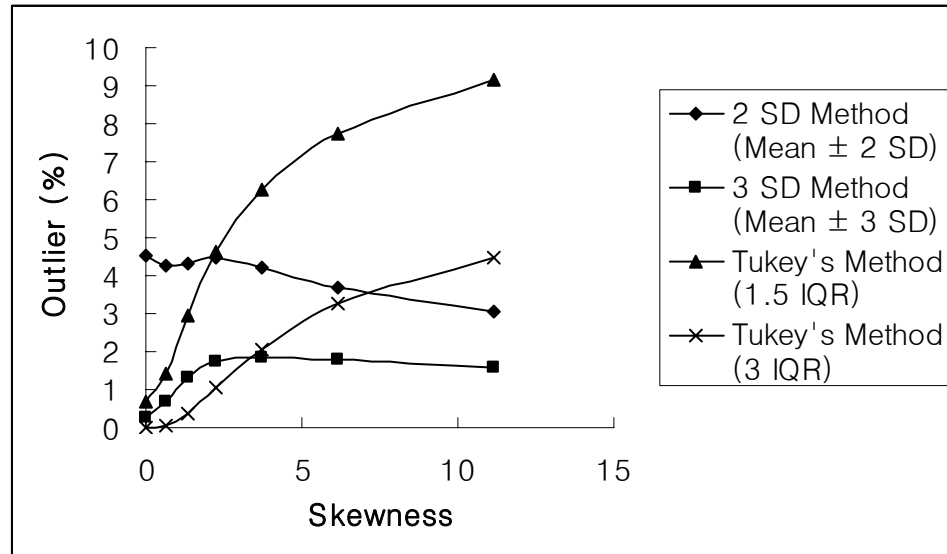


**Figure 1: Probability density function for a normal distribution according to the standard deviation.**

The boxplot which was developed by Tukey (1977) is another very helpful method since it makes no distributional assumptions nor does it depend on a mean or standard deviation.<sup>19</sup> The lower quartile (q1) is the 25th percentile, and the upper quartile (q3) is the 75th percentile of the data. The inter-quartile range (IQR) is defined as the interval between q1 and q3.

Tukey (1997) defined  $q1-(1.5*iqr)$  and  $q3+(1.5*iqr)$  as “inner fences”,  $q1-(3*iqr)$  and  $q3+(3*iqr)$  as “outer fences”, the observations between an inner fence and its nearby outer fence as “outside”, and anything beyond outer fences as “far out”.<sup>31</sup> High (2000) renamed the “outside” potential outliers and the “far out” problematic outliers.<sup>19</sup> The “outside” and “far out” observations can also be called possible outliers and probable outliers, respectively. This method is quite effective, especially when working with large continuous data sets that are not highly skewed.<sup>19</sup>

Although Tukey’s method is quite effective when working with large data sets that are fairly normally distributed, many distributions of real-world data do not follow a normal distribution. They are often highly skewed, usually to the right, and in such cases the distributions are frequently closer to a lognormal distribution than a normal one.<sup>21</sup> The lognormal distribution can often be applied to such data in a variety of forms, for instance, personal income, blood pressure, and assessment of toxic compounds in environmental analysis. In order to illustrate how the theoretical percentage of outliers changes according to the skewness of the data in the SD method ( $Mean \pm 2 SD$ ,  $Mean \pm 3 SD$ ) and Tukey’s method, lognormal distributions with the same mean (0) but different standard deviations (0.2, 0.4, 0.6, 0.8, 1.0, 1.2) are used for the data sets with different degrees of skewness, and the standard normal distribution is used for the data set whose skewness is zero. The computation of the mean, standard deviation, and skewness in a lognormal distribution is in Appendix A. According to Figure 2, the two methods show a different pattern, e.g., the outlier percentage of Tukey’s method increases, unlike the SD method. It shows that the results of outlier detection may change depending on the outlier detection methods or the distribution of the data.

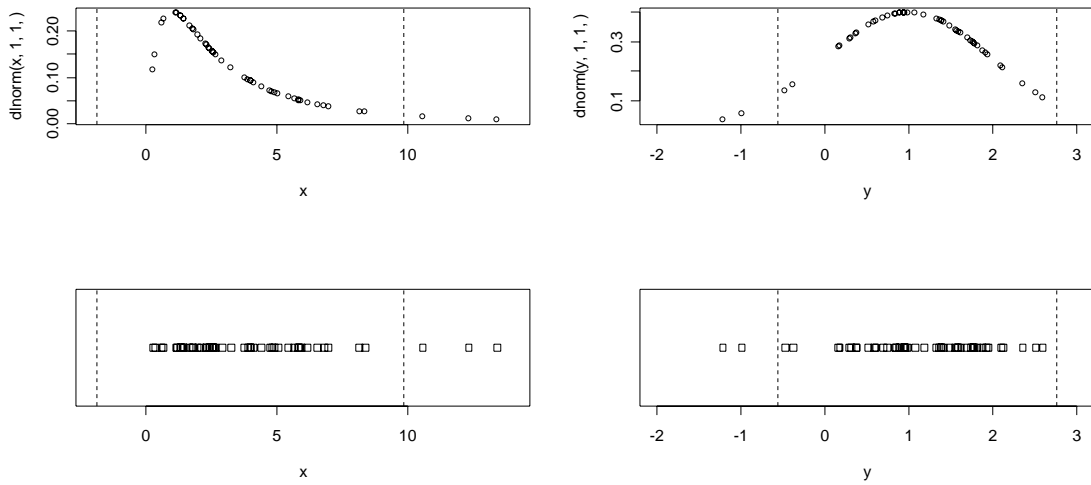


**Figure 2: Theoretical Change of Outliers' Percentage According to the Skewness of the Lognormal Distributions in the SD Method and Tukey's Method**

When data are highly skewed or in other respects depart from a normal distribution, transformations to normality is a common step in order to identify outliers using a method which is quite effective in a normal distribution. Such a transformation could be useful when the identification of outliers is conducted as a preliminary step for data analysis and it helps to make possible the selection of appropriate statistical procedures for estimating and testing as well.<sup>21</sup> However, if an outlier itself is a primary concern in a given study, as was shown in a previous example in the identification of medical practitioners who under- or over-utilize such medical equipment as x-ray instruments, a transformation of the data could affect our ability to identify outliers. For example, 50 random samples (x) are generated through statistical software R in order to show the effect of the transformation. The random variable X has a lognormal distribution (Mean=1, SD=1), and its logarithm,  $Y=\log(x)$ , has a normal distribution. If the observations which are beyond the mean by two standard deviations are considered outliers, the expected outliers before and after transformation are totally different. As shown in Figure 3, while three observations which have large values are considered as outliers in the original 50 random samples(x), after log transformation of these samples, two observations of small values appear to be outliers, and the former large valued observations are no longer considered to be outliers. The vertical lines in each graph represent cutoff values (Mean  $\pm$  2\*SD). Lower and

upper cutoff values are  $(-1.862268, 9.865134)$  and  $(-0.5623396, 2.763236)$ , respectively, in the lognormal data( $x$ ) and its logarithm( $y$ ).

Although this approach is not be affected by extreme values because it does not depend on the extreme observations after transformation, after an artificial transformation of the data, however, the data may be reshaped so that true outliers are not detected or other observations may be falsely identified as outliers.<sup>21</sup>



**Figure 3: Density Plot and Dotplot of the Lognormal Distribution (sample size=50) with Mean=1 and SD=1, and its Logarithm,  $Y=\log(x)$ .**

Several methods to identify outliers have been developed. Some methods are sensitive to extreme values like the SD method, and others are resistant to extreme values like Tukey's method. The objective of this paper is to review and compare several common and less common labeling methods for identifying outliers and to present information that shows how the average percentage of outliers changes in each method according to the degree of skewness and sample size of the data in order to help establish guidelines for the choice of outlier detection methods in skewed data when an outlier itself is a primary concern in a given study.

### 3.0 OUTLIER LABELING METHOD

This chapter reviews seven outlier labeling methods and gives examples of simple numerical computations for each test.

#### 3.1 STANDARD DEVIATION (SD) METHOD

The simple classical approach to screen outliers is to use the SD (Standard Deviation) method. It is defined as

2 SD Method:  $\bar{x} \pm 2 \text{ SD}$

3 SD Method:  $\bar{x} \pm 3 \text{ SD}$ , where the mean is the sample mean and SD is the sample standard deviation.

The observations outside these intervals may be considered as outliers. According to the Chebyshev inequality, if a random variable  $X$  with mean  $\mu$  and variance  $\sigma^2$  exists, then for any  $k > 0$ ,

$$P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$$

$$P[|X - \mu| < k\sigma] \geq 1 - \frac{1}{k^2}, \quad k > 0$$

the inequality  $[1 - (1/k)^2]$  enables us to determine what proportion of our data will be within  $k$  standard deviations of the mean<sup>3</sup>. For example, at least 75%, 89%, and 94% of the data are within 2, 3, and 4 standard deviations of the mean, respectively. These results may help us determine the likelihood of having extreme values in the data<sup>3</sup>. Although Chebychev's theorem is true for any data from any distribution, it is limited in that it only gives the smallest proportion of observations within  $k$  standard deviations of the mean<sup>22</sup>. In the case of when the distribution of a



random variable is known, a more exact proportion of observations centering around the mean can be computed. For instance, if certain data follow a normal distribution, approximately 68%, 95%, and 99.7% of the data are within 1, 2, and 3 standard deviations of the mean, respectively; thus, the observations beyond two or three SD above and below the mean of the observations may be considered as outliers in the data.

The example data set, X, for a simple example of this method is as follows:

3.2, 3.4, 3.7, 3.7, 3.8, 3.9, 4, 4, 4.1, 4.2, 4.7, 4.8, 14, 15.

For the data set,  $\bar{x} = 5.46$ ,  $SD = 3.86$ , and the intervals of the 2 SD and 3 SD methods are (-2.25, 13.18) and (-6.11, 17.04), respectively. Thus, 14 and 15 are beyond the interval of the 2 SD method and there are no outliers in the 3 SD method.

### 3.2 Z-SCORE

Another method that can be used to screen data for outliers is the Z-Score, using the mean and standard deviation.

$$Z_i = \frac{x_i - \bar{x}}{sd}, \text{ where } X_i \sim N(\mu, \sigma^2), \text{ and } sd \text{ is the standard deviation of data.}$$

The basic idea of this rule is that if X follows a normal distribution,  $N(\mu, \sigma^2)$ , then Z follows a standard normal distribution,  $N(0, 1)$ , and Z-scores that exceed 3 in absolute value are generally considered as outliers. This method is simple and it is the same formula as the 3 SD method when the criterion of an outlier is an absolute value of a Z-score of at least 3. It presents a reasonable criterion for identification of the outlier when data follow the normal distribution. According to Shiffler (1988), a possible maximum Z-score is dependent on sample size, and it is computed as  $(n-1)/\sqrt{n}$ . The proof is given in Appendix B. Since no z-score exceeds 3 in a sample size less than or equal to 10, the z-score method is not very good for outlier labeling, particularly in small data sets<sup>21</sup>. Another limitation of this rule is that the standard deviation can be inflated by a few or even a single observation having an extreme value. Thus it can cause a masking problem, i.e., the less extreme outliers go undetected because of the most extreme outlier(s), and vice versa. When masking occurs, the outliers may be neighbors. Table 3 shows

a computation and masking problem of the Z-Score method using the previous example data set, X.

**Table 3: Computation and Masking Problem of the Z-Score**

$i$	Case 1 ( $\bar{x}=5.46, sd=3.86$ )		Case 2 ( $\bar{x}=4.73, sd=2.82$ )	
	$x_i$	Z-Score	$x_i$	Z-Score
1	3.2	-0.59	3.2	-0.54
2	3.4	-0.54	3.4	-0.47
3	3.7	-0.46	3.7	-0.37
4	3.7	-0.46	3.7	-0.37
5	3.8	-0.43	3.8	-0.33
6	3.9	-0.41	3.9	-0.29
7	4	-0.38	4	-0.26
8	4	-0.38	4	-0.26
9	4.1	-0.35	4.1	-0.22
10	4.2	-0.33	4.2	-0.19
11	4.7	-0.20	4.7	-0.01
12	4.8	-0.17	4.8	0.02
13	14	<b>2.21</b>	14	<b>3.29</b>
14	15	<b>2.47</b>	-	-

For case 1, with all of the example data included, it appears that the values 14 and 15 are outliers, yet no observation exceeds the absolute value of 3. For case 2, with the most extreme value, 15, among example data excluded, 14 is considered an outlier. This is because multiple extreme values have artificially inflated standard deviations.

### 3.3 THE MODIFIED Z-SCORE

Two estimators used in the Z-Score, the sample mean and sample standard deviation, can be affected by a few extreme values or by even a single extreme value. To avoid this problem, the median and the median of the absolute deviation of the median (MAD) are employed in the

modified Z-Score instead of the mean and standard deviation of the sample, respectively (Iglewicz and Hoaglin, 1993).

$MAD = median\{|x_i - \tilde{x}|\}$ , where  $\tilde{x}$  is the sample median.

The modified Z-Score ( $M_i$ ) is computed as

$$M_i = \frac{0.6745(x_i - \tilde{x})}{MAD}, \text{ where } E(MAD) = 0.675 \sigma \text{ for large normal data.}$$

Iglewicz and Hoaglin (1993) suggested that observations are labeled outliers when  $|M_i| > 3.5$  through the simulation based on pseudo-normal observations for sample sizes of 10, 20, and 40.<sup>21</sup> The  $M_i$  score is effective for normal data in the same way as the Z-score.

**Table 4: Computation of Modified Z-Score and its Comparison with the Z-Score**

$i$	$x_i$	Z-Score	modified Z-Score
1	3.2	-0.59	-1.80
2	3.4	-0.54	-1.35
3	3.7	-0.46	-0.67
4	3.7	-0.46	-0.67
5	3.8	-0.43	-0.45
6	3.9	-0.41	-0.22
7	4	-0.38	0
8	4	-0.38	0
9	4.1	-0.35	0.22
10	4.2	-0.33	0.45
11	4.7	-0.20	1.57
12	4.8	-0.17	1.80
13	14	<b>2.21</b>	<b>22.48</b>
14	15	<b>2.47</b>	<b>24.73</b>

Table 4 shows the computation of the modified Z-Score and its comparison with the Z-Score of the previous example data set. While no observation is detected as an outlier in the Z-Score, two extreme values, 14 and 15, are detected as outliers at the same time in the modified Z-Score since this method is less susceptible to the extreme values.

