


## . <br> (12) LE

NAVAL CIVIL ENGINEERING LABORATORY Port Hueneme, California

Sponsored by CHIEF OF NAVAL MATERIAL

A REVIEW OF ADDED MASS AND FLUID INERTIAL FORCES

January 1982


An Investigation Conducted by
C. E. Brennan

360 Olive Tree Lane
Sierra Made, California

N62583-81-MR-554

Approved for public release; distribution unlimited


Unclassified

background including the definition of added mass, the struclure of the added mass matrix and other effects such as the influence of viscosity, fluid compressibility and the proximity of solid and free surface boundaries. Then the existing data base from experiments and potential flow calculations is reviewed. Approximate empirical methods for bodies of complex geometry are explored in a preliminary way. The possible dramatic effects of the proximity of the ocean bottom are further highlighted. The confused state of affairs regarding the possibly major effects of viscosity in certain regimes of frequency and Reynolds number is discussed. Finally a number of recommendations stemming from ocean engineering problems are put forward.

## CONTENTS

1. INTRODUCTION ..... 1
2. GENERAL EXPLLANATION OF ADDED MASS ..... 2
3. ANALYTICAL APPROACHESTO ADDED MASS. ..... 4
3.1 EXA M PLES: RECTILINEAR M OTION OF A SPHERE AND CYLINDER WITH
POTENTIAL FLOW ..... 4
3.2 RELATION TO DISPLACED MASS: VARIATION WITH DIRECTION OF
ACCELERA TION ..... 4
3.3 THE ADDED MASS MATRIX ..... 6
3.4 ADDED NASS MATRIX SYMMETRY AND SUPERPOSBILITY OF FLOW SOLU-
TIONS ..... 8
3.5 EVALUATION OF THE ADDED MASS MATRIX ..... 10
3.6 VELOCITY AND ACCELERATION OF THE FLUID RATHER THAN THE BODY ..... 10
3.7 THE EFFECT OF A NEARLY SOLID BOUNDARY ..... 13
38 THE EFFECT OF A NEA RLY FREE SURFACE ..... 14
3.9 THE EFFECT OF FLUID COM PRESSIBLITY ..... 15
4. REVIEW OF EXISTING DATA ON ADDED MASS ..... 16
4.1 THEORETICAL POTENTIAL FLOW ADDED MASSES ..... 16
4.2 SENSITIVITY TO THE GEOM ETRY OF THE BODY ..... 17
4.3 BODIES OF COM PLEX GEOM ETRY ..... 18
4.4 THE EFFECTS OF A NEARLY SOLID BOUNDARY ..... 19
4.5 VISCOUS EFFECTS ON ADDED MASS AND DRAG ..... 22
5. SUMMARY ..... 25
6. RECOMMENDATIONS ..... 27
7. REFERENCES ..... 28

## 1. INTRODUCTION

Whenever acceleration is imposed on a fluid flow either by acceleration of a body or by accelaration externally inmosed on the fluid, additional fluid forces will act on the surfaces in contact with the fluid. These fluid inertial forces can be of considerable importance in many oceen engineering problems. The purpose of this report is to review some of the characteristics of these fluid inertial forces and, in perticular, to evaluate the state of knowledge of the "added mass' matrices which are used to characterize the forces. The first part of the report (Section 3) is also intended to serve educational purposes. The second part (Section 4) reviews the existing data bose and some of the areas in which thare is either a lack of data or a data base which is contradictory. It is also intended to convey the limitations of the existing knowledge. Finally a number of suggestions for improvement in our present undartanding are listed in the conduding section.

Unlike many reviews, the author has not attempted to absorb every publication on the subject. Rather the time which would have been spent on such an effort, wes devoted to more concentrated analysis of the subject. Other excellent reviews of vanous aspects of unsteady fluid forces exist; in particular the reader is referred to the recent books by Blevins (Ref.18) and by Sarpkaya and Isaacson (Ref.17)


## 2. GENERAL EXPLANATION OFADDED EASS

Perhaps the simplest view of the phenomenon of "added mas" is that it determines the neowary work done to change the kinetic energy associated with the motion of the fluid. A ny motion of a fluid such as that which occurs when a body moves through it implies a certain positive, non-zero amount of kinetic energy associated with the fluid motions. This kinetic energy, $T$, can be simply represented by

$$
\begin{equation*}
T=\frac{\rho}{2} \int_{V}\left(u^{2}+u_{2}^{2}+u_{3}^{2}\right) d V=\frac{\rho}{2} \int_{V} u_{4} u_{4} d V \tag{1}
\end{equation*}
$$

where the $u_{i}(i=1,2,3)$ represent the Cartesian components of fuuid velocity and $V$ is the entire domain or volume of fluid. For simplicity we shall assume throughout that the fluid is incompressible with a density $\rho$.

If the motion of the body is one of steady rectilinear translation at velocity, $U$, through a fluid otherwise at rest then clearly the amount of kinetic energy, T, remains constant with time. Furthermore it is dear that $T$ will in some manner be proportional to the square of the velocity, $U$, of translation. Indeed if the flow is such that when $U$ is altered the velocity, $u$, at each point in fluid relative to the body varies in direct proportion to $U$ then $T$ could conveniently be expressed as

$$
\begin{equation*}
T=\rho \frac{I}{2} U^{2} \quad \text { uhere } \quad I=\int_{V} \frac{u_{i}}{U} \frac{u_{i}}{U} d V \tag{2}
\end{equation*}
$$

and the integral $I$ would be a simple invariant number. This is indeed the cose with some fluid flow solutions such as potential flow and low Reynolds number Stokes flow. However it may not be true for the complex, vortex shedding flows which occur at intermediate Reynolds numbers.

Now consider that the body begins to eccelerate or decelerate. Clearly the kinetic energy in the fuid will also begin to change as $U$ changes. If the body is accelerated then the kinetic energy will in all probability increase. But this energy must be supplied; additional work must be done on the fluid by the body in order to increase the kinetic energy of the fluid. A nd the rate of additional work required is simply the rate of change of $T$ with respect to time, $d T /$ tht. This additional work is therefore experienced by the body as an additional drag, $F$, such that the rate of additional work done, -FU is simply
equal to $d T$ dit. If the pattern of flow is not changing such that the integral $I$ remains constent it forlows that the "added drag', $F$. is simply

$$
\begin{equation*}
F=-\frac{1}{U} \frac{d T}{d t}=-p I \frac{d U}{d t} \tag{3}
\end{equation*}
$$

Now this force has the same form and sign as that required to accelerate the mass (m) of the body itself, namely $m \frac{d U}{d t}$. Consequently it is often convenient to visualize the mass of fuid $\rho I$, as an "added mass'. $M$. of fluid which is being accelerated with the body. Of course, there is no such identifable fluid mass; rather all of the fluid is accelerating to some degree such that the total kinetic energy of the fluid is increasing.

It is important to stress that $F$ is not the only drag force experienced by the body. During steady translation in a real viscous fluid there is a steady drag associated with the necessary work which must be done to balance the steady rate of dissipation of energy in the viscous fluid $\mathbf{W}$ hen the body accelerates there will be a smilar though not necessarily equal drag associated with the instantaneous value of U. Furthermore there may be delayed offeds associated with the entire previous history of translation (e.g. the Basset force. Ref.1. p.375)

## 2 ANALYTICAL APPROACHES TO ADDES MASS

## S1 EXAMPLES: RECTILINEAR MOTION OF A SPHERE AND CYIINDER WITR POTLSTIAL FLOW

In the preceding discussion the consequences of accaleration were illustrated by reference to siruple rectilinear motion of velocity, $U$. It should be clear that the methodology could be extended to more general motions and indeed this will be carried out in the following section. But prior to this it is worth illustrating how the integral, I, and therefore the added mass can be calculated for rectilinear motion. For the purposes of this example let us examine the idealized potential flows past a sphere and a cylinder. The geometry for both is as depicted in Fig.l. The sphere or cylinder of radius $R$ is assumed to be moving with time varying velocity $U(t)$ ( $t$ is time) in the positive $x$ direction. Polar coordinates $(r, v)$ are used where $x=r \cos v$

Fig. 1


The resulting fluid velocities $u_{r} u_{\infty}$ in the $r$ and $v$ directions are then given by a velocity potential, $\varphi$. such that

$$
\begin{equation*}
u_{r}=\frac{\partial \varphi}{\partial r} ; u_{d}=\frac{1}{r} \frac{\partial \varphi}{\partial v} \tag{4}
\end{equation*}
$$

and the appropriate velocity potentials in the two cases are

$$
\begin{align*}
& \varphi_{\text {qume }}=-\frac{U R^{3}}{2 r^{2}} \cos v  \tag{5}\\
& \varphi_{\text {quedr }}=-\frac{U R^{2}}{r} \cos v
\end{align*}
$$

The reader who is unfarniliar with these solutions may wish to satisfy himself that two solutions satisty (i) Laplaces equation, $\nabla^{2} \varphi=0$, in spherical and cylindrical coordinates respectively and (ii) the boundary condition that the relative velocity normal to the surfece of the body is zero ( $\left(\psi_{4}\right)_{r=R}=U$ 00s v $)$.