### 3.4 TUKEY'S METHOD (BOXPLOT)

Tukey's (1977) method, constructing a boxplot, is a well-known simple graphical tool to display information about continuous univariate data, such as the median, lower quartile, upper quartile, lower extreme, and upper extreme of a data set. It is less sensitive to extreme values of the data than the previous methods using the sample mean and standard variance because it uses quartiles which are resistant to extreme values. The rules of the method are as follows:

1. The IQR (Inter Quartile Range) is the distance between the lower (Q1) and upper (Q3) quartiles.
2. Inner fences are located at a distance 1.5 IQR below Q1 and above Q3 [ $Q1-1.5 \text{ IQR}$ ,  $Q3+1.5 \text{ IQR}$ ].
3. Outer fences are located at a distance 3 IQR below Q1 and above Q3 [ $Q1-3 \text{ IQR}$ ,  $Q3+3 \text{ IQR}$ ].
4. A value between the inner and outer fences is a possible outlier. An extreme value beyond the outer fences is a probable outlier. There is no statistical basis for the reason that Tukey uses 1.5 and 3 regarding the IQR to make inner and outer fences.

For the previous example data set,  $Q1=3.725$ ,  $Q3=4.575$ , and  $\text{IQR}=0.85$ . Thus, the inner fence is  $[2.45, 5.85]$  and the outer fence is  $[1.18, 7.13]$ . Two extreme values, 14 and 15, are identified as probable outliers in this method. Figure 4 is a boxplot generated using the statistical software STATA for the example data set.

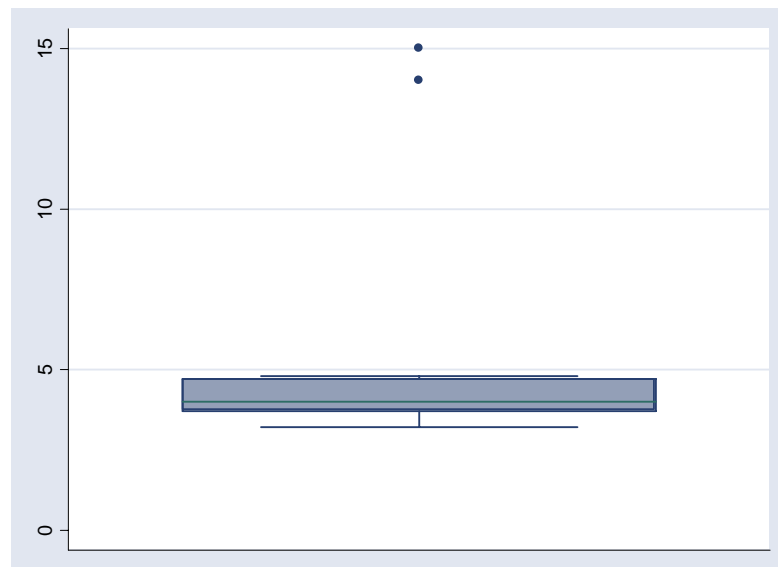
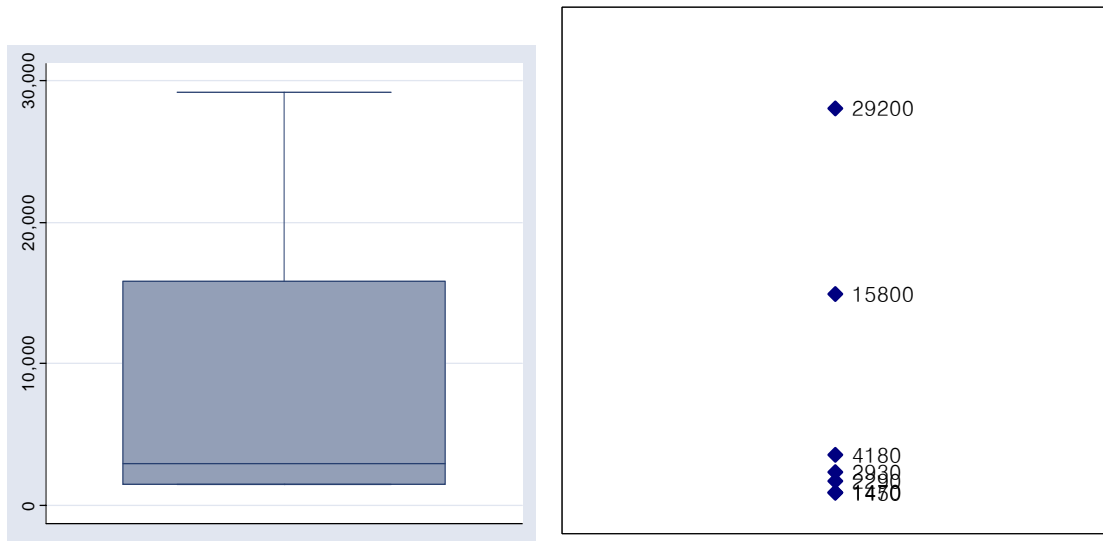


Figure 4: Boxplot for the Example Data Set

While previous methods are limited to mound-shaped and reasonably symmetric data such as the normal distribution<sup>21</sup>, Tukey’s method is applicable to skewed or non mound-shaped data since it makes no distributional assumptions and it does not depend on a mean or standard deviation. However, Tukey’s method may not be appropriate for a small sample size<sup>21</sup>. For example, let’s suppose that a data set consists of data points 1450, 1470, 2290, 2930, 4180, 15800, and 29200. A simple distribution of the data using a Boxplot and Dotplot are shown in Figure 5. Although 15800 and 29200 may appear to be outliers in the dotplot, no observation is shown as an outlier in the boxplot.



**Figure 5: Boxplot and Dotplot.** (Note: No outlier shown in the boxplot)

### 3.5 ADJUSTED BOXPLOT

Although the boxplot proposed by Tukey (1977) may be applicable for both symmetric and skewed data, the more skewed the data, the more observations may be detected as outliers,<sup>32</sup> as shown in Figure 2. This results from the fact that this method is based on robust measures such as lower and upper quartiles and the IQR without considering the skewness of the data. Vanderviere and Huber (2004) introduced an adjusted boxplot taking into account the medcouple ( $MC$ )<sup>32</sup>, a robust measure of skewness for a skewed distribution.

When  $X_n = \{x_1, x_2, \dots, x_n\}$  is a data set independently sampled from a continuous univariate distribution and it is sorted such as  $x_1 \leq x_2 \leq \dots \leq x_n$ , the  $MC$  of the data is defined as

$$MC(x_1, \dots, x_n) = \text{med} \frac{(x_j - \text{med}_k) - (\text{med}_k - x_i)}{x_j - x_i}, \text{ where } \text{med}_k \text{ is the median of } X_n, \text{ and } i$$

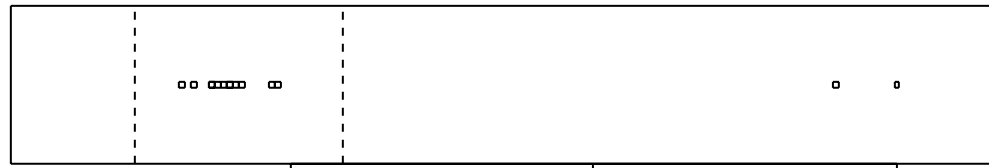
and  $j$  have to satisfy  $x_i \leq \text{med}_k \leq x_j$ , and  $x_i \neq x_j$ . The interval of the adjusted boxplot is as follows (G. Bray et al. (2005)):

$$\begin{aligned} [L, U] &= [Q_1 - 1.5 * \exp(-3.5MC) * IQR, Q_3 + 1.5 * \exp(4MC) * IQR] \text{ if } MC \geq 0 \\ &= [Q_1 - 1.5 * \exp(-4MC) * IQR, Q_3 + 1.5 * \exp(3.5MC) * IQR] \text{ if } MC \leq 0, \end{aligned}$$

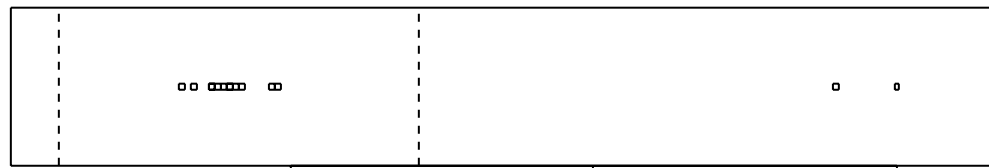
where  $L$  is the lower fence, and  $U$  is the upper fence of the interval. The observations which fall outside the interval are considered outliers.

The value of the  $MC$  ranges between -1 and 1. If  $MC=0$ , the data is symmetric and the adjusted boxplot becomes Tukey's box plot. If  $MC>0$ , the data has a right skewed distribution, whereas if  $MC<0$ , the data has a left skewed distribution.<sup>32</sup> A simple example for computation of  $MC$  and a brief comparison of classical and  $MC$  skewness are in Appendix C.

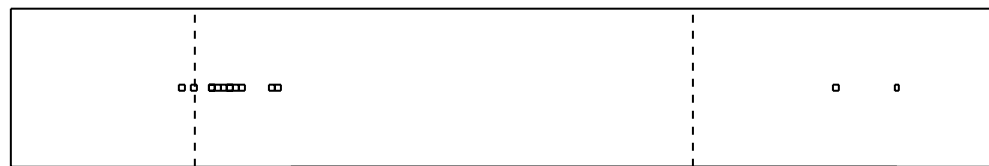
For the previous example data set,  $Q_1=3.725$ ,  $Q_3=4.575$ ,  $IQR=0.85$ , and  $MC=0.43$ . Thus, the interval of the adjusted boxplot is  $[3.44, 11.62]$ . Two extreme values, 14 and 15, and the two smallest values, 3.2 and 3.4, are identified as outliers in this method. Figure 6 shows the change of the intervals of two boxplot methods, Tukey's method and the adjusted boxplot, for the example data set. The vertical dotted lines are the lower and upper bound of the interval of each method. Although the example data set is artificial and is not large enough to explain their difference, we can see a general trend that the interval of the adjusted boxplot, especially the upper fence, moves to the side of the skewed tail, compared to Tukey's method.



Inner fences of Tukey Method ( $Q1-1.5*IQR$ ,  $Q3+1.5*IQR$ )



Outer fences of Tukey Method ( $Q1-3*IQR$ ,  $Q3+3*IQR$ )



Single fence of adjusted box plot ( $Q1-1.5 * \exp(-3.5MC) * IQR$ ,  $Q3+1.5 * \exp(4MC) * IQR$ )

**Figure 6: Change of the intervals of Two Different Boxplot Methods  
(Tukey's Method vs. the Adjusted Boxplot)**

Vanderviere and Huber (2004) computed the average percentage of outliers beyond the lower and upper fence of two types of boxplots, the adjusted Boxplot and Tukey's Boxplot, for several distributions and different sample sizes. In the simulation, less observations, especially in the right tail, are classified as outliers compared to Tukey's method when the data are skewed to the right.<sup>32</sup> In the case of a mildly right-skewed distribution, the lower fence of the interval may move to the right and more observations in the left side will be classified as outliers compared to Tukey's method. This difference mainly comes from a decrease in the lower fence and an increase in the upper fence from  $Q1$  and  $Q3$ , respectively.<sup>32</sup>

### 3.6 MAD<sub>E</sub> METHOD

The MAD<sub>e</sub> method, using the median and the Median Absolute Deviation (MAD), is one of the basic robust methods which are largely unaffected by the presence of extreme values of the data set.<sup>11</sup> This approach is similar to the SD method. However, the median and MAD<sub>e</sub> are employed in this method instead of the mean and standard deviation. The MAD<sub>e</sub> method is defined as follows;

2 MAD<sub>e</sub> Method: Median  $\pm$  2 MAD<sub>e</sub>

3 MAD<sub>e</sub> Method: Median  $\pm$  3 MAD<sub>e</sub>,

where MAD<sub>e</sub>=1.483×MAD for large normal data.

MAD is an estimator of the spread in a data, similar to the standard deviation<sup>11</sup>, but has an approximately 50% breakdown point like the median<sup>21</sup>. The notion of breakdown point is delineated in Appendix D.

$$\text{MAD} = \text{median} (|x_i - \text{median}(x)|_{i=1,2,\dots,n})$$

When the MAD value is scaled by a factor of 1.483, it is similar to the standard deviation in a normal distribution. This scaled MAD value is the MAD<sub>e</sub>.

For the example data set, the median=4, MAD=0.3, and MAD<sub>e</sub>=0.44. Thus, the intervals of the 2 MAD<sub>e</sub> and 3 MAD<sub>e</sub> methods are [3.11, 4.89] and [2.67, 5.33], respectively.

Since this approach uses two robust estimators having a high breakdown point, i.e., it is not unduly affected by extreme values even though a few observations make the distribution of the data skewed, the interval is seldom inflated, unlike the SD method.

### 3.7 MEDIAN RULE

The median is a robust estimator of location having an approximately 50% breakdown point. It is the value that falls exactly in the center of the data when the data are arranged in order.



That is, if  $x_1, x_2, \dots, x_n$  is a random sample sorted by order of magnitude, then the median is defined as:

Median,  $\tilde{x} = x_m$  when  $n$  is odd

$\tilde{x} = (x_m + x_{m+1})/2$  when  $n$  is even, where  $m = \text{round up}(n/2)$

For a skewed distribution like income data, the median is often used in describing the average of the data. The median and mean have the same value in a symmetrical distribution.

Carling (1998) introduces the median rule for identification of outliers through studying the relationship between target outlier percentage and Generalized Lambda Distributions (GLDs). GLDs with different parameters are used for various moderately skewed distributions<sup>12</sup>. The median substitutes for the quartiles of Tukey's method, and a different scale of the IQR is employed in this method. It is more resistant and its target outlier percentage is less affected by sample size than Tukey's method in the non-Gaussian case<sup>12</sup>. The scale of IQR can be adjusted depending on which target outlier percentage and GLD are selected. In my paper, 2.3 is chosen as the scale of IQR; when the scale is applied to normal distribution, the outlier percentage turns out to be between Tukey's method of 1.5 IQR and that of 3 IQR, i.e., 0.2 %.

It is defined as:

$[C_1, C_2] = Q2 \pm 2.3 \text{ IQR}$ , where  $Q2$  is the sample median.

For the example data set,  $Q2=4$ , and  $\text{IQR}=0.85$ . Thus, the interval of this method is [2.05, 5.96].

#### 4.0 SIMULATION STUDY AND RESULTS FOR THE FIVE SELECTED LABELING METHODS

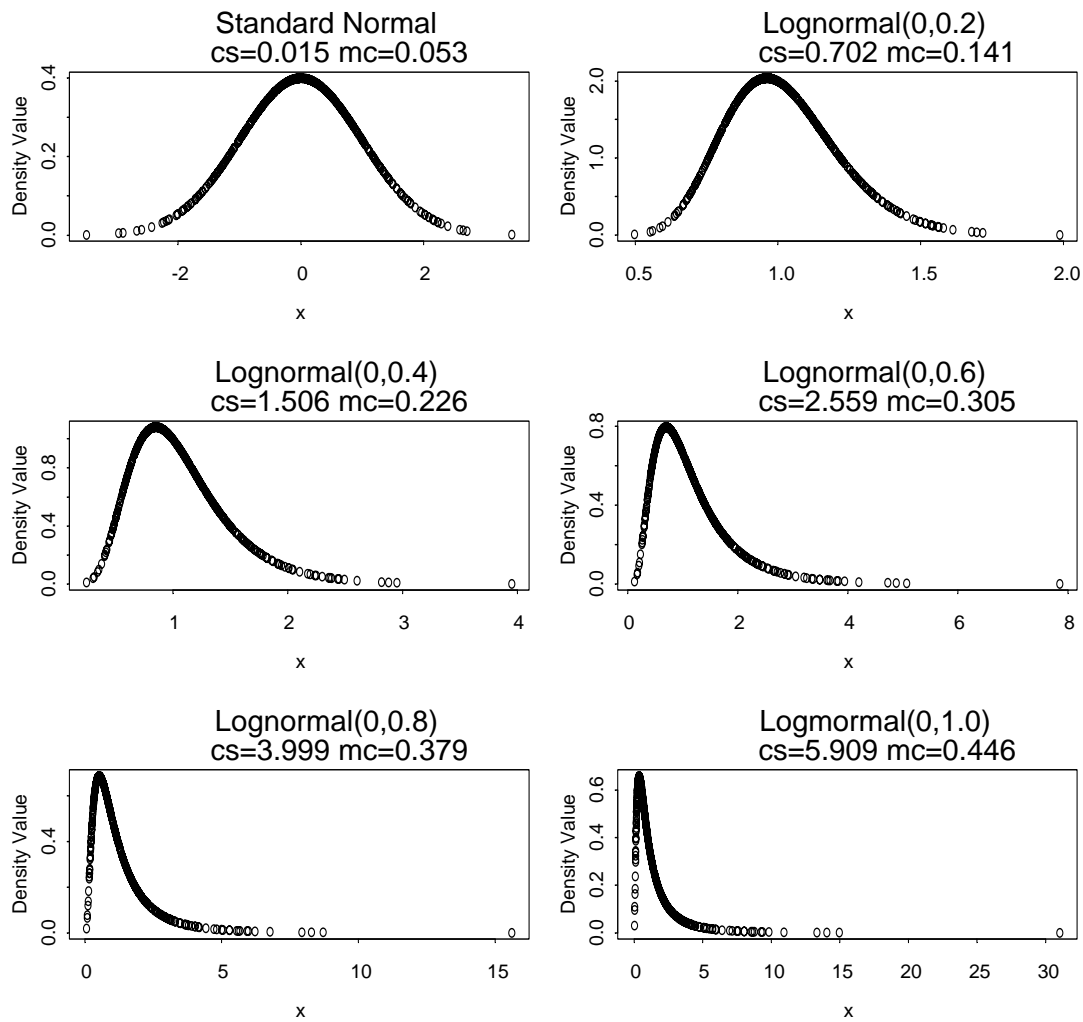
Most intervals or criteria to identify possible outliers in outlier labeling methods are effective under the normal distribution. For example, in the case of a well-known labeling method such as the 2 SD and 3 SD methods and the Boxplot (1.5 IQR), the expected percentages of observations outside the interval are 5%, 0.3%, and 0.7%, respectively, under large normal samples. Although these methods are quite powerful with large normal data, it may be problematic to apply them to non-normal data or small sample sizes without information about their characteristics in these circumstances. This is because each labeling method has different measures to detect outliers, and expected outlier percentages change differently according to the sample size or distribution type of the data.

The purpose of this simulation is to present the expected percentage of the observations outside of the interval of several labeling methods according to the sample size and the degree of the skewness of the data using the lognormal distribution with the same mean and different variances. Through this simulation, we can know not only the possible outlier percentage of several labeling methods but also which method is more robust according to the above two factors, skewness and sample size. The simulation proceeds as follows:

Five labeling methods are selected: the SD Method, the MADe Method, Tukey's Method (Boxplot), Adjusted Boxplot, and the Median Rule. The Z-Score and modified Z-Score are not considered because their criteria to define an outlier are based on the normal distribution.

Average outlier percentages of five labeling methods in the standard normal (0,1) and lognormal distributions with the same mean and different variances (mean=0, variance= $0.2^2$ ,  $0.4^2$ ,  $0.6^2$ ,  $0.8^2$ ,  $1^2$ ) are computed. For each distribution, 1000 replications of sample sizes 20 and 50, 300 replications of the sample size 100, and 100 replications of the sample sizes 300 and 500 are considered. To illustrate the shape of each distribution, i.e., the degree of skewness of the data,

500 random observations were generated from the distributions, and their density plots and skewness are as shown in Figure 7.



**Figure 7: Standard Normal Distribution and Lognormal Distributions**  
( $cs$ =classical skewness ,  $mc$ =medcouple skewness)

Figures 8 and 9 visually show the characteristics of the five labeling methods according to the sample size and skewness of the data using the lognormal distribution. All the values of the Figures including their standard error of the average percentage are reported in Table 5. The results of this simulation are as follows:

1. The 2 MADE method classifies more observations as outliers than any other method. This method approaches the 2 SD method in large normal data; however, as the data increases in skewness, the difference in outlier percentages between the MADE method and the SD method

becomes larger since the location and scale measures such as the median and MADe become the same as the mean and standard variance of the SD method when data follows a normal distribution with a large sample size. The MADe, Tukey's method, and the Median rule increase in the total average percentages of outliers the more skewed the data, while the SD method and adjusted boxplot seldom change over different sample sizes.

2. The Median rule classifies less observations than Tukey's 1.5 IQR method and more observations than Tukey's 3 IQR method.

3. The decrease range of the total outlier percentage of the adjusted boxplot is larger than other methods as the sample size increases.

4. Most methods except the adjusted boxplot show similar patterns in the average outlier percentages on the left side of the distribution. They decrease in left outlier percentage rapidly, especially in 2 MADe and 2 SD methods, the more skewed the data; however, the adjusted boxplot decreases slowly in sample sizes over 300. Different patterns of the adjusted boxplot, e.g., increase in left outlier percentage in small sample sizes, may be due to the following:

- The left fence of the interval may move to the right side because of the MC skewness and a few observations may be distributed outside the left fence by chance.

- Although the number of the observations is small, the ratio in a small sample size could large. This may affect an increase in the average of the percentage of outliers on the left of the distribution.

- The adjusted boxplot may still detect observations on the left side of the distribution in right skewed data, especially mildly skewed data; however, the average percentages are quiet low.

5. The MADe, Tukey's method, and the Median rule increase in the percentage of outliers on the right side of the distribution as the skewness of the data increases while the SD method and adjusted boxplot seldom change in each sample size (the SD method increases slightly and plateaus). The right fence of the intervals of both methods, the SD method and adjusted boxplot, move to the right side of the distribution as the skewness of the data increases. Since the adjusted boxplot takes into account the skewness of the data, its right fence of the interval moves more to the side of the skewed tail, here the right side of the distribution, as the skewness increases. On the other hand, the interval of the SD method is just inflated because of the extreme values.

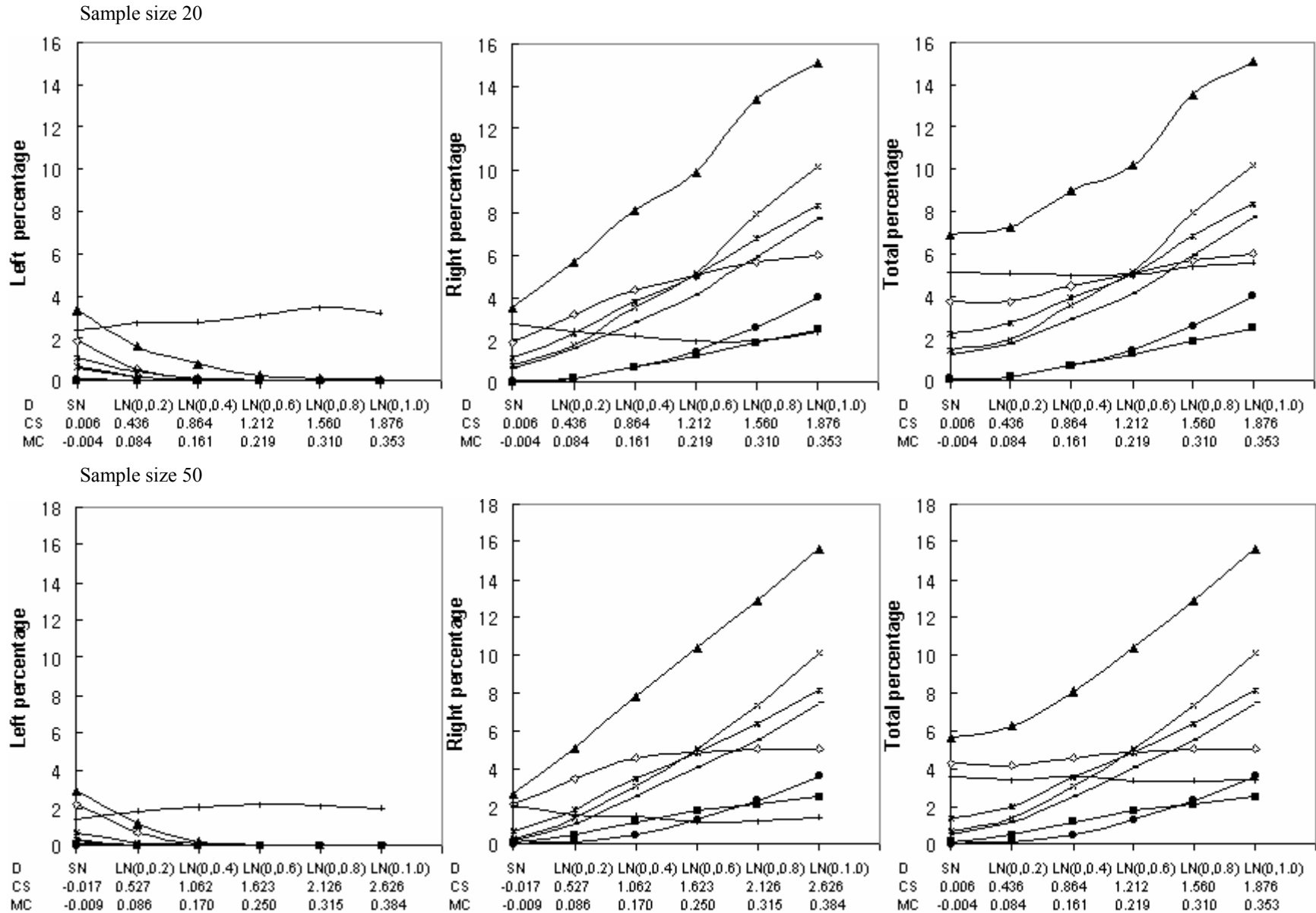
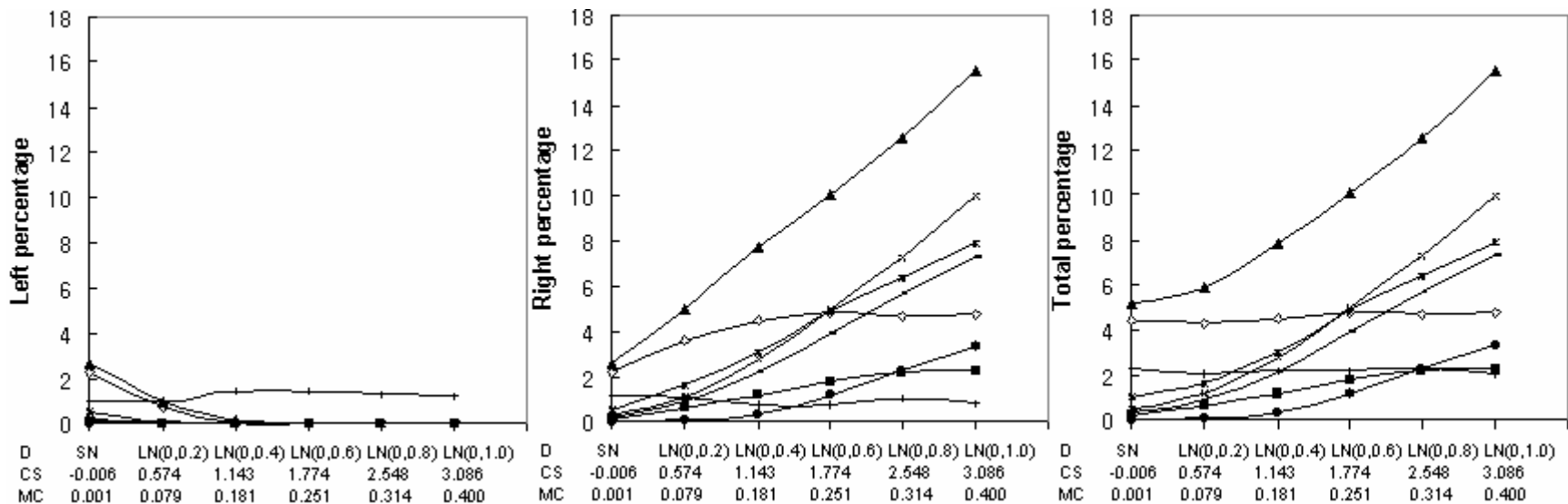