It follows that these flows are of the type in which $\boldsymbol{u}_{i}$ is directly proportionai to $U$ and consequently the integrals. $I$. can be evaluated as

$$
\begin{align*}
& \text { Sphere: } I=\int_{R}^{-\pi} \int_{0}^{\pi}\left[\left(\frac{1}{U} \frac{\partial \varphi}{\partial r}\right]^{2}+\left(\frac{1}{U r} \frac{\partial \varphi}{\partial v}\right)^{2}\right] 2 \pi r^{2} \sin v d v i r=\frac{2}{3} \pi R^{3}  \tag{7}\\
& \text { therefore } M=\frac{2}{3} \pi R^{3} \rho \\
& \text { Cybinder: } I=\int_{R}^{-2 \pi} \int_{0}^{2 \pi}\left[\left(\frac{1}{U} \frac{\partial \varphi}{\partial r}\right)^{2}+\left(\frac{1}{U r} \frac{\partial \varphi}{\partial v}\right)^{2}\right] r d v c r=\pi R^{2} \quad \text { per writ length }  \tag{B}\\
& \text { therefore } M=\pi R^{2} \rho \text { per writ length }
\end{align*}
$$

Note that the added mass, $M$, of the cylinder is equal to the mass of the fluid displeced by the body, whereas the added mass of the sphere is one half of the displaced mass

These are algebraically the simplest potential flows for which the value of $I$ can be evaluated. However. it is conceptually simple to visualize how $I$. uld be evaluated for any flow provided the flow solution (the $u^{\prime} / U$ values) are available. Note that, in effect, one need only have available the solution for the steady flow in the direction under consideration This considerahly simplifies the added mass colculation for rectilinear motion. Later we shall examine the more general case of arbitrary motion

## 32 resation to displaced mass; variation with direction of acceleration

In the preceding section it was noted that in the ideal case of potential flow around a circular cylinder in rectilinear motion the added mass is equal to the mass of fluid displased by the cylinder. This should be regarded as coincidental. There is no general correlation between added mass and displased fluid mass. As we have seen the added mass for a sphere is one half of the displaced fuid mass. Further more the idealized potential flow past an infnitely thin flat plate (zero displaced fluid mess) accolereted normal to itself has an added mass equal to the mass of a circular cylinder of fluid with a diameter equal to the width of the plate.

Thus the displaced fluid mass may not even be a good first approximation to the added mess (except
for the case of the circular cylinder). Furthermore we shall see that, in general, the value of the added mass depends on the direction of acceleration For example, the ideelized potential flow solution for the infinitely thin flat plate accelerated in a tangential rather than a normal direction yields zaro added mass rather than the value described above. A review of the available data suggests that a better (but still very crude) first approximation to the added mass of a body for a given direction of acceleration would be the mass of the fluid volume obtained by taking the projected area of the body in that direction and evaluating one half of the volume of the sphere with the same projected area (see Sections 4.2, 4.3). An improvement on this is included in Section 4.2.

One other complication will emerge in the following section when the complete added mass matrix is defined, namely that the forve on the body due to acceleration is not neoessarily in the same direction as the acceleration. For an unsymmetric body acceleration in one direction can give rise to an "added mass' effect resulting in a force which has a component in a direction perpendicular to the direction of acceleration. If, for exarmple, one were lifting a body from the ocean bottom by means of a cable then an increase in the lift rate could produce a lateral . stion of the body.

## 33 THE ADDED MASS YATREX

Up to this point, most of the examples and discussion have centered on simple rectinear motion However in general the response of a body to an additional force applied at some point and in some direction will not be confined to motion in that same direction Instead there will be a general induced acceleration of the body consisting of three translation accelerations, $A_{j}, j=1,2,3$ in three perpendicular directions and three angular accelerations, $A_{j}, j=4,5,6$. Then the added mass matrix $M_{i j}, i=1 \rightarrow 6, j=1 \rightarrow 6$ provides a method of expressing the relationship between the six force components. $F_{i}$. imposed on the body by the inertial effects of the fluid due to the six possible components of acceleration:

$$
\begin{equation*}
F_{i}=-M_{\psi} A_{j} \tag{9}
\end{equation*}
$$

The matrix $M_{i j}$ must have added to it the inertial matrix due to the mass of the body in order to complete the formulation of the inertial forces. If the center of mass of the body is chosen es the origin the
body mass matrix is symmetric and contains only seven different, non-zero values, namely the mass and the six different components of the moment-of-inertia matrix [Yih, p.102]. However one cannot in general relate any of the 36 different components of the added mass matrix nor prove that any of them are zero except in specific cases or for specific kinds of flow. Consequently an externally applied additional force will in general create acceleration in all six components of velocity and angular velocity. Thankfully it is rarely necessary to have to handle 36 different added mass coefficients. For potential flow one can show [Yih. p. 100] that the added mass matrix must be symmetric since the system is then conservative the symmetry also follows from the theorem of reciprocity. This redures the number of coefficients to $2 l$. However no further reduction is possible except for bodies with geometric symmetries

The smplifications introduced by geometric symmetries of the body are fainy easily established. Consider for example a body with a single plane of symmetry, for example an airplane. It is clearly convenient to select axes such that this plane of symmetry corresponds say, the $x_{3}=0$ plane. Then any acceleration confined to this plane, namely any $\cdots m b i n a t i o n ~ o f ~ A_{1}, A_{2}$ and $A_{8}$ will produce no added mass force $F_{3} . F_{4}$ or $F_{5}$ : the only possible non-zero forces will be $F_{1}, F_{2}$ and $F_{8}$. It follows that for such a body the following 9 components of the added mass matrix will be zero:

$$
\begin{equation*}
M_{6 j}=0 \text { for } i=3,4,5 ; j=1,2,6 \tag{10}
\end{equation*}
$$

If in addition the flow is assumed to be potential such that the matrix is symmetric then $M_{x}=0$ for the same domans of $i$ and $J$. The number of non-zero values required to define the matrix is 12 namely

$$
\begin{equation*}
M_{\mathbf{t}}, i=1 \rightarrow 6 \quad \text { and } \quad M_{12} . M_{94}, M_{95} . M_{45} M_{16} \text { and } M_{\mathbf{2 x}} \tag{11}
\end{equation*}
$$

Bodies which have two planes of symmetry (for example a hemisphere) yield a further reduction in the number of non-zero values. Suppose axes are chosen such that both $x_{2}=0$ and $x_{3}=0$ are planes of symmetry. Then not only must (10) be true but also

$$
\begin{equation*}
M_{v}=0 \text { for } i=2,4,6 ; j=1,3,5 \tag{12}
\end{equation*}
$$

and again, assuming potential flow $M_{\mu}=0$ for the same domains. Then the only non-zero values which
need evaluation are

$$
\begin{equation*}
M_{4}, i=1 \rightarrow 6 \text { and } M_{20}, M_{50} \tag{13}
\end{equation*}
$$

The last two, which with $M_{\epsilon 2}=M_{z 8}$ and $M_{6 s}=M_{s 0}$ represent the only non-zero off-diegonal terms, correspond to the moment about the $x_{3}$ axis generated by acceleration in the $x_{k}$ direction and the moment about the $x_{2}$ axis generated by acceleration in the $x_{9}$ direction. In other words since the body is not symmetric about the $x_{2} x_{3}$ plane linear acceleration in either the $x_{2}$ or $x_{3}$ direction will cause pitching moments in the $x_{1} x_{2}$ or $x_{1} x_{3}$ planes.

A few simple bodies such as a sphere, circular cylinder, cube, rectangular box, etc hawe three planes of symmetry. By following the same procedure used above it is clear that the only possible non-zaro elements are

$$
\begin{equation*}
M_{*}, i=1 \rightarrow 6, M_{16}, M_{10,} M_{24}, M_{20}, M_{34}, M_{30} \tag{14}
\end{equation*}
$$

and that if potential flow is assumed all of the off-diagonal terms are zero. Only in this simple case of three axes of symmetry and symmetry of the mi". $x$ (see below for conditions on this) does the matrix become purely diagonal so that there are no secondary induced accelerations.

It remains to discuss the precise flow conditions under which the natrix can be assumed to be symmetric and then finally to indicate how all of the elements could be evaluated.

## 34 ADDED UASS YATRIX SYMYETRY AND SUPERPOSIBILITY OF FLOW SOLUTIONS

The astute reader will have reoognized that the mere definition of $M_{v}$ in Eq.( $\theta$ ) requires certaln assumptions conceming the nature of the flow and the ability to lineariy superpose the effects (i.e. forces) of acceleration in the six directions. The question of the minimum preconditions necessary in order to write Equation (9) is one which will not be addressed here. It is however deer that these preconditions are met as soon as one makes the assumptions necessary to evaluate $M_{\psi}$. To the euthors knowledge the only evaluations which exist require that the fluid flow is superposable in the following sense: that the total induced fluid velocity can be obtained by linear addition of the fluid velocities caused by each of the components of the body motion or velocity. For this to be true requires thet
both the equations used to solve for the fuid fow and the boumdery conditions be lineer. This is not true in general of the Navier-Stokes equations for fluid motion and therefore superposablity is not in general, applicable. However there are two models of fluid flow which do satisty this condition namely (i) the potential flow model for high Reynolds flow [Yih, p.100] and (ii) the Stokes flow model for asymptotically small Reynoids numbers. In both cases the equations of motion can be put in linear form Furthermore provided if one is dealing with rigid or undeformable boundaries the boundary conditions are also linear. Oniy in these two limiting cases can the added mass matrix be regarded as an exact representation of the relation between fluid inertial force and body acceleration. In other types of flow it could however be regarded as a reasonable first approximation. Case (ii) above is of interest in flows such as occur in slurries or suspensions: however we shall from here on conflie our remerks to case (i) which is of greater practical importance in ocean engineering.