Figure 8: Change in the Outlier Percentages According to the Skewness of the Data

Sample100



Sample size 300

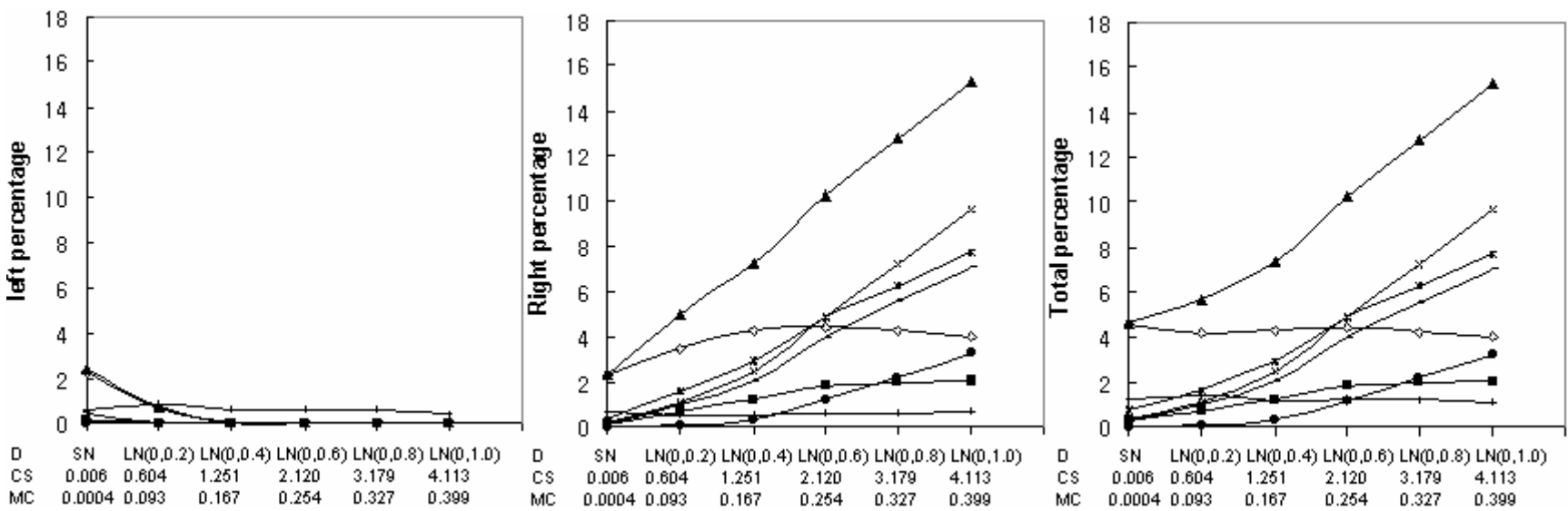


Figure 8 (continued)

Sample size 500

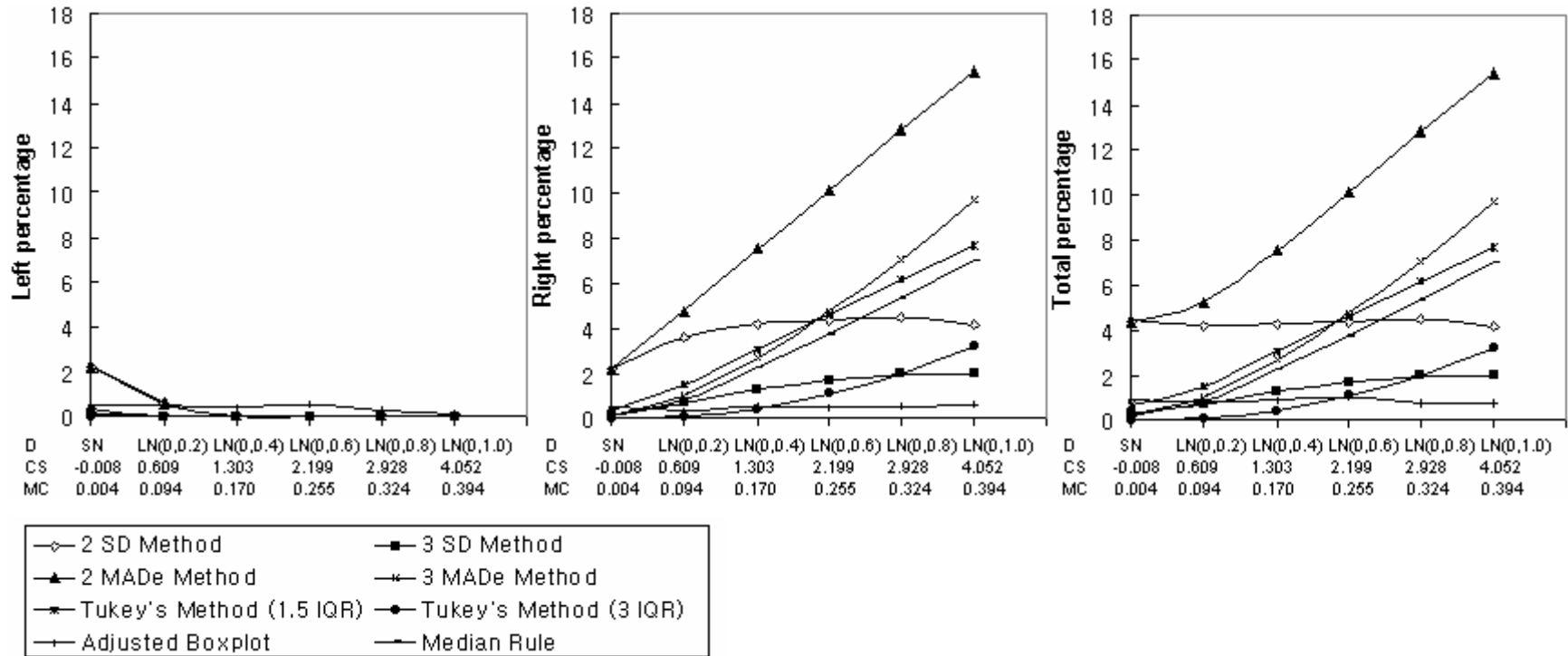


Figure 8 (continued)

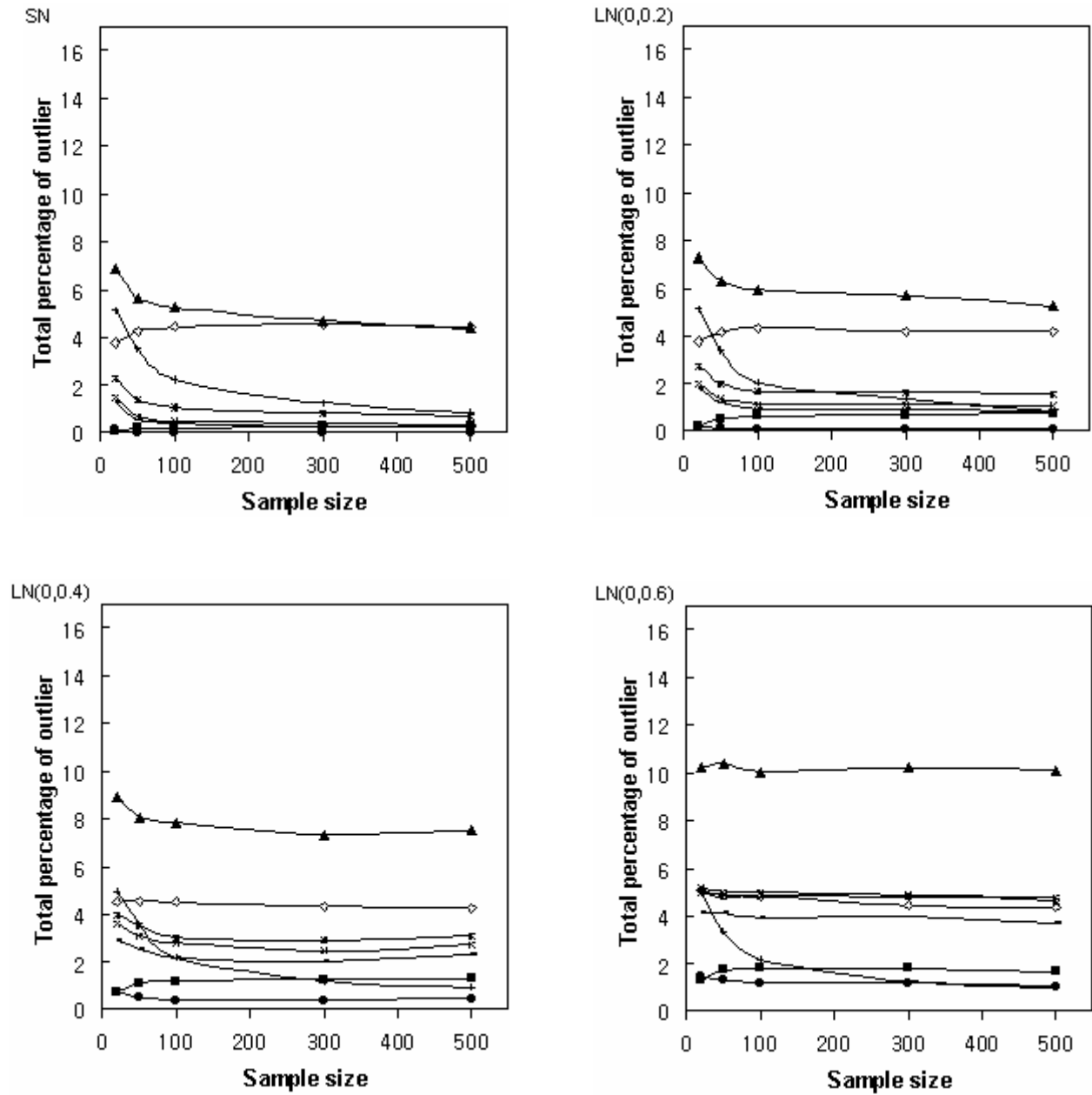


Figure 9: Change in the Total Percentages of Outliers According to the Sample Size



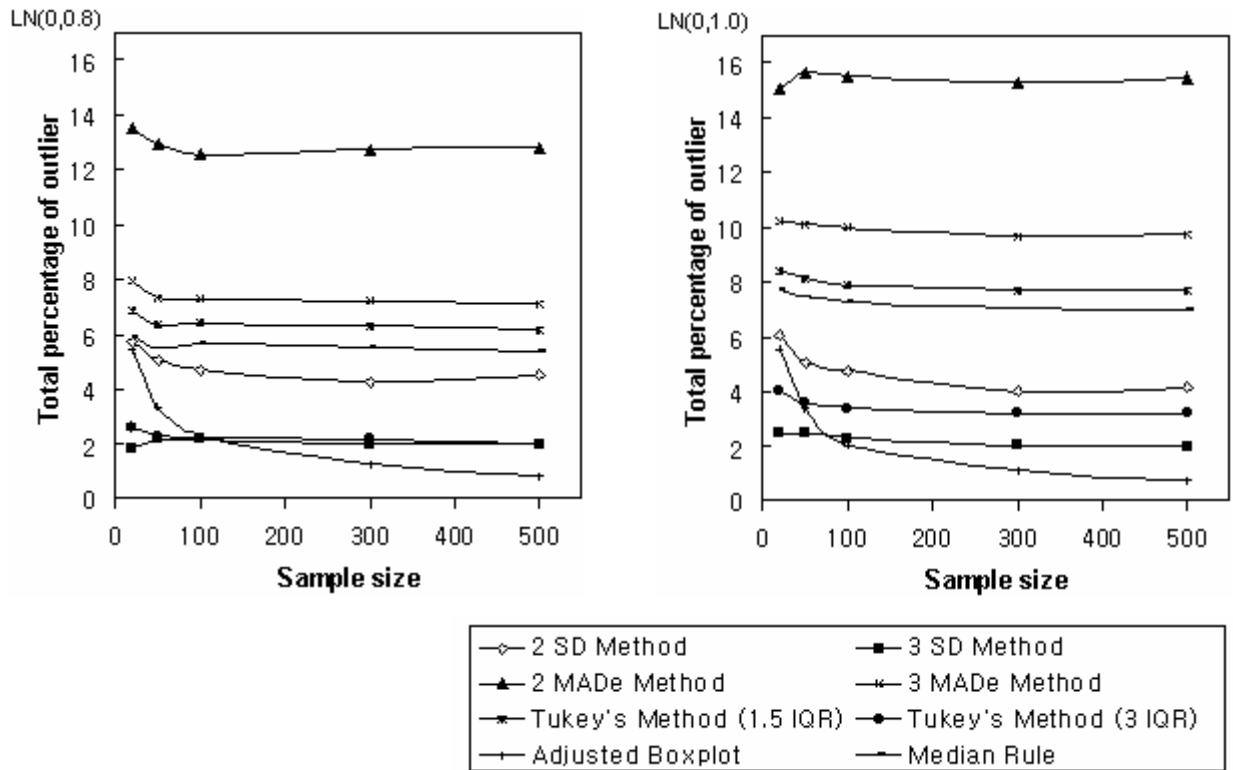


Figure 9 (continued)

**Table 5: The Average Percentage of Left Outliers, Right Outliers and the Average Total Percent of Outliers for the Lognormal Distributions with the Same Mean and Different Variances (mean=0, variance=0.2<sup>2</sup>, 0.4<sup>2</sup>, 0.6<sup>2</sup>, 0.8<sup>2</sup>, 1.0<sup>2</sup>) and the Standard Normal Distribution with Different Sample Sizes.**

Distribution	n	CS	MC	SD Method						MADe Method					
				Mean ± 2 SD			Mean ± 3 SD			Median ± 2 MADe			Median ± 3 MADe		
				Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)
SN	20	0.006 (0.015)	-0.004 (0.007)	1.865 (0.080)	1.87 (0.083)	3.735 (0.101)	0.03 (0.012)	0.025 (0.011)	0.055 (0.016)	3.35 (0.150)	3.53 (0.156)	6.88 (0.241)	0.685 (0.066)	0.77 (0.073)	1.455 (0.109)
	50	-0.017 (0.010)	-0.009 (0.005)	2.176 (0.053)	2.09 (0.052)	4.266 (0.063)	0.086 (0.013)	0.076 (0.012)	0.162 (0.017)	2.948 (0.095)	2.676 (0.088)	5.624 (0.141)	0.366 (0.032)	0.296 (0.027)	0.662 (0.045)
	100	-0.006 (0.013)	0.001 (0.006)	2.26 (0.066)	2.19 (0.060)	4.45 (0.079)	0.093 (0.017)	0.113 (0.020)	0.207 (0.026)	2.637 (0.115)	2.573 (0.109)	5.21 (0.184)	0.233 (0.032)	0.253 (0.036)	0.487 (0.055)
	300	0.006 (0.015)	0.0004 (0.006)	2.267 (0.073)	2.307 (0.060)	4.573 (0.086)	0.117 (0.019)	0.14 (0.021)	0.257 (0.026)	2.347 (0.121)	2.31 (0.099)	4.657 (0.173)	0.167 (0.029)	0.18 (0.028)	0.347 (0.042)
	500	-0.008 (0.010)	0.004 (0.005)	2.266 (0.051)	2.17 (0.047)	4.436 (0.059)	0.13 (0.016)	0.138 (0.017)	0.268 (0.025)	2.2 (0.078)	2.17 (0.080)	4.37 (0.133)	0.146 (0.019)	0.148 (0.019)	0.294 (0.029)
LN (0, 0.2)	20	0.436 (0.016)	0.084 (0.007)	0.555 (0.050)	3.195 (0.092)	3.75 (0.095)	0 (0)	0.2 (0.031)	0.2 (0.031)	1.615 (0.110)	5.67 (0.183)	7.285 (0.227)	0.195 (0.034)	1.765 (0.108)	1.96 (0.119)
	50	0.527 (0.012)	0.086 (0.005)	0.71 (0.037)	3.434 (0.055)	4.144 (0.059)	0 (0)	0.508 (0.028)	0.508 (0.028)	1.16 (0.060)	5.092 (0.114)	6.252 (0.141)	0.03 (0.008)	1.334 (0.057)	1.364 (0.059)
	100	0.574 (0.018)	0.079 (0.006)	0.723 (0.052)	3.57 (0.073)	4.293 (0.076)	0.003 (0.003)	0.623 (0.038)	0.627 (0.038)	0.93 (0.077)	4.96 (0.139)	5.89 (0.168)	0.017 (0.009)	1.113 (0.065)	1.13 (0.067)
	300	0.604 (0.020)	0.093 (0.006)	0.676 (0.044)	3.49 (0.071)	4.167 (0.081)	0 (0)	0.657 (0.035)	0.657 (0.035)	0.737 (0.060)	4.947 (0.160)	5.683 (0.185)	0 (0)	1.09 (0.068)	1.09 (0.068)
	500	0.609 (0.015)	0.094 (0.004)	0.594 (0.035)	3.602 (0.064)	4.196 (0.065)	0 (0)	0.71 (0.029)	0.71 (0.029)	0.524 (0.040)	4.73 (0.116)	5.254 (0.132)	0 (0)	1.024 (0.051)	1.024 (0.051)
LN (0, 0.4)	20	0.864 (0.020)	0.161 (0.007)	0.095 (0.022)	4.385 (0.090)	4.48 (0.090)	0 (0)	0.715 (0.055)	0.715 (0.055)	0.795 (0.091)	8.15 (0.197)	8.945 (0.225)	0.07 (0.030)	3.51 (0.141)	3.58 (0.144)
	50	1.062 (0.017)	0.170 (0.005)	0.04 (0.009)	4.522 (0.055)	4.562 (0.054)	0 (0)	1.132 (0.037)	1.132 (0.037)	0.234 (0.025)	7.816 (0.127)	8.05 (0.133)	0.002 (0.002)	3.068 (0.084)	3.07 (0.085)
	100	1.143 (0.027)	0.181 (0.007)	0.02 (0.008)	4.46 (0.073)	4.48 (0.073)	0 (0)	1.157 (0.044)	1.157 (0.044)	0.107 (0.023)	7.743 (0.168)	7.85 (0.173)	0 (0)	2.763 (0.110)	2.763 (0.110)
	300	1.251 (0.033)	0.167 (0.006)	0.007 (0.005)	4.297 (0.065)	4.303 (0.066)	0 (0)	1.247 (0.046)	1.247 (0.046)	0.033 (0.014)	7.3 (0.158)	7.333 (0.163)	0 (0)	2.467 (0.094)	2.467 (0.094)
	500	1.303 (0.025)	0.170 (0.005)	0.002 (0.002)	4.244 (0.056)	4.246 (0.057)	0 (0)	1.296 (0.032)	1.296 (0.032)	0.014 (0.005)	7.518 (0.149)	7.532 (0.151)	0 (0)	2.684 (0.074)	2.684 (0.074)
LN (0, 0.6)	20	1.212 (0.024)	0.219 (0.007)	0 (0)	5.035 (0.084)	5.035 (0.084)	0 (0)	1.3 (0.069)	1.3 (0.069)	0.24 (0.042)	9.965 (0.216)	10.205 (0.224)	0.005 (0.005)	5.150 (0.164)	5.155 (0.165)
	50	1.623 (0.024)	0.250 (0.005)	0 (0)	4.868 (0.056)	4.868 (0.056)	0 (0)	1.74 (0.038)	1.74 (0.038)	0.034 (0.011)	10.39 (0.140)	10.424 (0.140)	0 (0)	5.008 (0.105)	5.008 (0.105)
	100	1.774 (0.039)	0.251 (0.006)	0 (0)	4.793 (0.074)	4.793 (0.074)	0 (0)	1.777 (0.050)	1.777 (0.050)	0.01 (0.01)	10.047 (0.170)	10.057 (0.171)	0 (0)	4.963 (0.133)	4.963 (0.133)
	300	2.120 (0.063)	0.254 (0.007)	0 (0)	4.413 (0.086)	4.413 (0.086)	0 (0)	1.817 (0.051)	1.817 (0.051)	0 (0)	10.23 (0.178)	10.23 (0.178)	0 (0)	4.877 (0.146)	4.877 (0.146)
	500	2.199 (0.064)	0.255 (0.005)	0 (0)	4.368 (0.068)	4.368 (0.068)	0 (0)	1.68 (0.047)	1.68 (0.047)	0 (0)	10.124 (0.145)	10.124 (0.145)	0 (0)	4.724 (0.106)	4.724 (0.106)

**Table 5 (continued)**

Distribution	n	CS	MC	SD Method						MADe Method					
				Mean $\pm$ 2 SD			Mean $\pm$ 3 SD			Median $\pm$ 2 MADe			Median $\pm$ 3 MADe		
				Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)
LN (0, 0.8)	20	1.560 (0.024)	0.31 (0.007)	0.005 (0.005)	5.71 (0.081)	5.715 (0.081)	0 (0)	1.86 (0.076)	1.86 (0.076)	0.095 (0.031)	13.4 (0.218)	13.495 (0.224)	0.005 (0.005)	7.915 (0.191)	7.92 (0.191)
	50	2.126 (0.030)	0.315 (0.005)	0 (0)	5.05 (0.058)	5.05 (0.058)	0 (0)	2.132 (0.038)	2.132 (0.038)	0.006 (0.004)	12.932 (0.143)	12.938 (0.144)	0 (0)	7.33 (0.120)	7.33 (0.120)
	100	2.548 (0.058)	0.314 (0.007)	0 (0)	4.693 (0.084)	4.693 (0.084)	0 (0)	2.177 (0.049)	2.177 (0.049)	0 (0)	12.577 (0.183)	12.577 (0.183)	0 (0)	7.283 (0.151)	7.283 (0.151)
	300	3.179 (0.152)	0.327 (0.007)	0 (0)	4.25 (0.103)	4.25 (0.103)	0 (0)	1.967 (0.053)	1.967 (0.053)	0 (0)	12.75 (0.193)	12.75 (0.193)	0 (0)	7.217 (0.160)	7.217 (0.160)
	500	2.928 (0.096)	0.324 (0.005)	0 (0)	4.48 (0.076)	4.48 (0.076)	0 (0)	1.942 (0.041)	1.942 (0.041)	0 (0)	12.84 (0.132)	12.84 (0.132)	0 (0)	7.09 (0.120)	7.09 (0.120)
LN (0, 1.0)	20	1.876 (0.026)	0.353 (0.007)	0 (0)	6.03 (0.078)	6.03 (0.078)	0 (0)	2.455 (0.079)	2.455 (0.079)	0.05 (0.025)	15.065 (0.217)	15.115 (0.218)	0.02 (0.016)	10.175 (0.195)	10.195 (0.195)
	50	2.626 (0.034)	0.384 (0.005)	0 (0)	5.048 (0.060)	5.048 (0.060)	0 (0)	2.486 (0.037)	2.486 (0.037)	0 (0)	15.65 (0.152)	15.65 (0.152)	0 (0)	10.124 (0.133)	10.124 (0.133)
	100	3.086 (0.079)	0.400 (0.006)	0 (0)	4.76 (0.096)	4.76 (0.096)	0 (0)	2.273 (0.051)	2.273 (0.051)	0 (0)	15.527 (0.190)	15.527 (0.190)	0 (0)	9.963 (0.165)	9.963 (0.165)
	300	4.113 (0.183)	0.399 (0.007)	0 (0)	4.027 (0.108)	4.027 (0.108)	0 (0)	2.013 (0.060)	2.013 (0.060)	0 (0)	15.26 (0.211)	15.26 (0.211)	0 (0)	9.67 (0.187)	9.67 (0.187)
	500	4.052 (0.155)	0.394 (0.005)	0 (0)	4.15 (0.085)	4.15 (0.085)	0 (0)	1.992 (0.046)	1.992 (0.046)	0 (0)	15.43 (0.137)	15.43 (0.137)	0 (0)	9.732 (0.126)	9.732 (0.126)

Distribution	n	CS	MC	Tukey's Method						Adjusted Boxplot			Median Rule		
				Q1-1.5 IQR / Q3+1.5 IQR			Q1-3 IQR / Q3+3 IQR			Q1-1.5exp(-3.5mc) / Q3+1.5exp(4mc)			Q2 $\pm$ 2.3 IQR		
				Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)
SN	20	0.006 (0.015)	-0.004 (0.007)	1.1 (0.083)	1.17 (0.089)	2.27 (0.137)	0.065 (0.019)	0.04 (0.014)	0.105 (0.026)	2.39 (0.135)	2.75 (0.153)	5.14 (0.178)	0.615 (0.061)	0.685 (0.067)	1.3 (0.102)
	50	-0.017 (0.010)	-0.009 (0.005)	0.704 (0.043)	0.66 (0.041)	1.364 (0.065)	0.006 (0.003)	0.002 (0.002)	0.008 (0.004)	1.462 (0.090)	2.07 (0.110)	3.532 (0.121)	0.292 (0.028)	0.236 (0.024)	0.528 (0.039)
	100	-0.006 (0.013)	0.001 (0.006)	0.537 (0.049)	0.51 (0.049)	1.047 (0.078)	0.003 (0.003)	0.003 (0.003)	0.007 (0.005)	1.05 (0.101)	1.18 (0.107)	2.23 (0.125)	0.18 (0.026)	0.183 (0.028)	0.363 (0.042)
	300	0.006 (0.015)	0.0004 (0.006)	0.42 (0.045)	0.363 (0.038)	0.783 (0.061)	0 (0)	0 (0)	0 (0)	0.59 (0.077)	0.647 (0.093)	1.237 (0.098)	0.127 (0.025)	0.13 (0.024)	0.257 (0.035)
	500	-0.008 (0.010)	0.004 (0.005)	0.342 (0.029)	0.354 (0.030)	0.696 (0.047)	0 (0)	0.002 (0.002)	0.002 (0.002)	0.564 (0.071)	0.468 (0.057)	0.892 (1.172)	0.112 (0.016)	0.11 (0.016)	0.222 (0.023)