When the flow is linearly superposable, it is convenient to define $U_{i v}$ as the induœed fluid velocity ceused by unit velocity of the body in the $j$ direction ( $j=1 \rightarrow 8$ ). Induded here are both the translation components, $j=1,2,3$, and rotational compunents, $j=4,5,6$ of body motion Then if the body velocitues are denoted by $U_{j,} j=1 \rightarrow 6$, it follows that the fluid velocity

$$
\begin{equation*}
u_{i}=u_{i} v_{j} \tag{15}
\end{equation*}
$$

Consequently one can write Equation (1) as

$$
\begin{equation*}
T=\frac{1}{2} A_{\boldsymbol{k}} U_{j} U_{k} \tag{16}
\end{equation*}
$$

where the matrix $A_{k}$ is composed of elements

$$
\begin{equation*}
A_{\star}=\rho \int u_{\psi} u_{\psi_{k}} d V=M_{\star} \tag{17}
\end{equation*}
$$

It can be shown [Yih, p.102] that the matrix $A_{k}$ is in fact the added mass matrix $M_{*}$. It is certainly dear that the diagonal terms $A_{11} A_{22} A_{33}$ are identical to the added masses evaluated in Section 3. (To establisth this define the direction $x$ of Section 3 as either $x_{1} x_{2}$ or $x_{3}$ then $u_{i}$ and $u_{k}$ are identical and equal to the velocity $\frac{u_{4}}{U}$ used in Section 3.)

Furthermore it is ciear from this eveluation that the added maes metrix must be symmetic since reversing $j$ and $k$ in Equation (18) does not change the value of the integrel. Hence superposeblity implies symmetry of the added mass matrix.

## 26 EVALUATION OF THE ADDED MASS MATREX

The expression (17) will permit the evaluation of the entire added mess matrix Indeed it should be particularly noted that use of this result only requires the solution of steady fow problerns stince $w$ is the fluid velocity due to unit velocity of motion of the body in the $j$ direction. Consequently the solution of six steady flows for $j=1 \rightarrow 6$ allows evaluation of all 2 distinct velues in the added mess matrix. Hence one can make use of existing methods for solving steady flows around bodies of quite complex geometry in order to evaluate the added mass matrix. References 1.2,3,4 and 9 provide informetion on these existing methods.

One other form of Equation (17) can also be valuable in dealing with potential flows. Then if $\varphi_{\mathrm{J}}$ represents the velocity potential of the steady flow due to unit motion of the body in the j-direction then it follows that

$$
\begin{equation*}
w_{w}=\frac{\partial \varphi_{j}}{\partial x_{i}} \tag{18}
\end{equation*}
$$

Substitution into Equation (17) and application of Green's theorem leads to

$$
\begin{equation*}
A_{\star}=-\rho \int_{S} \varphi_{j} \frac{\partial \varphi_{k}}{\partial n} d S \tag{19}
\end{equation*}
$$

where $n$ is the outwand nomal to the surface, $S$, which represents the body surfece. In many cases of steady potential flows around complex bodies it is dearly easier to evaluate the surface integral in (19) than the volume integral in (17). Indeed the form (19) is ideally suited for use with potental tow codes such as the Douglas-Neuman code.

## 36 VELOCITY AND acceleration of the rlud rather than the body

All of the preceding discussion was centered on the inertial forces due to acceleration of a body in a flud This review would be incomplete without some comment on the cases in which the fuld fer
from the body is either (i) moving with a constant, uniform velocity or (ii) accelerating.

Examine case (i) first. It was implicitly assumed in all the preceding sections that the fluid far from the body was at rest. Otherwise clearly the integral defining $T$ (Eq(1)) would have an infinite value and the subsequent analysis would be meaningless. If, as in case (i). the fluid far from the body has some uniform constant velocity denoted by $W_{i}$ then it is clear that since the inertial force cannot be altered by a simple Galilean transformation it follows that the proper deffition of $T$ under these circumstances is

$$
\begin{equation*}
T=\frac{\rho}{2} \int_{V}\left(u_{i}-W_{i}\right)\left(u_{4}-W_{i}\right) d V \tag{20}
\end{equation*}
$$

The value of this integral is then finite and the conundrum resolved. In other words the appropriate $u$ to be used in Eq.(1) is the velocity of the fluid relatiae to the flide velocity fer from the bady provided the latter is constant with time. This leads to no alteration in fluid inertial forces A rigorous expression for the forces would be

$$
\begin{equation*}
F_{i}=-M_{i} \frac{d}{d i}\left(U_{j}-W_{j}\right)=-M_{i} \frac{d U_{j}}{d t} \tag{21}
\end{equation*}
$$

but since the time derivative of $W_{j}$ is zero the onginal relation (9) ic recovered.

However case (ii) in which $W_{j}$ is a function of time is more complex. It is important to identify the fluid incrial forces in this case for two reasons. First it is of practical importance in andyaing, for example, ccean wave forces on structures. Secondly, many of the important experiments on unsteedy forces are performed using an acoelerating fluid rather than an accelerating body (e.g. Refs. 10 and 11). We begin by visualizing a case (i) flow with a constant, uniform fluid velocity. $W_{j}(j=1 \rightarrow 3)$, fer from a body whose center of volume is moving at a velocity, $U_{j}(j=1 \rightarrow 6)$. The body is also accelerating with components, $A_{j}$. The flow satisfies the Navier-Stokes equations for fluid motion * and the solid body boundary conditions. The fluid inertial forces in this case are given by Eq.(21). Now consider a slightly different flow whose velocities are identical to those of the first flow but in which an additional uniform acceleration $j=1,2.3$ is applied globally to both the fluid and body. Now the actual acceleration of the body is $\left(A_{j}+\frac{d N_{j}}{d t}\right)$. The Navier-Stokes equations of fluid motion and
the solid body boundery conditions are identical for the two flows exoegt thet where the preaure, in appears in the equations for the first flow, the exprestion $p-\rho x_{j} \frac{d W_{j}}{d t}$ appeers in the equations for the second flow. Consequently the stresses and forces which the fluid exerts on the body are identical except for an additional contribution in the second flow due to the additional pressure, $\rho x_{j} \frac{d W_{j}}{d t}$ When this is integrated over the surface of the body the additional force on the body turns out to be $\rho V_{D} \frac{d W_{j}}{d t}$ where $V_{D}$ is the volume of fluid displaced by the body. Consequently the inertial force is

$$
\begin{equation*}
F_{i}=-M_{i} A_{j}+\rho V_{D} \frac{d W_{i}}{d t} \tag{22}
\end{equation*}
$$

But as stated previously the acceleration of the body in the second flow is now $A_{j}+d W_{g}$ tit and hence in the case of the second flow

$$
\begin{equation*}
\frac{d U_{j}}{d t}=A_{j}+\frac{d W_{j}}{d t} \tag{23}
\end{equation*}
$$

where $d W_{j}$ ftit is the acceleration of the fluid fan rom the body. Substitution for $A_{j}$ in Eq.(22) produces the final required result for the second flow:

$$
\begin{equation*}
F_{i}=-M_{i j} \frac{d U_{j}}{d t}+\left(M_{i j}+\rho V_{D} \delta_{i j}\right) \frac{d W_{j}}{d t} \quad j=1.2 .3 \tag{24}
\end{equation*}
$$

where $\delta_{\psi}$ is the Kronecker delta ( $\delta_{\psi j=1}$ for $i=j, \delta_{\psi}=0$ for $i \neq j$ ).

Therefore the "added mass' associated with the ftide acceleration, $d W$, 人立 in the second flow is the sum of the true added mass, $M_{y}$. and a diagonal matrix with components equal to the mass of the displaced fluid. $\rho V_{D}$

However we must now examine more closely the general validity of Eq.(24). The first and second flows described above were assumed to have identical fluid velocity flelds at the moment et which the fonces were considered This will not be true in general for solutions of the Navier-Stokes equation even though the body velocities and far fleld fluid velocities are identical. In general the solutions to the Navier-Stokes equations will also depend on all of the previous time history of the body and far
field fluid motions and consequently the two nows will rot in general have identical tuld velocty telds. There are however two important exceptions to this and in both ceses Eq(24) will be true. First if the viscous effects are neglected then the fluid has no memory and the fluid velocity ffelds will be identical; thus Eq.(24) holds for potental flows. Secondly if the previous history of the relotive velocity, $\left(U_{j}-W_{j}\right)$ is identical in the two flows then (24) will hold regardless of viscous effects.