**Table 5 (continued)**

Distribution	n	CS	MC	Tukey's Method						Adjusted Boxplot			Median Rule		
				Q1-1.5 IQR / Q3+1.5 IQR			Q1-3 IQR / Q3+3 IQR			Q1-1.5exp(-3.5mc)/ Q3+1.5exp(4mc)			Q2 ±2.3 IQR		
				Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)
LN (0, 0.2)	20	0.436 (0.016)	0.084 (0.007)	0.415 (0.052)	2.29 (0.113)	2.705 (0.137)	0 (0)	0.21 (0.033)	0.21 (0.033)	2.725 (0.146)	2.395 (0.143)	5.12 (0.177)	0.19 (0.035)	1.575 (0.098)	1.765 (0.111)
	50	0.527 (0.012)	0.086 (0.005)	0.146 (0.020)	1.806 (0.067)	1.952 (0.075)	0 (0)	0.108 (0.015)	0.108 (0.015)	1.864 (0.103)	1.548 (0.091)	3.412 (0.118)	0.028 (0.008)	1.114 (0.052)	1.142 (0.054)
	100	0.574 (0.018)	0.079 (0.006)	0.063 (0.012)	1.6 (0.076)	1.663 (0.078)	0 (0)	0.063 (0.015)	0.063 (0.015)	0.95 (0.101)	1.1 (0.103)	2.05 (0.125)	0.003 (0.003)	0.913 (0.055)	0.917 (0.055)
	300	0.604 (0.020)	0.093 (0.006)	0.02 (0.009)	1.587 (0.086)	1.607 (0.086)	0 (0)	0.077 (0.016)	0.077 (0.016)	0.82 (0.103)	0.543 (0.068)	1.363 (0.097)	0 (0)	0.94 (0.064)	0.94 (0.064)
	500	0.609 (0.015)	0.094 (0.004)	0.012 (0.006)	1.512 (0.060)	1.524 (0.060)	0 (0)	0.036 (0.009)	0.036 (0.009)	0.472 (0.070)	0.356 (0.039)	0.828 (0.066)	0 (0)	0.838 (0.044)	0.838 (0.044)
LN (0, 0.4)	20	0.864 (0.020)	0.161 (0.007)	0.145 (0.033)	3.805 (0.131)	3.95 (0.139)	0 (0)	0.755 (0.063)	0.755 (0.063)	2.785 (0.153)	2.16 (0.130)	4.945 (0.175)	0.025 (0.011)	2.87 (0.120)	2.895 (0.121)
	50	1.062 (0.017)	0.170 (0.005)	0.01 (0.004)	3.496 (0.088)	3.506 (0.088)	0 (0)	0.506 (0.034)	0.506 (0.034)	2.038 (0.110)	1.504 (0.087)	3.542 (0.119)	0 (0)	2.538 (0.076)	2.538 (0.076)
	100	1.143 (0.027)	0.181 (0.007)	0 (0)	3.03 (0.105)	3.03 (0.105)	0 (0)	0.373 (0.038)	0.373 (0.038)	1.437 (0.143)	0.717 (0.071)	2.153 (0.143)	0 (0)	2.143 (0.092)	2.143 (0.092)
	300	1.251 (0.033)	0.167 (0.006)	0 (0)	2.903 (0.095)	2.903 (0.095)	0 (0)	0.363 (0.040)	0.363 (0.040)	0.587 (0.098)	0.553 (0.061)	1.14 (0.098)	0 (0)	2.047 (0.083)	2.047 (0.083)
	500	1.303 (0.025)	0.170 (0.005)	0 (0)	3.078 (0.077)	3.078 (0.077)	0 (0)	0.41 (0.030)	0.41 (0.030)	0.402 (0.059)	0.514 (0.048)	0.916 (0.065)	0 (0)	2.254 (0.067)	2.254 (0.067)
LN (0, 0.6)	20	1.212 (0.024)	0.219 (0.007)	0.01 (0.007)	5.005 (0.151)	5.015 (0.152)	0 (0)	1.48 (0.086)	1.48 (0.086)	3.075 (0.169)	1.94 (0.117)	5.015 (0.181)	0 (0)	4.145 (0.139)	4.145 (0.139)
	50	1.623 (0.024)	0.250 (0.005)	0 (0)	4.82 (0.095)	4.82 (0.095)	0 (0)	1.27 (0.050)	1.27 (0.050)	2.178 (0.123)	1.154 (0.069)	3.332 (0.125)	0 (0)	4.098 (0.087)	4.098 (0.087)
	100	1.774 (0.039)	0.251 (0.006)	0 (0)	4.873 (0.128)	4.873 (0.128)	0 (0)	1.15 (0.069)	1.15 (0.069)	1.397 (0.134)	0.767 (0.077)	2.163 (0.136)	0 (0)	3.897 (0.119)	3.897 (0.119)
	300	2.120 (0.063)	0.254 (0.007)	0 (0)	4.81 (0.132)	4.81 (0.132)	0 (0)	1.193 (0.061)	1.193 (0.061)	0.633 (0.128)	0.593 (0.066)	1.227 (0.132)	0 (0)	3.97 (0.119)	3.97 (0.119)
	500	2.199 (0.064)	0.255 (0.005)	0 (0)	4.59 (0.093)	4.59 (0.093)	0 (0)	1.07 (0.048)	1.07 (0.048)	0.52 (0.074)	0.496 (0.058)	1.016 (0.078)	0 (0)	3.702 (0.082)	3.702 (0.082)
LN (0, 0.8)	20	1.560 (0.024)	0.31 (0.007)	0.01 (0.01)	6.815 (0.162)	6.825 (0.163)	0 (0)	2.595 (0.113)	2.595 (0.113)	3.46 (0.177)	1.955 (0.122)	5.415 (0.191)	0 (0)	5.91 (0.157)	5.91 (0.157)
	50	2.126 (0.030)	0.315 (0.005)	0 (0)	6.366 (0.102)	6.366 (0.102)	0 (0)	2.28 (0.068)	2.28 (0.068)	2.134 (0.122)	1.218 (0.074)	3.352 (0.127)	0 (0)	5.484 (0.098)	5.484 (0.098)
	100	2.548 (0.058)	0.314 (0.007)	0 (0)	6.397 (0.131)	6.397 (0.131)	0 (0)	2.227 (0.084)	2.227 (0.084)	1.28 (0.147)	0.99 (0.093)	2.27 (0.158)	0 (0)	5.65 (0.128)	5.65 (0.128)
	300	3.179 (0.152)	0.327 (0.007)	0 (0)	6.267 (0.137)	6.267 (0.137)	0 (0)	2.153 (0.075)	2.153 (0.075)	0.59 (0.125)	0.64 (0.062)	1.23 (0.119)	0 (0)	5.553 (0.134)	5.553 (0.134)
	500	2.928 (0.096)	0.324 (0.005)	0 (0)	6.166 (0.113)	6.166 (0.113)	0 (0)	1.974 (0.068)	1.974 (0.068)	0.242 (0.056)	0.532 (0.052)	0.774 (0.063)	0 (0)	5.388 (0.103)	5.388 (0.103)

**Table 5 (continued)**

Distribution	n	CS	MC	Tukey's Method						Adjusted Boxplot			Median Rule		
				Q1-1.5 IQR / Q3+1.5 IQR			Q1-3 IQR / Q3+3 IQR			Q1-1.5exp(-3.5mc)/ Q3+1.5exp(4mc)			Q2 ±2.3 IQR		
				Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)	Left (%)	Right (%)	Total (%)
LN (0, 1.0)	20	1.876 (0.026)	0.353 (0.007)	0 (0)	8.37 (0.166)	8.37 (0.166)	0 (0)	4.005 (0.133)	4.005 (0.133)	3.185 (0.179)	2.385 (0.134)	5.57 (0.197)	0 (0)	7.695 (0.163)	7.695 (0.163)
	50	2.626 (0.034)	0.384 (0.005)	0 (0)	8.126 (0.110)	8.126 (0.110)	0 (0)	3.596 (0.083)	3.596 (0.083)	1.968 (0.121)	1.412 (0.076)	3.38 (0.127)	0 (0)	7.464 (0.107)	7.464 (0.107)
	100	3.086 (0.079)	0.400 (0.006)	0 (0)	7.887 (0.144)	7.887 (0.144)	0 (0)	3.357 (0.120)	3.357 (0.120)	1.2 (0.135)	0.847 (0.074)	2.047 (0.138)	0 (0)	7.263 (0.143)	7.263 (0.143)
	300	4.113 (0.183)	0.399 (0.007)	0 (0)	7.723 (0.158)	7.723 (0.158)	0 (0)	3.23 (0.101)	3.23 (0.101)	0.423 (0.114)	0.687 (0.062)	1.11 (0.116)	0 (0)	7.04 (0.148)	7.04 (0.148)
	500	4.052 (0.155)	0.394 (0.005)	0 (0)	7.682 (0.120)	7.682 (0.120)	0 (0)	3.186 (0.075)	3.186 (0.075)	0.134 (0.042)	0.616 (0.052)	0.75 (0.058)	0 (0)	6.986 (0.116)	6.986 (0.116)

(standard error of the average percentage of outliers)

## 5.0 APPLICATION

In this chapter the five selected outlier labeling methods are applied to three real data sets and one modified data set of one of the three real data sets. These real data sets are provided by Gateway Health Plan, a managed care alternative to the Department of Public Welfare’s Medical Assistance Program in Pennsylvania. These data sets are part of Primary Care Provider (PCP)’s basic information which is needed to identify providers (PCPs) associated with Member Dissatisfaction Rates (MDRs = the number of member complaints/PCP practice size) that are unusually high compared with other PCPs of similar sized practices<sup>23</sup>. Case 1 (data set 1) is “visit per 1000 office med”, and its distribution is not very different from the normal distribution. Case 2 (data set 2) is “Scripts per 1000 Rx”, and its distribution is mildly skewed to the right. Case 3 (data set 3) is “Svcs per 1000 early child im”, and its distribution is highly skewed to the right because of one observation which has an extremely large value. Case 4 (data set 4) is the data set which is modified from the data set 3 by means of excluding the most extreme value from the data set 3 to see the possible effect of the one extreme outlier over the outlier labeling methods. Figure 10 shows the basic statistics and distribution of each data set (Case 1-Case 4).

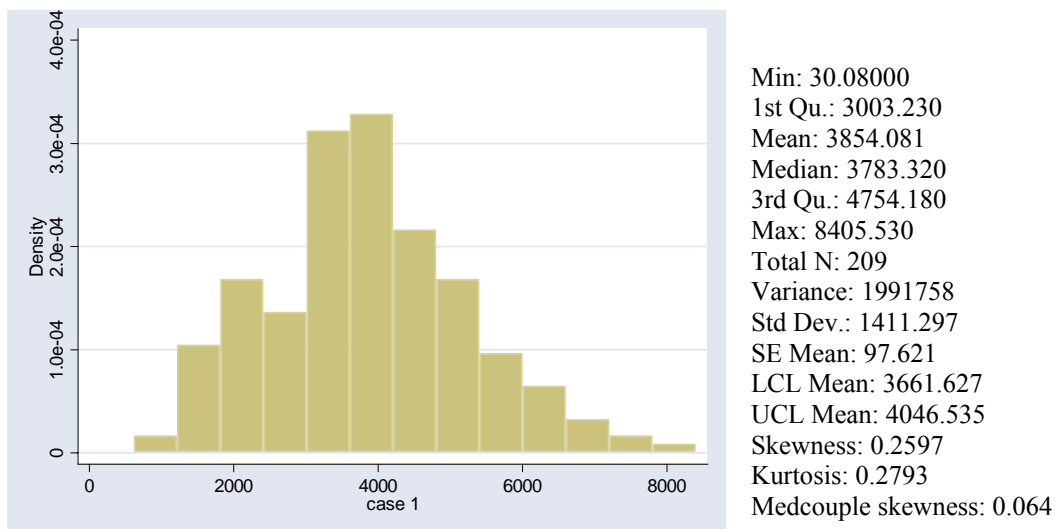
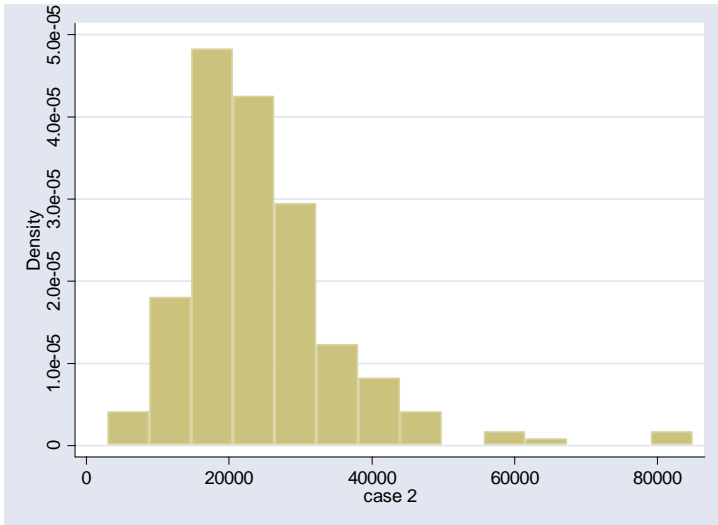
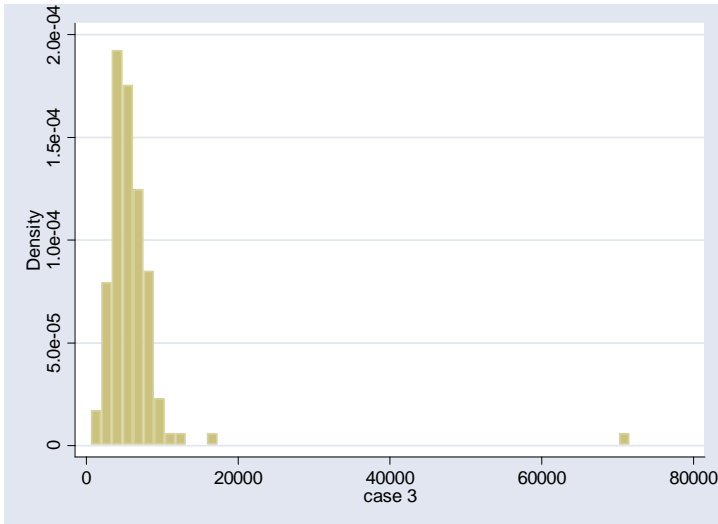


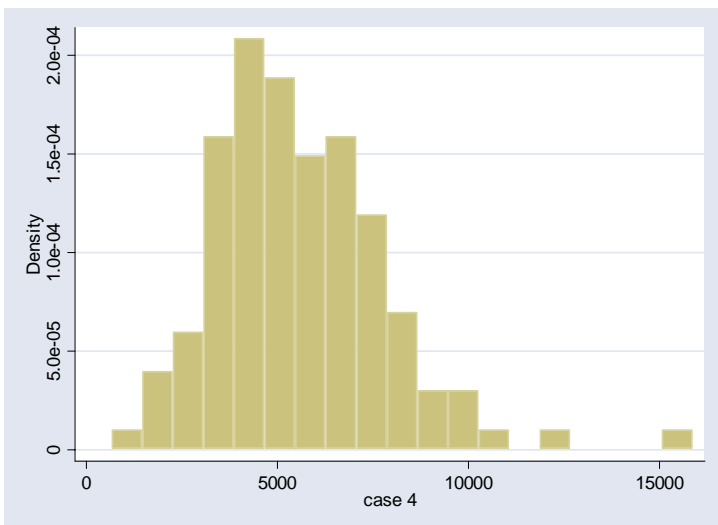
Figure 10: Histogram and Basic Statistics of Case 1-Case 4



Min: 3065.420  
 1st Qu.: 17700.58  
 Mean: 24574.59  
 Median: 22428.26  
 3rd Qu.: 29387.86  
 Max: 85018.02  
 Total N: 209.0000  
 Variance: 132580400  
 Std Dev.: 11514.35  
 SE Mean: 796.4646  
 LCL Mean: 23004.41  
 UCL Mean: 26144.77  
 Skewness: 1.9122  
 Kurtosis: 6.8409  
 Medcouple skewness: 0.187



Min: 681.82  
 1st Qu.: 4171.395  
 Mean: 6153.383  
 Median: 5395.580  
 3rd Qu.: 7016.580  
 Max: 72000  
 Total N: 127  
 Variance: 39593430  
 Std Dev.: 6292.331  
 SE Mean: 558.354  
 LCL Mean: 5048.417  
 UCL Mean: 7258.349  
 Skewness: 9.252583  
 Kurtosis: 96.8279  
 Medcouple skewness: 0.121



Min: 681.820  
 1st Qu.: 4165.698  
 Mean: 5630.791  
 Median: 5370.060  
 3rd Qu.: 6985.942  
 Max: 16000  
 Total N: 126  
 Variance: 4948672  
 Std Dev.: 2224.561  
 SE Mean: 198.1796  
 LCL Mean: 5238.569  
 UCL Mean: 6023.013  
 Skewness: 1.008251  
 Kurtosis: 3.036602  
 Medcouple skewness: 0.119

Figure 10 (continued)

Table 6 shows the left, right, and total number of outliers identified in each data set after applying the five outlier labeling methods. Sample programs for Case 4 are given in APPENDIX E.