Therefore, in summary, the fluid inertial forces due to any combination of body or fer feld fluid acceleration ( $\alpha U_{j} / a t$ or $d W$, $A t$ ) can be exactly represented by Eq.(24) if either (i) viscous effects arr neglected or (ii) the matrix $M_{i j}$ represents the fluid inertial forces for the case in which the fluid is at rest far from the body and the entire previous history of the relative motion $\left(U_{f}-W_{j}\right)$ is identical to that of the flow under consderation. The latter is indeed the case when companing two cases, for example, in the frst of which the far field fluid motion is sinusoidal in time and the body at rest and in the second of which the far field fluid motion is at rest and the body moves sinusoidally. Consequently the "added mass" in the experiments of Keulegan and Carpenter (Ref.10) in which the far fleld fluid is accelerated sinusoidally should yield $\left(M_{i i}+\rho l_{D}\right)$ whereas the experiments of Skop. Ramberg and Ferer (Ref. 15) in which the body is accelerated should yield. $M_{\mathrm{e}}$. To transfer results from one case to the other requires the addition or subtraction of the displaced ma: For the examples of Section 3.1 the values of $\left(M_{i}+\rho V_{D}\right)$ would be $2 \rho \pi R^{2}$ per unit length in the case of the cytinder and $2 \rho \pi R^{9}$ in the case of the sphere. Sometimes the total $\left(M_{i}+\rho V_{D}\right)$ is referred to as the added mass and this can result in some confusion Strictly speaking the term added mess should be reserved for $M_{v}$ only. or in other words the case in which the body is accelerating and not the far field fluid

## 37 THE EFFTCT OF A NEARLY SOLID BOUNDARY

The effects on the added mess due to the proximity of a solid boundary will be addressed in more detail later (see Section 4.4). It is generally true that the presence of the boundary tends to increase the added mass (see Tables I $\rightarrow \mathrm{V}$ ) and sometimes this increase can be very large. Here we merely remark that the preceding theoretical results are equally applicable in the presence of a solid boundary with the following addenda:
A. The redudions due to geometric symmetries disarssed in Section 3.3 only apply to total geometric symmetries of both the body and solid boundary.
B. Potential flows with a plane solid boundary can be modelled by refecting the now and body in the plane and treating the total flow due to the body and its image. Equivalence of the two problems allows the transference of added mass coefficients from one to the other. As an example of this see the cases of two cylinders and a cylinder plus a plane boundery in Table 11.

### 3.8 THE EFTECT OF A NEARLY FREE SURFACE

Unlike the presence of a solid boundary, a free surface boundary adds considerably to the complexity of the problem This is due to the fact that, in general, the boundary condition is non-linear and hence superposability is not satisfled. As a consequence the dynarric behavior of bodies near a free surface is a specialized area in which the literature is also rather specialized because of the complexity of the fluid flow problems. Though this subject is outside the scope of this report it is necessary to meke a few brief remarks and in particular, to identify the conditions under which one must account for free surface effects.

In the case of floating bodies the reader is referred to excellent reviews of the analytical techniques by Wehausen (Ref.12), Newman (Ref.13) and Oglvie (Ref.14). Submerged bodies are only slightly easser to handle. Some data on submerged bodies is given in Table III. It should be stressed that these examples are only pertinent to the inertial forces generated when accelerating the bodies fromnest. A ny prior translation or rotational motion of those bodies would have generated free surface waves which would in turn affect the unsteady loading on the body. This represents the major complication introduced by the presence of a free surface. It is however clear that if the body motion is suffiently siow (characterized by a velocity, $U$, say) then the waves created will be negligibly small and these prior history effects would also be small. This requires that the Froude number, U $\operatorname{(gd})^{\frac{1}{2}} \ll 1$.

The results of Table III do allow one to estimate what constitutes proximity to a free surfece providing the above conditions hold. It can be seen that the free surface has little effect (less than 5\%) provided the ratio of the depth of the body to the body dimension is greater than about 4. For lesser
depths the added mass first increases as the acceleration of the fluid between the free surface and the body increases but then decreases when the depth is less than about one body dimension becuse less fluid is being accelerated.

## 39 THE EPTECT OF FLUID CONPRESSIBILITY

Generally the effects of the compressibility of the water on the added mass can be neglected in most ocean engineering applications. This is because the compressibility does not begin to aflect the fuld flow until the Mach number ratio of the typical fluid velocity to the velocity of sound. $c_{1}$ in the fluid exceeds a value of at least 0.1 . In unsteady flows one must also consider a parameter computed as the typical acceleretion times the typical body dimension and divided by c. Again one would not normally expect any compressibility effect if this is less than 0.1 .

Such conditions are almost always satisfled in ocean engineening applications. However it is possible that the presence of a large quantity of bubbles in the water could sufficiently reduce the sonic velocity, c. to such an extent that the added mass would be altered by the compresibility of the water/gas mixture

## 4. REVIET OF EISTING DATA ON ADDED HASS

## 41 THEORETICAL POTENTIAL MOT ADDED MASSIES

By far the largest category of analytical results for added mess are thoee colculated for bocties in en infnite fluid domain assuming the flow to be potential. The majority of these results are obtained by methods analogous to those described in Section 3. Bodies for which the steady flows cen be generted by superposition of an array of potential fow singularities (sources, sinks, doublets, potentiel vortices, etc) are particularly compatible with the use of expression (18). Such methods ere described in Ref. 9 and in many mechanics texts (e.g. Ref. 1, p.104). A particularly useful tabulation of many of the aveilable results is given in a paper by Patton (Ref.8) and his Tables 1 and 2 are reproduced here as Tables I and III. Note that the "iird column of these tables contains the added mass denoted by $m_{n}$; the values given comespond to the diegonal terms in the added mass matrix, $M_{*}$, the direction of acceleration likurg specinied in the second colurnn (No off-diagonal components of the added mass matrix are listssi. itme results are also listed for bodies on or near to a solid or free surface and comment on these is delasted until later. Patton has induded both theoretical potential flow added messes and experimerially determined added masses in Tables I and III. These are distinguished by the letters $T$ and $E$ in the fourth column of these tables. A nother excellent source of tabulated added masses is given in a DTM B report by Kennard (Ref.9). Kennard's tables for added mass coefficents are attached to this report as Tables II, IV and $V$.

Though not exhaustive Tables I through IV provide a substantial reference list of added messes. It could be argued with some justification that these tables are more than adequate for most enginearing purposes provided the body under analysis is not in close proximity to a solid or free surfece. The remainder of this report will concentrate on the limitations of this analytical knowledge in terms of boundry effects and reel fluid effects (e.g. viscous effects). However before procseding to these discussions two further points should be made.

First Tables I through IV could be supplemented by the potential flow methods desaribed in Section 3 and detailed in many references (e.g. Ref.9). Modern potential now computer programe
(e.g.Douglas-Neuman code) for steady flows could readily be adapted for this purpose as discussed in Section 3.5. The capability to do this might be important in circurnstances where accunate added massses are required for bodies of unusual or complex geometry or in circumstances where the offdiagonal terms in the added mass matrix are deemed important (the tables contain virtually no information on off-diagonal terms).

The second point is that approximate values for the conventional or diagonal added mass terme for bodies of complex geometry (for example, an airplane) can be obtained by combining the added masses for each component of the structure (wings, fusilage, tail, etc). Such a strategy is outlined in Section 4.3.

## 42 SENSITIVITY TO THE GEOM ETRY OF THE BODY

The diagonal terms in the added mass matrix (i.e the conventional added masses) are relatively independent of the precise geometry of a body. For example. when accelerated normal to their longitudinal axes. cylinders with any elliptical cross-section have an added mass equal to that of a circular cylinder with the same width normal to the direction of acceleration under consideration (see Table I). Cylinders with more irregular rectangular or diarnond shaped cross-sections deviate somewhat from this rule; however the deviations are rather unpredictable. Compare for example the fact that the rectangular and diamond shapes in Table I show opposite trends as the cross-section becomes more streamlined in the direction of acceleration $W$ hen the ratio of cross-sectional dimension in the direction of acceleration to that normal to the direction of acceleration is about 5 the rectangular shape has increased its added mass by a factor of 2 whereas the diamond shape has decreased its added mass by a factor of $40 \%$. The unsubstantiated opinion of the author is that the experimental values would show less deviation due to the effects of fow separation

Despite these doviations, a reasonable first approximation to the translational added mass, $M_{\text {a }}$. for two dimensional bodies (large aspect ratio of length, $L$ to cross-sectional dimension, $2 a$ ) would be the mass of a cylinder of fluid whose diameter is the same as the width. $2 \mathrm{a}_{3}$. of the projected area in the direction of acceleration, x:

$$
\begin{equation*}
M_{*} \approx \rho \pi\left(a_{k}\right) l \tag{25}
\end{equation*}
$$

Consider therefore the following empinical approximation for axbltrary three dimenaiond bodies; that the added mass for a particular direction of acceleration, $x_{1}$. is given by the volume obtained by rotating the projected area of the body in that direction about an axis defined by the smeller of the two principal dimensions, $2 a_{4}, 2 b_{h}$, of the projected area For an elliptical projected area this would yield

$$
\begin{equation*}
M_{i n}=\frac{4}{3} \rho \pi b_{i}\left(a_{i}\right)^{2}, \quad b_{i}>a_{i} \tag{26}
\end{equation*}
$$

where there is no implied summation over the index $i$ This would yield a reasonably conservative approximation for the preceding case of the cylinders. However it would substantially overestimate the added mass for a body like a sphere which has a small aspect ratio. Then the above estimate would be twice the potential flow value. Perhaps a better empirical approximation would be

$$
\begin{equation*}
M_{i}=\frac{4}{3} \frac{\rho \pi\left(a_{i}\right)^{2}\left(b_{i}\right)^{2}}{\left(b_{i}+a_{i}\right)} \tag{27}
\end{equation*}
$$

which would then predict both the cylinders and we sphere correctly. Testing this against the deta for a prolate ellipsoid accelerating 'broadside on" (see Table IV) we find a value of the added mass using Eq.(27) which is within $7 \%$ of the exact value. Further improvennnts could dearly be made but are probably of minor value considering the other uncertainties discussed below.