**Table 6: Interval, Left, Right, and Total Number of Outliers According to the Five Outlier Methods**

Case 1 (Data set 1): N=209

Method	Interval	Left (%)	Right (%)	Total (%)
<b>2 SD Method</b>	(1031.49, 6676.67)	2 (0.96)	6 (2.87)	8 (3.83)
<b>3 SD Method</b>	(-379.809, 8087.97)	0 (0)	1 (0.48)	1 (0.48)
<b>Tukey's Method (1.5 IQR)</b>	(376.81, 7380.61)	1 (0.48)	2 (0.96)	3 (1.44)
<b>Tukey's Method (3 IQR)</b>	(-2249.62, 10007.03)	0 (0)	0 (0)	0 (0)
<b>Adjusted Boxplot</b>	(905.41, 8149.69)	1 (0.48)	1 (0.48)	2 (0.96)
<b>2 MADe Method</b>	(1310.52, 6256.12)	4 (1.91)	11 (5.26)	15 (7.18)
<b>3 MADe Method</b>	(74.12, 7492.52)	1 (0.48)	2 (0.96)	3 (1.44)
<b>Median Rule</b>	(-243.87, 7810.51)	0 (0)	1 (0.48)	1 (0.48)

Case 2 (Data set 2): N=209

Method	Interval	Left (%)	Right (%)	Total (%)
<b>2 SD Method</b>	(1545.88, 47603.30)	0 (0)	8 (3.83)	8 (3.83)
<b>3 SD Method</b>	(-9968.47, 59117.65)	0 (0)	4 (1.91)	4 (1.91)
<b>Tukey's Method (1.5 IQR)</b>	(169.66, 46918.78)	0 (0)	8 (3.83)	8 (3.83)
<b>Tukey's Method (3 IQR)</b>	(-17361.26, 64449.70)	0 (0)	3 (1.44)	3 (1.44)
<b>Adjusted Boxplot</b>	(8580.85, 66385.47)	5 (2.39)	2 (0.96)	7 (3.35)
<b>2 MADe Method</b>	(5361.49, 39495.03)	4 (1.91)	20 (9.57)	24 (11.48)
<b>3 MADe Method</b>	(-3171.90, 48028.42)	0 (0)	6 (2.87)	6 (2.87)
<b>Median Rule</b>	(-4452.48, 49309.00)	0 (0)	5 (2.39)	5 (2.39)

Case 3 (Data set 3): N=127

Method	Interval	Left (%)	Right (%)	Total (%)
<b>2 SD Method</b>	(-6431.28, 18738.04)	0 (0)	1 (0.79)	1 (0.79)
<b>3 SD Method</b>	(-12723.61, 25030.38)	0 (0)	1 (0.79)	1 (0.79)
<b>Tukey's Method (1.5 IQR)</b>	(-96.38, 11284.36)	0 (0)	3 (2.36)	3 (2.36)
<b>Tukey's Method (3 IQR)</b>	(-4364.16, 15552.13)	0 (0)	2 (1.57)	2 (1.57)
<b>Adjusted Boxplot</b>	(1373.94, 13932.44)	1 (0.79)	2 (1.57)	3 (2.36)
<b>2 MADe Method</b>	(1142.27, 9648.89)	1 (0.79)	6 (4.72)	7 (5.51)
<b>3 MADe Method</b>	(-984.39, 11775.55)	0 (0)	3 (2.36)	3 (2.36)
<b>Median Rule</b>	(-1148.35, 11939.51)	0 (0)	3 (2.36)	3 (2.36)



**Table 6 (continued)**

Case 4 (Data set 4): N=126

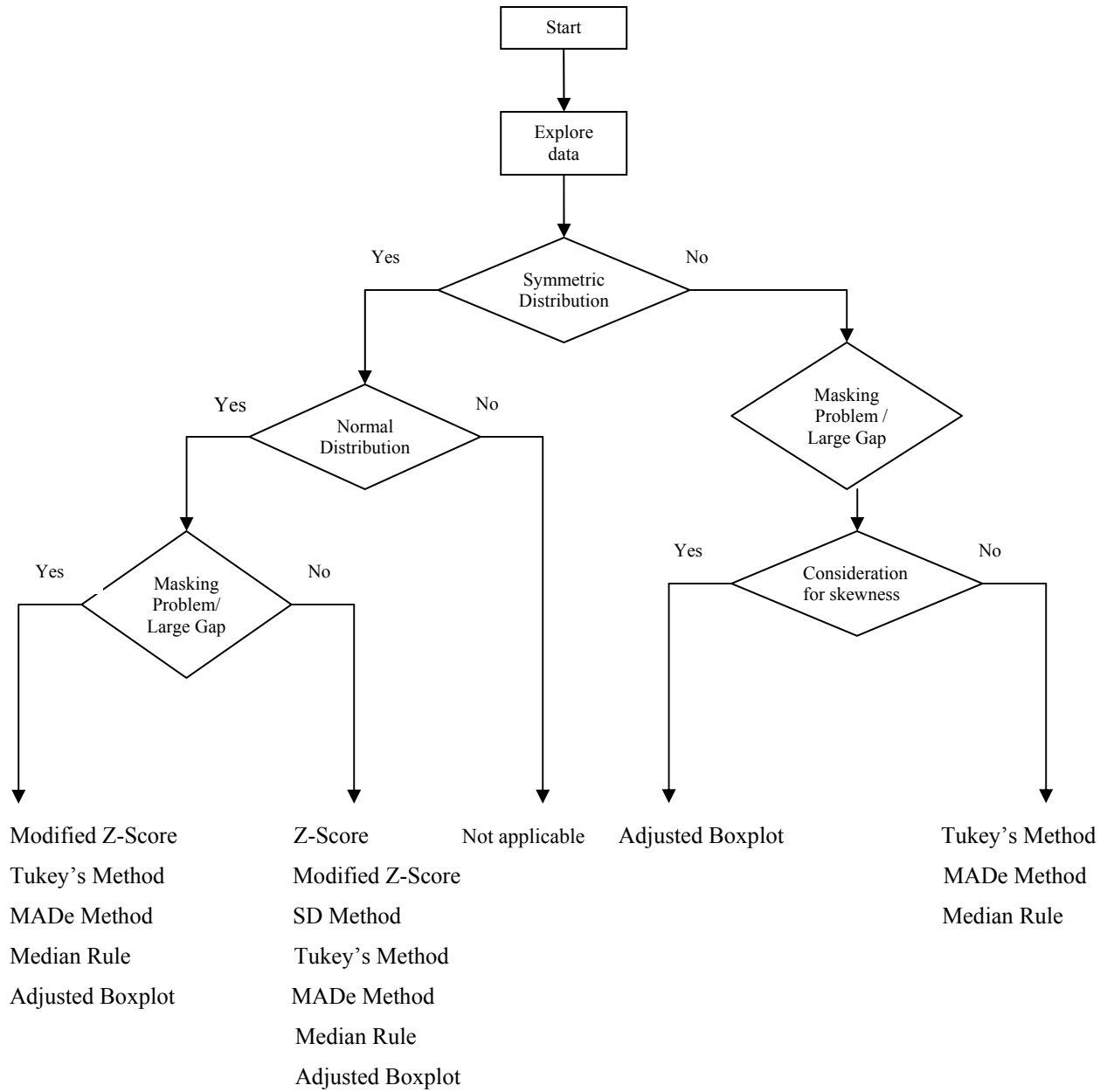
<b>Method</b>	<b>Interval</b>	<b>Left (%)</b>	<b>Right (%)</b>	<b>Total (%)</b>
<b>2 SD Method</b>	(1181.67, 10079.91)	1 (0.79)	4 (3.17)	5 (3.97)
<b>3 SD Method</b>	(-1042.89, 12304.47)	0 (0)	1 (0.79)	1 (0.79)
<b>Tukey's Method (1.5 IQR)</b>	(-64.67, 11216.31)	0 (0)	2 (1.59)	2 (1.59)
<b>Tukey's Method (3 IQR)</b>	(-4295.04, 15446.68)	0 (0)	1 (0.79)	1 (0.79)
<b>Adjusted Boxplot</b>	(1375.92, 13793.89)	1 (0.79)	1 (0.79)	2 (1.59)
<b>2 MADe Method</b>	(1139.14, 9600.99)	1 (0.79)	5 (3.97)	6 (4.76)
<b>3 MADe Method</b>	(-976.33, 11716.45)	0 (0)	2 (1.59)	2 (1.59)
<b>Median Rule</b>	(-1116.50, 11856.62)	0 (0)	2 (1.59)	2 (1.59)

Overall, the results of the applications show similar patterns to those in the simulation study. First, when data are skewed, the difference of the average percentage of outliers between the 2 SD method and the 2 MADe method increases. Second, the 2 MADe method classifies more observations as outliers than any other method does. Third, in the mildly right skewed data set, Case 2, in which the adjusted boxplot is utilized, the number of the left outliers is larger than that of the right outliers. Finally, the interval of the Median rule is between Tukey's method with 1.5 IQR and Tukey's method with 3 IQR.

As was shown in the results of Case 3 and Case 4, such methods with robust measures as the MADe method, Tukey's method, the Median rule, and the Adjusted Boxplot are less affected by the extreme value than the SD method, and the interval of the SD method becomes much narrower after the single extreme value is excluded from data set 3 than other methods. With regard to the 2 SD method, while one observation is found in Case 3, five observations are detected as outliers in Case 4. That is, when there is a large gap between extreme values and the rest of values as shown in the data set 3, such outlier labeling methods with mean and standard deviation as the SD method and Z-Score may not detect the possible outliers which other methods could detect. In the case of the two skewness measures, i.e., classical and medcouple skewness, classical skewness, unlike medcouple skewness, is highly affected by even a few extreme values. The classical skewness in data set 3 was 9.25, but it decreased to 1.008 in data set 4 with the most extreme value—which was included in the data set 3—excluded, whereas the medcouple skewness decreased only a little.

## 6.0 RECOMMENDATIONS

Figure 11 shows a decision making flowchart at to which outlier labeling method can be used in different data situations. First, it is necessary to understand the data characteristics (explore data step). When a data set consists of such subgroups as sex and income, it may be necessary to check if its research variables have different characteristics according to the subgroups. For example, in the case of detecting outliers in the adult height variable, it may be necessary to adjust for sex since the distribution of height can vary by sex. In such a case, an appropriate approach may be to stratify by sex. All the labeling methods in this paper can be applicable if a data set has a normal distribution without a possible masking problem or large gap between the majority of the data and extreme values. If the data set has a normal distribution with a possible masking problem or large gap between the majority of the data and extreme values, the Z-score and SD method may be inappropriate to use since these methods are highly sensitive to extreme values. The methods for the data whose distribution is symmetric but not normal, e.g., a bimodal distribution, are beyond the purview of this study. Tukey's method, the MADe method, the Median rule, and the Adjusted Boxplot may be appropriate when a data set is skewed, such as in a lognormal distribution; however, among these four methods, the Adjusted Boxplot especially takes into account the skewness of the data<sup>32</sup>.



**Figure 11: Flowchart of Outlier Labeling Methods**

## 7.0 DISCUSSION AND CONCLUSIONS

As shown in the simulation study, each method has different measures to detect outliers and shows different behaviors according to the skewness and sample size of the data. The SD methods use less robust measures, such as the mean and standard deviation, which are highly affected by extreme values. Thus, their intervals have a tendency to be inflated as the data increases in skewness, and consequently the average percentages of outliers change less than other types of methods such as the MADe, Tukey's method and the Median rule. Three methods such as the MADe, Tukey's method and the Median rule show similar patterns in skewed data since they employ robust measures to build their intervals. The total average percentages of outliers for these methods increase when data are skewed. Although the basic idea of the adjusted boxplot is similar to Tukey's method, it is different in that the adjusted boxplot has skewness measure to take into consideration. Thus, the total average percentage of outliers for the adjusted boxplot seldom changes, even decreases very slightly, when data are skewed. In addition, the range of the percentages declines more rapidly than other methods as the sample size increases. The total average percentage of outliers for the method, consequently, becomes smaller than other methods as data becomes skewed and the sample size gets large.

The simulation results reported in Table 5 may not be an exact index of the outlier percentage for each method according to the skewness and the sample size of the data as the real-world data may not follow the same distributions employed in the simulation study as was shown in Chapter 5. However, understanding the general features which the methods show would be helpful in choosing the outlier labeling methods in normal or skewed data.

There can be a gap between the majority and a small fraction of the data in a skewed data set. In general, when the observations located in the small fraction apart from the majority of the data are considered target outliers, the likelihood of defining them as outliers can increase as the distance of the gap increases. However, if the gap is not large enough, detecting outliers may

have different results depending on the methods. In such a case, it may be hard to generalize how large the gap in each method should be in order to identify the observations in the small fraction as outliers since data are diversely distributed. Another method to detect outliers is the formal test based on specific distribution assumptions. This test defines the target outliers first, and then examines whether or not the outliers are true. Some formal tests may define all of the observations in the small fraction as outliers, whereas others may define only some of the last observations in the tail of data distribution as outliers. Selection of formal tests mainly depends on the number and the type of target outliers and the type of data distribution.<sup>1</sup> In the future, formal tests in various distributions will be reviewed, compared, and discussed.

## APPENDIX A

### THE EXPECTATION, STANDARD DEVIATION AND SKEWNESS OF A LOGNORMAL DISTRIBUTION

Let  $X$  denote a random variable having a lognormal distribution, and then its natural logarithm,  $Y = \log(X)$ , has a normal distribution. Aitchison and Brown (1957) note that when  $Y$  has mean value  $E(Y) = \mu$ , and variance  $Var(Y) = \sigma^2$ , the expected value and standard deviation of the original variable  $X$  are as follows:

$$E(X) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$
$$STDEV(X) = \sqrt{\exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2)}$$

It is usually denoted by  $X \sim LOGN(\mu, \sigma^2)$ , i.e.,  $X \sim LOGN(\mu, \sigma^2)$  if and only if  $Y = \log(X) \sim N(\mu, \sigma^2)$ . The skewness of  $X$  can be denoted as follows:

$$SKEW(X) = [\exp(\sigma^2) + 2]\sqrt{\exp(\sigma^2) - 1}$$

Simple example:

If  $X$  is a lognormal random variable with parameters  $\mu$  and  $\sigma$ , its natural logarithm,  $Y = \log(X)$ , follows  $N(\mu, \sigma^2)$ . When  $\mu=0$  and  $\sigma=1$  for  $Y$ , the corresponding mean, standard deviation, and skewness of  $X$  can be determined from the following:

$$E(X) = \exp\left(\mu + \frac{\sigma^2}{2}\right) = \exp(0 + 0.5) = 1.649$$

$$STDEV(X) = \sqrt{\exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2)} = \sqrt{\exp(2) - \exp(1)} = 2.161$$

$$SKEW(X) = [\exp(\sigma^2) + 2]\sqrt{\exp(\sigma^2) - 1} = [\exp(1) + 2]\sqrt{\exp(1) - 1} = 6.185$$

We may compute the theoretical cutoff value of the SD method using this information. For example, when a certain variable,  $X$ , follows  $LOGN(0,1)$ , the theoretical lower and upper cutoff value of the 2 SD method in the variable are  $1.649 \pm 2*2.161$ .

## APPENDIX B

### MAXIMUM Z SCORES

Shiffler (1988) showed that the maximum Z-Score depends on sample size  $n$ . Let  $x_1, x_2, \dots, x_{n-1}, x_n$  be an ordered random sample of size  $n$  from a population with unknown mean and variance, and let  $\bar{x}_{n-1}$  be zero. The sample variance of the sample is presented as follows:

$$\begin{aligned}
 S_n^2 &= \frac{\sum_{i=1}^n (x_i - \bar{x}_n)^2}{n-1} \\
 &= \frac{\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2 / n}{n-1} = \frac{\sum_{i=1}^{n-1} x_i^2 + x_n^2 - x_n^2 / n}{n-1} \\
 &= \frac{\sum_{i=1}^{n-1} x_i^2}{n-1} + \frac{x_n^2}{n} \\
 &= \frac{n-2}{n-1} \cdot \frac{\sum_{i=1}^{n-1} (x_i - \bar{x}_{n-1})^2}{n-2} + \frac{x_n^2}{n} \\
 &= \frac{n-2}{n-1} \cdot S_{n-1}^2 + \frac{x_n^2}{n}
 \end{aligned}$$

Now, the Z-Score of the sample is maximized when  $S_n$  is minimized. Here, when  $S_{n-1}^2$  is zero,  $S_n$  has the smallest value. That is, the maximum Z-Score can be presented as follows:

$$Z_{\max} = \frac{(x_n - \bar{x})}{S_n} = \frac{(x_n - x_n/n)}{S_n}$$



$$= \frac{(x_n - x_n/n)}{x_n/\sqrt{n}} = (n-1)/\sqrt{n}$$

It shows that no matter how large  $x_n$  is, the maximum Z-Score of the sample depends on sample size  $n$ . The smallest achievable value for the negative Z-Score is  $-(n-1)/\sqrt{n}$ <sup>28</sup>. For several samples size  $n$ , the maximum absolute Z-Score is as follows:

$N$	$Z_{\max}$
3	1.16
5	1.79
<b>10</b>	<b>2.85</b>
<b>11</b>	<b>3.02</b>
15	3.61
18	4.01

## APPENDIX C

### CLASSICAL AND MEDCOUPLE (MC) SKEWNESS

Skewness is a measure of the symmetry of data distribution. Classical skewness, using the third moment of the distribution, i.e.,  $\sum_i (x_i - \bar{x})^3$ , where any variable  $x$ , is commonly used. It is defined as

$$\text{Classical skewness} = \frac{\sum_{i=1}^N (x_i - \bar{x})^3}{(N-1)s^3},$$

where  $s$  is the sample standard deviation and  $N$  is the sample size. If the value of skewness is negative, the distribution of the data is skewed to the left, and if the value of skewness is positive, the distribution of the data is skewed to the right. Any symmetric data has a zero value of skewness.

Another type of skewness is the medcouple ( $MC$ ), a robust alternative to classical skewness<sup>10</sup>, introduced by Brys et al. (2003). When  $X_n = \{x_1, x_2, \dots, x_n\}$  is a data set, independently sampled from a continuous univariate distribution, and it is sorted such as  $x_1 \leq x_2 \leq \dots \leq x_n$ , the  $MC$  of the data is defined as follows:

$MC = \text{med } h(x_i, x_j)$ , where the kernel function  $h$  is given by:

$$h(x_i, x_j) = \frac{(x_j - \text{med}_k) - (\text{med}_k - x_i)}{x_j - x_i}, \text{ where } \text{med}_k \text{ is the median of } X_n, \text{ and } i \text{ and } j \text{ have}$$

to satisfy  $x_i \leq \text{med}_k \leq x_j$ , and  $x_i \neq x_j$ . The value of the  $MC$  ranges between -1 and 1. If  $MC=0$ , the data is symmetric. If  $MC>0$ , the data has a right skewed distribution, whereas if  $MC<0$ , the data has a left skewed distribution.<sup>32</sup> While classical skewness is highly affected by one or more

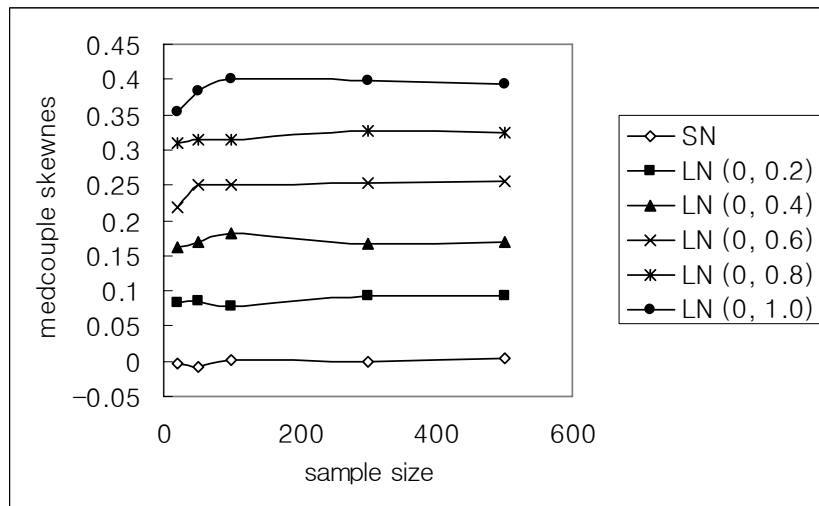
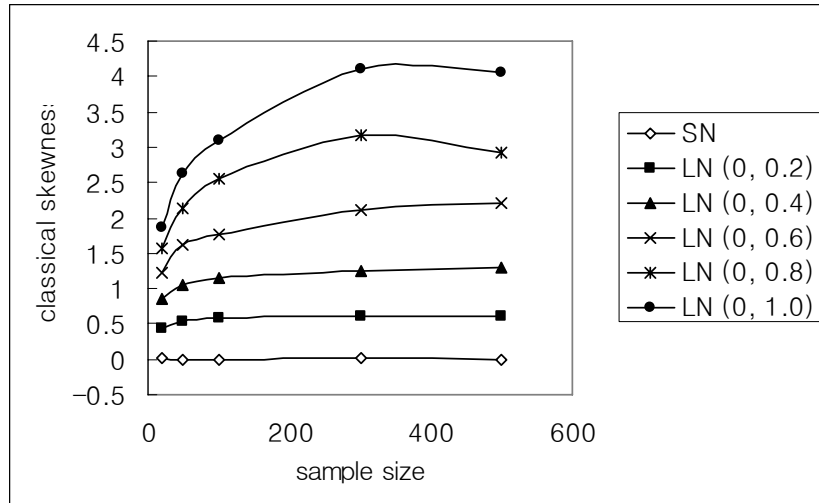
extreme values of a data set since it is based on the third moments of distribution, *MC* is robust to the extreme values.<sup>10</sup> Suppose that an example data set consists of 1, 2, 3, 4, 5, 6, 7, 10, 15, 16 and the computation of the kernel function  $h(x_i, x_j)$  for the data set is as follows:

(median = 5.5)

$x_i \backslash x_j$	6	7	10	15	16
1	-0.800	-0.500	0.000	0.357	0.4
2	-0.750	-0.400	0.125	0.462	0.500
3	-0.667	-0.250	0.286	0.583	0.615
4	-0.500	0.000	0.500	0.727	0.750
5	0.000	0.500	0.800	0.900	0.909

Thus,  $MC = \text{median } h(x_i, x_j) = 0.357$ . Several properties of the *MC* including other types of robust skewness are presented well in Brys et al. (2003, 2004).

Figure 12 shows that *MC* skewness is more robust than classical skewness as the sample size increases, especially in skewed data. The skewness is the average value for repetition in the previous simulation study. Classical skewness in skewed data increases and becomes flat while the *MC* seldom changes over different sample sizes, regardless of skewed data. This is because more extreme values are generated from skewed distributions as the sample size gets large, and classical skewness is sensitive to the extreme values.



**Figure 12: Change of the Two Types of Skewness Coefficients According to the Sample Size and Data Distribution.** (Note: This results came from the previous simulation. All the values are in Table 5 )

## APPENDIX D

### BREAKDOWN POINT

The notion of breakdown point was introduced by Hodges (1967) and Hampel (1968, 1971). It is a robustness measure of an estimator such as the mean and median or a related procedure using the estimators. The breakdown point of an estimator generally can be defined as the largest percentage of the data that can be changed into arbitrary values without distorting the estimator<sup>21</sup>. For example, if even one observation of a univariate data set is moved to infinity, the estimators of the data set such as the mean and variance go to infinity. Thus, the breakdown point of these estimators is zero. In contrast, the breakdown point of the median is approximately 50% and it varies slightly according whether the sample size  $n$  is odd or even. The exact breakdown point of the median is  $50(1-1/n)$  % and  $50(1-2/n)$  % for odd sample size  $n$  and even sample size  $n$ , respectively<sup>21</sup>. Therefore, if the breakdown point of an estimator is high, the estimator is robust.

## APPENDIX E

### PROGRAM CODE FOR OUTLIER LABELING METHODS

```
##SPLUS 2000 Professional  
##Data set of Case 4 in Chapter 5, Application, is used.
```

#### ##2SD METHOD

```
#interval  
sd2l_mean(case4)-2*stdev(case4)  
sd2l  
sd2u_mean(case4)+2*stdev(case4)  
sd2u  
#number of outliers  
sd2lrr_ifelse(case4<sd2l,1,0)  
sum(sd2lrr)  
sd2urr_ifelse(case4>sd2u,1,0)  
sum(sd2urr)
```

#### ##3SD METHOD

```
#interval  
sd3l_mean(case4)-3*stdev(case4)  
sd3l  
sd3u_mean(case4)+3*stdev(case4)  
sd3u  
#number of outliers  
sd3lrr_ifelse(case4<sd3l,1,0)  
sum(sd3lrr)  
sd3urr_ifelse(case4>sd3u,1,0)  
sum(sd3urr)
```

#### ##MADE

```
median(case4)  
made_1.4826*(median(abs(median(case4)-case4)))
```

```

##2MADE METHOD
#interval
made2l_median(case4)-2*made
made2l
made2u_median(case4)+2*made
made2u
#number of outliers
made2lrr_ifelse(case4<made2l,1,0)
sum(made2lrr)
made2urr_ifelse(case4>made2u,1,0)
sum(made2urr)

##3MADE METHOD
#interval
made3l_median(case4)-3*made
made3l
made3u_median(case4)+3*made
made3u
#number of outlier
made3lrr_ifelse(case4<made3l,1,0)
sum(made3lrr)
made3urr_ifelse(case4>made3u,1,0)
sum(made3urr)

sortf_sort(case4)
sortfi_sortf[1:63]
sortfj_sortf[64:126]
medk_median(case4)
c_matrix(0,63,63)
for (j in 1:63) {
for (i in 1:63) {
c[i,j]_((sortfj[j]-medk)-(medk-sortfi[i]))/(sortfj[j]-sortfi[i])
}}

##MC (medcouple skewness)
mc_median(c,na.rm=T)

## CLASSICAL SKEWNESS
clasicskew_mean((case4 - mean(case4))^3)/((mean((case4- mean(case4))^2))^1.5)

q1_quantile(case4,0.25)
q2_quantile(case4,0.5)
q3_quantile(case4,0.75)
iqr_q3-q1

##ADJUSTED BOXPLOT

```

```

#interval
adjl_q1-1.5*exp(-3.5*mc)*iqr
adjl
adju_q3+1.5*exp(4*mc)*iqr
adju
#number of outliers
adjlrr_ifelse(case4<adjl,1,0)
sum(adjlrr)
adjurr_ifelse(case4>adju,1,0)
sum(adjurr)

## TUKEY'S METHOD
#inner fence
tukey1.5l_q1-1.5*iqr
tukey1.5l
tukey1.5u_q3+1.5*iqr
tukey1.5u
#outer fence
tukey3l_q1-3*iqr
tukey3l
tukey3u_q3+3*iqr
tukey3u
#number of outliers (inner fence)
tukey1.5lrr_ifelse(case4<tukey1.5l,1,0)
sum(tukey1.5lrr)
tukey1.5urr_ifelse(case4>tukey1.5u,1,0)
sum(tukey1.5urr)
#number of outliers (outer fence)
tukey3lrr_ifelse(case4<tukey3l,1,0)
sum(tukey3lrr)
tukey3urr_ifelse(case4>tukey3u,1,0)
sum(tukey3urr)

##MEDIAN RULE
#interval
medianl_q2-2.3*iqr
medianl
medianu_q2+2.3*iqr
medianu
#number of outliers
medianlrr_ifelse(case4<medianl,1,0)
sum(medianlrr)
medianurr_ifelse(case4>medianu,1,0)
sum(medianurr)

```



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