## 43 BODIES OF COMPLEX GEON BTRY

The result (27) of the previous section suggests an extension for the purposes of evaluating the added mass for a body of complex geometry (an airplane). Though it would require further detailed analysis and testing it would not be unreasonable to suggest that a complex body be considered disassembled into its principal component parts (wings, fuselage, tail) and that the added masses for each component be evaluated for three perpendicular directions of acceleration using the technique outlined in the previous section. Then we must ask whether it is approximately correct to simply add the added masses for the components in each of the three directions. From an engineering point of view it seems reasonable to do this. However it is very difficult to give any quantitative measure of the error in such en estimate
due to the interaction of the components. The case of two percllel cylinders touching eech other which is detalled in Table II. provides a particular harsh test. For directions parallel to and normal to the plane of the axes of the cylinders the errors would be $35.5 \%$ and $120 \%$ reapectively. But this simply dernonstrates that the two cylinders together should be treated es a single component; then the errors are significantly smaller namely $20 \%$ and $14 \%$ respectively.

Much more reasonable tests are provided by the winged objects in Table III and we shall, in perticular examine the values given in Item 3. Table III. Taking the individual components (two fiet plates and an ellipsoid) and using the tabulated added masses of these individual components in the case $N=0.5$ one arnives at a value of $K$ of 1.293 . This is within $5 \%$ of the actual value colculleted nemely 1.24

Further tests would be needed to establish confidence limits on this superposition method but it does not seem unlikely that one could confidently predict potential flow added masses for complex bodies to within $\pm 30 \%$ using the methods outined above and empirical formules such es represented by Eq.(27).

## 44 THE EFFECTS OF A NEARLY SOLID BOUNDARY

The discussion in Sections 4.2 and 4.3 wes confined to bodies remote from a solid boundary. It is deer from the various examples given in the tables that the presence of a solid boundery can ceasee substan= tial increase in the added mass. This is due to the necessary increase in the fluid accoelerations primerily in the region between the fuid and the boundary. For example from Table II it is seen that the mass for a craular cylinder (redius, a) is increased by a factor $a^{2} / R h^{2}$ for a well at a distance $h$ from the center of the cylinder. The result presented is only approximate and requires ǎ << 1. If the body is brought closer to the boundary the added mass increases at an even grester rote because one develops a "film" or namrow gep between the body and the well in which the fluid eccelection cen be very large indeed

From a practical engineering point of view there is a peucty of data for these extrume conditions of dose proxdrity of a body to a solid boundary. A simple example will illustrete some of the drumetic

## ,

effects of the proximity of a solid surface on the inertial forces required to move a body away from that surfacf Consider the two dimensional problem of a flat plate of width. $2 a$ lying on an ocean floor. A vertically upward force, $F$. per unit length of the plate is applied at the center of the plate to lift it AWay from the floor. Due to this force the plate has nsen to a uniform height, $h(t)$, above the floor at time $t$ The velocity and acceleration of the plate in the upward direction are therefore ch/fet and $d^{2} h$ di $^{2}$ (see Fig. 2).

Figure 2.


This probiem could be visualized as charactertstic of eny fainly nat object lying on the oceen foor. Typically only portions of the undersurface would be in contact with the ocean floor. However one could visualize that prior to application of the force there is a typical average separation distance, $h_{0}$. between the undersurface and the object. Silting up of the object couid. of course, make $h_{0}$ very small. In any case some $h_{0}$ would be partinent to the moment, $t=0$, when the lift force is applied.

We concentrate here on the dynamics of the body while the separation. $h$ is very small compered with the lateral dimension. 2 m , of the object because it will be seen that these ere the most critical conditions. Then upward velocity of the plate, ahtit, will generste much larger horizontal velocities, 4. (Fig.2), in the gap than vertical velocities and hence continuity of mass in the gap requires

$$
\begin{equation*}
h u=-\frac{d h}{d t} x \tag{28}
\end{equation*}
$$

and the momenturn equation for the fluid in the gap in the absence of frictional or viscous forces yields

$$
\begin{equation*}
\frac{1}{\rho} \frac{\partial p}{\partial x}+2 u \frac{\partial u}{\partial x}+\frac{1}{h} \frac{\partial}{\partial t}(h u)=0 \tag{28}
\end{equation*}
$$

where $p(x, t)$ pressure at any point. $x$, in the gap. Substitution and integration yield the following form for the pressure distribution in the gap

$$
\begin{equation*}
p=p_{E}+\frac{\rho}{2}\left(a^{2}-x^{2}\right) h \frac{\partial^{2}}{\partial t^{2}}\left(\frac{1}{h}\right) \tag{30}
\end{equation*}
$$

where $p_{E}$ is the pressure at the edges. $x= \pm a$ While the gap is small $p_{E}$ will be approxdmately equal to the ambient pressure. $p_{A}$, and the pressure on the top-side will deviate much less from $p_{A}$ than the pressure $p$ in the gap. Consequently by integration using the relation for $\boldsymbol{p}$ one can obtain the downward inertial force or added mass force, F. imposed by the fluid on the plate (par unit plate length)

$$
\begin{equation*}
F=\frac{2}{3} \rho \frac{\mathbf{a}^{3}}{h}\left\{\frac{\partial^{2} h}{\partial t^{2}}-\frac{2}{h}\left(\frac{\partial h}{\partial t}\right)^{2}\right\} \tag{31}
\end{equation*}
$$

Compare this with the known inertial force on the plate in the absence of the solid boundery namely $\rho \pi a^{2} \partial^{2} h / \partial t^{2}$ where $\partial^{2} h / \theta t^{2}$ is again used to represent the vertically upward acceleration. It is dear that the magniturie of the former given by Eq.(31) will typically be very much larger than the latter by the large factor, a/h of course the preceding analysis ceases to be valid when $h$ approaches a But it is dear that as the plate is raised the added mass per unit length begins at a very large value of the order of $\rho a^{3} h_{0}$ and will rapidly decrease with $h$ so that it asymptotically approaches a value of the order of $\rho a^{2}$ when $h$ is of the order $a$.

It is of interest to examine briefly the consequences of this bebavior of the fuild inertial forces. If, for simplicity one neglects the mass of the plate itself, then the upward force applied to the plate by the cable will be equal and opposite to the fluid inertial force. For illustrative purpose suppose a constant upward cable force is applied. Then integration of the equation of motion represented by Eq.(31) if F is now visualized as the cable force yields a time history $h(t)$, given by

$$
\begin{equation*}
h=\frac{h_{0}}{\cos \lambda t} ; \lambda=\left(\frac{3 F}{2 \rho a^{3}}\right)^{\frac{1}{2}} \tag{32}
\end{equation*}
$$

where the initial conditions $h=h_{0}$ and $d h \neq 0$ at $t=0$ have been used. It is readily seen that this leads to a kind of "catastrophic' release from the bottom in which the upward acceleration increases with time. It is unlikely therefore that a constant uplift force could be maintained under these circumstances. Consequently the actual initial motion would be dependent on other factors such as the cable elasticity.

The author has, as yet. encountered little in the way of analysis of such problems and suggests this as an area deserving further study both experimentally and analytically.

### 4.5 VISCOUS BFFTECTS ON ADDED MASS AND DRAG

The previous sections of this chapter have deliberately avoided reference to a further complication caused by the viscous effects in the flow around the body. These viscous effects on both the fluid inertial and fluid drag forces have been the subject of a number of detailed studies as represented for example by Refs. 10, 11 and 15. The essence of the complication is th: : certair. ismes of flow the viscous processes of flow separation and vortex shedding cause radical modifications to the forces expected on the basis of simple addition of fluid inertial and fluid drag forces. The latter approximation is embodied in what is known as M orison's equation (Ref.16) in which the total force on the body, $F^{T}$. is expressed as

$$
\begin{equation*}
F_{i}^{T}=-M_{i j} \frac{d U_{j}}{d t}-\frac{1}{2} \rho A C_{v}\left(\left\langle U_{j}\right|\right\}^{2} \tag{33}
\end{equation*}
$$

where $C_{\psi}$ is a lift and drag coefflcient matrix, and $A$ is a typical area for the body. This equation is
normally quoted for only one direction at a time and is written as

$$
\begin{equation*}
\left.F^{r}=-M \frac{d U}{d t}-\frac{1}{2} \rho A C_{D} l U\right]^{2} \tag{34}
\end{equation*}
$$

where $C_{D}, U$ and $d U /$ de refer to a drag coefficient, the velocity and acceleretion in line with the force. It ruight be expected that both $M$ and $C_{D}$ would be independent of the specific motion under consideration. However Keulegan and Carpenter (Ref.10) have observed experimentally that this was not the case and that substantial changes in $M$ and $C_{D}$ occurred as the rate of acceleration represented by the period. $T$, of their sinusoidal motion was increased such that $U_{L} T / D$ became of order one. Here $U_{N}$ is the typical velocity (the peak velocity of the sinusoidal motion) and $D$ is the cylinder or plate width It is signiffcant that all of their data was obtained within a range of Reynolds numbers, $U_{N} D N$ ( $\nu$ is the knematic viscosity) between 5000 and 30,000 . Even the steady flows pest bodies in this Reynolds number regime experience substantial unsteadiness due to flow seperation and vortex shedding.

Keulegan and Carpenter found that the "effective" value of the added mass for cylinders was close to the potential flow value ( $\rho \pi D^{2} / 4$ per unit length) for $U_{M} T / D$ below about 5 but decreased rapidly with increasing $U_{N} T / D$ becoming negative for a range of $U_{N} T / D$ between 10 and 20! (Note that we have subtracted the displaced fluid mass from their results to get the true added mass in line with the discussion of Section 3.8) W ith further increase in $U_{N} T / D$ positive values similar to those for $U_{H} T / D<5$ are recovered. The drag coefficient. $C_{D}$. shows a large increase for the same range of $U_{N} T / D$ betweeen 10 and 20 . Flat plates exhibited a different pathological behavior of the added mass and drag coeffleent. No systematic variations with Reynolds number, $U_{M} D N$. could be detected

Skop. Ramberg and Ferer (Ref.15) have also carned out experiments on sinusoidolly oscillating flows except that the body rather than the fluid is accelerated. Their results do not agree with those of Keulegan and Carpenter. For values of $U_{\nu} T / D$ between 1 and 12 they found that the fluid inertial force egreed very well with the potential flow value. Moreover the variations in the effective drag coefficient could be accurately predicted by considering the instantaneous Reynolds number at each point during the cycle, using some appropriate form for the corresponding instantaneous drag and
thereby synthesizing the overell drag coefficient.

The results of Skop, Ramberg and Ferer camot be readily reconciled with those of Keulegan and Carpenter. The Reynolds numbers for the Skop, Ramberg and Ferer experiments are in the renge between 230 and 40,000 and are therefore similar to those of Keulegan and Carpenter. It is quite ciaer that further detailed measurements using more sophisticated measurement and data analysis techniques are needed to resolve this question. Though it has little value, I have formed the very tentative opinion that the experiments and data reduction techniques used by Skop, Ramberg and Ferer are superior to those of Keulegan and Carpenter and therefore I would place more confidence in their results. On the other hand the data of Keulegan and Carpenter is much more widely known and used; this I believe may be unfortunate.

For the present it is necessary for engineering purposes to be aware that pathological behavior of the fluid inertial forces might ocour for body motions whose typical amplitude is greater than about half of the body dimension

Before leaving this subject it is of value to Secord a few of the resuits of the experiments carried out by Sarpkaya (Ref.11). He oscillated a cylinder in a direction normal to the direction of an onooming stream of fluid and observed pathological behavior for $\bar{V} T / D$ (where $\bar{V}$ is now the steady stream velocity) between about 3 and 10. Furthemore, in one of the few experimental meesurements of offdiagonal terms in the force matrix he observed the oscillations in the force on the body perpendicular to the direction of oscillatory motion to be less than $7 \%$ of the steady drag in that direction.

## 5. SUEPARY

The anslytical beckground of the added mass matrix describing fuid forces due to accelaction of the body or the fluid has been reviewed. It is shown that the use of this concept is rigorounly juatifed only in the case of linearly superposable fuid motions with rigid boundaries In the context of oceen engineering problems this restricts the analysis to that of potential flow and indeed. almost all of the theoretical predictions are computed from potential now analyis. For emplical enginearing purposes the concept has also been used for real flows with boundary layers, seperation and vortex sheciding.

The majority of potential flow calculations of added mass are for bodies accelercted in an infinite domain of incompressible. inviscid fluid. M any of these are included in Tables I to IV. These tebles provide a substantial reference list which may be more than adequate for many engineering purposes provided the body is not in close proximity to a solid or free surfece boundary. Furthermore since the added mass is generally rather insensitive to the detailed geometry of the body we have some praliminary suggestions as to how the added mass for a body of complex geometry might be estimated As detailed in Sections 4.2 and 4.3 the first step is to decompose the body into major components. The added mass of each may then be estimated for each direction of acceleration from the principal dimansions $\left(2 a_{i}, 2 b_{i}\right)$ of the projected area in that direction and the appruximate formula (27). The added mass for each component in each direction would then be arithmetically summed. 1 believe it might be possible to make predictions within $\pm 30 \%$ by this means. If better eccuracy is required then we heve indicated how modem potential flow codes (.e.g Douglar-Neuman code) designed to calculate steady nows might be utilized to obtain better results.

Data is scarcer for the cases when a solid boundary or free surface is close to the body. In general modifications are required if such a boundary is within four major body dimensions Free sarface effects are quite complex and are not covered by this report. However it is shown that a solid surface (ocean bed) can have a dramatic effect particularly when a body is being reised from the oceen floor.

Data is also scarce for the off-diagonal elements of the added mass matrix. Virtually no results for these elements appear in the tables. Consequently it is almost impossible to assess whether these
interection terms are important in prectical problems. Agein, however, use of the aforementioned computer codes would permit better evaluation of the need to consider the off-diegonal terms

The relationship between the forces when the fluid is accelerating past the body as opposed to the reverse is discussed in Section 3.6. It is shown that a relation can only be firmly established if either (i) superposability is possible (e.g. potential flow) or (ii) if the entire previous history of the relative velocity is identical in the two cases. Then the appropriate fluid mass in the case of fluid acoeleration is equal to the added mass plus the displaced fluid mass.

Finally it is clear that viscous effects in the form of boundary layer separation and particularly vortex shedding could possibly cause radical departures from the theoretical, potential flow predictions. The data on this is limited and contradictory. For the present one can only point out that pathological behavior might occur in certain ranges of frequency (or typical time of acceleration) and Reynolds number.

## a RECOMMRNATIONS

It seems appropriate to suggest several arees of engineering importance in which further analyticol, empincal and experimental studies would provide valuable information
A. There is a relative paucity of good experimental data in the open literature which can be used to evaluate the real ftuid effects of viscosity. The data which does exdst is often contradictory. Such experiments are not easy and are frought with pittalls. However both meesurement techniques end data processing methods have substantially improved in the last five years. It therefore seems appropriate to suggest further expenimental programs which might help to provide some solid information that the engineer could use. At the present time there is little concrete knowledge which the engineer could use with confidence.
B. The theoretical predictions of added mass from potential flow provide a good deta bese for use in estimating the diagonal terms in the added mass matrix. This data base would be utilized to produce empirical methods for use with bodies of complex geometry. This could result in a simple and useful computer code for this purpose.
C. There are however very few calculated values for the off-diagonal terms in added mass metrices. I therefore recommend that in order to build up some data base for off-diegonal terms and in order to allow more accurate evaluation of the diagonal terms for bodies of cormplex geometry some of the modem potential fow computer codes (e.g. Douglas-Neuman) be adapted to evaluate the entire added mass matrix.
D. The dramatic effects which can occur during separation of a body from the ocean floor should be further investigated both analytically and experimentally.

## 7. REFLRENCES

1. Yih. C-S. 1909. Fluid Mechanics M oGraw-Hill Book Co.
2. Lamb, H. 1932. Hydrodynamics, Dover Pub., 6th Edition
3. Birkhoff, G. 1960. Hydrodynamics. Princeton Univ. Press,
4. M unk M. 1963. Fluid M echanics, Part II, A erodynamic Theory, Vol. 1 edited by $W$.Durund Dover Pub., NY.
5. Patton. K.t. 1985. Tables of hydrodynamic mess factors for translational motion ASME Paper, 65WA UNT-Z.
6. W endel, K. 19 . Hydrodynamic masses and hydrodynarmic moments of inertia. DTMB Translation, No.280. Kinsler, L.E. and Frey, A.R. 1962. Fundamentals of Acousticas J.W iley and Sons, NY.
7. Patton. K.T. 1065. An experimental investigation of hydrodynamic mass and mechanical impedances. MS Thesis, Univ. of Rhode Island
8. Kennard E.M. 1967. Inrotational fiow of frictionless fuid mostly of invariable dansity. DTMB Report 2299.
9. Keulegan, G.H. and Carpenter, L.H. 1958. Forces on cylinders and plates in an oscillating tuid. J.Res.Nat.Bur.Standerds, Vol.60, No.5, pp. 423-440.
10. Sarpkaya. T. 1977. Transverse oscillation of a circular cylinder in uniform flow. Navel Pootgreduste School, M onterey, Report NPS-69SL, $77071-\mathrm{R}$.
11. Wehausen, J.V. 1971. M otion of floating bodies Ann Rev.Fluid M ech., Vol.3, pp.237-288.
12. Newman. J.N. 1970. Ann Rev. Fluid Mech, Vol.2, pp.67-94.
13. Ogilvie, T.F. 1964. Recent progress toward the understanding and prediction of ship motions. Sth Symp. Naval Hydrodynamica, Bergen, pp.3-128.
14. Skop. R.A., Ramberg. S.E. and Ferer, K.M. 1976. Added mase and damping forces on drouler cylinders Naval Research Laboratory. W aehington. D.C.. Report NRL 7970.
15. M orison, J.R., O'Brien, M.P., Johnson, J.W . and Scheof. S.A. The forces exarted by surfece waves on piles J.Petrol.Tech. A.I.M.E., Vol.180, p 149.
16. Sarplaya. T. and Isaacson, M. 1981. Mechanics of wave forces on offshore structures. Van Nor trand Reinhold Co.. New York.
17. Blevins, R.D. 1977. Flow-induced vibrations. Van Nostrand Reinhold Co., New York

TABLE I
(From Reference 5)

ADDED (HYDRODYNAMIC) MASSES FOR TWO-DIMENSIONAL POTENTIAL FLOWS ; Reference numbers are given under SOURCE.


TABLE I (continued)


TABLE II
(From Reference 9)

ADDED MASSES FOR TWO-DIMENSIONAL POTENTIAL FLOWS
(See Reference 9 or TABLE $V$ for notation)

Circular cylinder in translation perpendicular to its axis:


Elliptic cylinder in translation parallel to an axis, called the $a$-axis, either $a>b$ as shown or $b>a$ :


$$
\begin{array}{ll}
T_{1}=\frac{1}{2} \rho \pi b^{2} U^{2}, & \text { from Equation [84I], } \\
H_{1}^{\prime}=\rho \pi a b, & k=b / a
\end{array}
$$

Plane lamina in translation perpendicular to its faces:

$$
\prod_{\alpha}^{u} \int \begin{aligned}
& T_{1}=\frac{1}{2} \rho \pi a^{2} U^{2}, \quad \text { as in Equation [86b], } \\
& k M_{1}^{\prime}=\rho \pi a^{2} .
\end{aligned}
$$

$\underline{\text { Elliptic cylinder rotating about its axis: }}$


$$
\begin{aligned}
& T_{1}=\frac{1}{16} \rho \pi\left(a^{2}-b^{2}\right)^{2} \omega^{2}, \quad \text { as in Equation [108z] } \\
& T_{1}=\frac{1}{4} \rho \pi a b\left(a^{2}+b^{2}\right), \quad k=\frac{\left(a^{2}-b^{2}\right)^{2}}{2 a b\left(a^{2}+b^{2}\right)}
\end{aligned}
$$

Plane lamina rotating about its central axis:


$$
\begin{aligned}
& T_{1}=\frac{1}{16} \rho \pi a^{4} \omega^{2}, \quad \text { as in Equation }\left[106 a^{\prime}\right] \\
& k I_{1}^{\prime}=\frac{1}{8} \rho \pi a^{4} .
\end{aligned}
$$

## Plane lamina rotating about one edge:



$$
\begin{gathered}
\left.T_{1}=\frac{9}{16} \rho \pi a^{4} \omega^{2}, \quad \text { as in Equation [106b }{ }^{\prime}\right] \\
\text { with } \beta=1,
\end{gathered}
$$


about a generator

Fluid inside elliptic-cylindrical shell rotating about its axis:


$$
\begin{array}{ll}
T_{1}=\frac{1}{8} \rho \pi a b \frac{\left(a^{2}-b^{2}\right)^{2}}{a^{2}+b^{2}} \omega^{2}, & \text { as in Equation [105m] } \\
I_{1}^{\prime}=\frac{1}{8} \rho \pi a b\left(a^{2}+b^{2}\right), & k=\left(\frac{a^{2}-b^{2}}{a^{2}+b^{2}}\right)^{2}
\end{array}
$$

Fluid inside semicircular cylindrical shel. otating about axis of the semicircle

$$
\begin{array}{ll}
T_{1}=\frac{\pi}{4}\left(\frac{8}{\pi^{2}}-\frac{1}{2}\right) \rho a^{4} \omega^{2}, \quad a s \text { in Equation [102e], } \\
l_{1}^{\prime}=\frac{\pi}{4} \rho a^{4}, & k=2\left(\frac{8}{\pi^{2}}-\frac{1}{2}\right)=0.621 .
\end{array}
$$

Fluid inside equilateral triangular prism rotating about its central axis:


$$
\begin{array}{ll}
T_{1}=\frac{1}{80 \sqrt{3}} \rho s^{4} \omega^{2}, & \text { as in Equation [103k] } \\
I_{1}=\frac{1}{16 \sqrt{3}} \rho s^{4}, & k=\frac{2}{5}
\end{array}
$$

Lamina bent in form of circular arc, in translation at angle $\theta$ with chord:


$$
T_{1}=\frac{1}{2} \rho \pi\left(b^{2} \sin ^{2} \theta+\frac{d^{2}}{2}\right) U^{2}, \text { as in Equation [78r]. }
$$

TABLE II (continued)
Cylinder with contour consisting of two similar circular arcs; see Section 89.


$$
\text { Cross-sectional area } S=\frac{c^{2}}{\sin ^{2} \theta}[2(1-f) \pi+\sin 2 \theta]
$$

1. Translation parallel to chord AB. $r_{1}=\frac{1}{2} \rho k S U^{2}$,

$$
k=\frac{2 \pi}{3}\left(\frac{1}{f^{2}}-1\right) \frac{c^{2}}{s}-1
$$

2. Translation perpendicular to chord AB: $T_{1}=\frac{1}{2} \rho k S U^{2}, \quad k=\frac{2 \pi}{3}\left(\frac{1}{2 f^{2}}+1\right) \frac{c^{2}}{S}-1$.

Cylinder with contour formed by two similar parabolic ares meeting perpendicularly; see Section 91(d):

$$
M_{1}^{\prime}=\frac{1}{3} \rho h^{2}, \quad T_{1}=\frac{1}{2} k M_{1}^{\prime} U^{2} ;
$$

1. Translation parallel to chord $\mathrm{AB}: k=\frac{4 K^{4}}{\pi^{3}}-1=0.525$.
2. Translation perpendicular to chord $\mathrm{AB}: k=\frac{8 K^{4}}{\pi^{3}}-1=2.049$.

Here $K=1.8541$, the complete elliptic integral of modulus $\sqrt{1} / \overrightarrow{2}$.
Cylinder whose contour is formed by four equal semicircles:


$$
\begin{array}{ll}
M_{1}=\frac{1}{4}(2+\pi) \rho h^{2} ; & \text { for translation in any direction } \\
T_{1}=\frac{1}{2} k H_{1} U^{2}, & k=\frac{\pi}{2+\pi} K^{2}-1=1.100 .
\end{array}
$$

For $K$, see the preceding case. See Section 91(e).
Double circular cylinder, each cylinder of radius a; see Section 90:


$$
M_{1}^{\prime}=2 \rho \pi a^{2}
$$

1. Translation parallel to line of axes $\mathrm{AB}: T_{1}=\rho \pi a^{2} U^{2}\left(\frac{\pi^{2}}{6}-1\right), \quad k=\frac{\pi^{2}}{6}-1=0.645$.
2. Translation perpendicular to line of ares $\mathrm{AB}: T_{1}=\rho \pi a^{2} U^{2}\left(\frac{\pi^{2}}{3}-1\right), \quad k=\frac{\pi^{2}}{3}-1=2.290$.

TABLE II (continued)
Cylinder of radius a sliding along fixed plane wall; see Section 90.

$$
\begin{aligned}
& T_{1}=\frac{1}{2} \rho \pi a^{2} U^{2}\left(\frac{\pi^{2}}{3}-1\right) . \\
& M_{1}^{\prime}=\rho \pi a^{2}, \quad k=\frac{\pi^{2}}{3}-1=2.290 .
\end{aligned}
$$

Cylinder of rhombic cross-section, in translation along a diagonal; see Section 91(c).

$$
\begin{aligned}
& M_{1}^{\prime}=\rho s^{2} \sin \theta \\
& r_{1}=\frac{1}{2} k M_{1} U^{2}, \quad k=\frac{2 \theta}{\sin \theta} \frac{\Gamma(3 / 2)}{\Gamma\left(1-\frac{\theta}{2 \pi}\right) \Gamma\left(\frac{1}{2}+\frac{\theta}{2 \pi}\right)}-1 .
\end{aligned}
$$



Here $\theta$ is in radians and $\Gamma$ stands for the gamma function.
Rectangular cylinder in translation parallel to a side; see Section 91 (b) for references.

$M_{1}^{\prime \prime}=k M_{1}^{\prime}=$ apparent increase in mass,
$M_{10}^{\prime \prime}=\rho \pi w^{2} / 4$ or $M_{1}^{\prime \prime}$ for a plane lamina of width $w$.

| $h / w$ | $=0$ | 0.025 | 0.111 | 0.298 | 0.676 | 1.478 | 3.555 | 9.007 |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $M_{1}{ }^{\prime} / M_{10}^{\prime \prime}=1$ | 1.05 | 1.16 | 1.29 | 1.42 | 1.65 | 2.00 | 2.50 | 3.50 |

Circular cylinder with symmetrical fins:

$$
\begin{array}{ll}
T_{1}=\frac{1}{2} k M_{1} U^{2}, & \text { as in Equation [91g], } \\
M_{1}=\frac{1}{4} \rho \pi D^{2}, & k=1+\left(\frac{h}{D}-\frac{D}{h}\right)^{2} .
\end{array}
$$

Cylinder of radius a in translation and instantaneously coaxial with enclosing fixed cylinder of radius b:


$$
\begin{array}{ll}
T_{1}=\frac{1}{2} \rho \pi a^{2} U^{2} \frac{b^{2}+a^{2}}{b^{2}-a^{2}}, & \text { as in Equation (104f] } \\
M ;=\rho \pi a^{2}, & k=\frac{b^{2}+a^{2}}{b^{2}-a^{2}} .
\end{array}
$$

## TABLE II (continued)

Cylinder of radius $a$ in translation in any direction across axis of enclosing fixed square cylinder of side $s, a / s$ small; see Section 91(l).


$$
\begin{aligned}
& T_{1}=\frac{1}{2} \rho \pi a^{2} U^{2}\left(1+6.88 \frac{a^{2}}{s^{2}} \cdots\right), \\
& M_{1}^{\prime}=\rho \pi a^{2}, \quad k=1+6.88 \frac{a^{2}}{s^{2}} \ldots . .
\end{aligned}
$$

Cylinder of radius; a in translation in any direction near a fixed infinite wall, $a / h$ small:


$$
\begin{aligned}
& T_{1}=\frac{1}{2} \rho \pi a^{2} U^{2}\left(1+\frac{a^{2}}{2 h^{2}} \cdots\right), \quad \text { as in Equation [95g] } \\
& M_{1}^{\prime}=\rho \pi a^{2},
\end{aligned} \quad k=1+\frac{a^{2}}{2 h^{2}}+\cdots, ~ l
$$

(Only the force required to accelerate the cylinder is considered here.)
Cylinder of radius a moving symmetrically between fixed infinite walls $h$ apart, $a / h$ rather small!:


Plane lamina of width $b$ moving symmetrically between fixed infinite rigid walls $h$ apart, $b_{i}$ i rather small:


$$
T_{1}=\frac{1}{2} \rho U^{2} \frac{\pi b^{2}}{4}\left(1+\frac{\pi^{2} b^{2}}{24 h^{2}} \ldots . .\right), \quad \text { as in Equation [651] }
$$

TABLE III
(From Reference 5)

ADDED MASSES FOR THREE-DIMENSIONAL POTENTIAL FLOWS
Reference numbers are given under SOURCE.

| 209\% . | trais Slaficial Dincct: |  | SOURC ${ }_{\text {a }}$ |
| :---: | :---: | :---: | :---: |
| :- EAT Phat: <br>  | 1.rtast | $m_{1} \cdot \frac{b}{i} n=$ <br> Elicet al fryueney ui Osciatision ea <br> Mydudyanemic Mass of <br> a Circular Dise <br> dimensional frequency | 2,7 <br> Ire quency of sound ? |
| E:tin:icui Dise | A Shown |  | 4 |
| iiseto:sular fle!es | Vertical | $\begin{aligned} & m_{n}=K \pi \rho \frac{a^{2}}{h} b \\ & \frac{b / a}{1.0} \frac{K}{.173} \\ & 1.5 \\ & 2.0 \\ & 2.30 \\ & 2.5 \\ & 3.053 \\ & 3.5 \\ & 1.00 \\ & i .0 \\ & \sim \end{aligned}$ | 8 |

TABLE III (continued)

| Souy sinat |  |  | Sư:e |
| :---: | :---: | :---: | :---: |
| Tric iguerpiates | Vertical |  |  |
|  |  | $m_{n}-\frac{\rho}{3} \times \frac{(1199}{(5)}{ }^{\prime}=$ | 8 |


| 12. ejetes cif revoiution Spheres | Veresal | $m_{n}=\frac{2}{3} \pi p n^{3}$ | $1,2$ |
| :---: | :---: | :---: | :---: |
|  | Vertesal <br>  <br>  <br> 9.8 <br> 1.00 <br> 1.50 <br> 2.00 <br> 2.51 <br> 2.02 <br> 3.09 <br> 4.90 <br> 6.01 <br> 6.07 <br> 8.01 <br> 0.92 <br> 0.97 |  | $2$ |

TABLE III (continued)
$E$


TABLE III (continued)

\begin{tabular}{|c|c|c|c|}
\hline liony minpe \& TRA:ISLATEM:3L biscors: \& moncorshec has \& SOUCE \\
\hline  \& Y'16....: \&  \& 8 \\
\hline Ellipsoialiser o Free Surface \& Vertical \& \[
\begin{aligned}
\& \mathrm{m}_{\mathrm{h}}=\mathrm{K} \cdot \frac{1}{j} \pi \rho \mathrm{a} \mathrm{~b}^{2} \\
\& \mathrm{a} / \mathrm{b}=2.00 \\
\& \frac{\mathrm{~s} / 2 \mathrm{~b}}{1.00} \quad \frac{\mathrm{~K}}{2.00} \quad \frac{.13}{.005}
\end{aligned}
\] \& 8
\(E\) \\
\hline \begin{tabular}{l}
3. BODIES OF AROITRAEY SHIPE \\
Ellipsoid with Attacthed Rectanguler Flat Plotes
\end{tabular} \& Vertical \&  \& 8
E \\
\hline \begin{tabular}{l}
Elliprotd wisi Alfecied Zectenculer Fle: Flatos \\

\end{tabular} \& Vertical \&  \& 8

$E$ <br>
\hline
\end{tabular}

TABLE III (continued)

\begin{tabular}{|c|c|c|c|}
\hline - borrcibuts \& Tfa:SLATMOAL DiAECT:O:: \&  \& 30JRCE \\
\hline \begin{tabular}{l}
5!ramiru. Evir \\

\end{tabular} \& \begin{tabular}{l}
Verrical \\
ca oi Bosty Ma
\end{tabular} \& \begin{tabular}{l}
\[
\left\{\begin{array}{c}
m_{:}-1.124 \rho\left[\frac{1}{3} \pi \mathrm{ad}^{-}\right] \\
d=\frac{c+b}{2}
\end{array}\right.
\] \\
una : Sorizontal Section.
\end{tabular} \& 8 \\
\hline \begin{tabular}{l}
Strumbined Lojy \\
Area of Horicunta! "'Tail" - 20r: of
\end{tabular} \& \begin{tabular}{l}
Vortical \\
rea of Body Max
\end{tabular} \& \begin{tabular}{l}
\[
\begin{gathered}
m_{h}=6: 2 p\left[\frac{4}{3} \pi \mathrm{ad}^{2}\right] \\
d=\frac{c+b}{2}
\end{gathered}
\] \\
mum llorizontal Section.
\end{tabular} \& 8 \\
\hline \begin{tabular}{l}
"Toposdo" Type 0:dy
\[
\frac{3}{3}-5.0
\] \\
A:c, of Lomimen:al "Tail" - 10\% of
\end{tabular} \& \begin{tabular}{l}
Vectesi \\
Aren of Body Max
\end{tabular} \& \begin{tabular}{l}
\[
\infty_{!}-.818 \pi \rho b^{2}(2 n)
\] \\
mum Horirontal Section.
\end{tabular} \& 8

$E$ <br>
\hline
\end{tabular}

TABLE III (continued)


TABLE IV
(From Reference 9)

ADDED MASSES FOR THREE-DIMENSIONAL POTENTIAL FLOWS
(See Reference 9 or TABLE $V$ for notation)

## Sphere in translator motion



$$
\begin{array}{ll}
T=\frac{\pi}{8} \rho a^{3} U^{2}, & \text { as in Equation [127f], } \\
M=\frac{4}{3} \pi \rho a^{3}, & k=\frac{1}{2} .
\end{array}
$$

Sphere moving perpendicularly to infinite rigid plane boundary, $a / h$ small:

$$
\begin{aligned}
& \text { with } \alpha=0 \text {, } \\
& M^{\prime}=\frac{4}{3} \pi \rho a^{3}, \quad k=\frac{1}{2}\left(1+\frac{3}{8} \frac{a^{3}}{h^{3}}+\ldots . .\right) .
\end{aligned}
$$

Only the force required to accelerate the sphere is considered here; see Section 130.
Sphere moving parallel to infinite rigid plane boundary, $a / h$ small:

$$
\begin{array}{ll}
1 U & T=\frac{\pi}{3} \rho a^{3}\left(1+\frac{3}{16} \frac{a^{3}}{h^{3}}+\ldots . .\right) \\
U^{2}, \quad \text { as in Equation [130a] } \\
\text { with } a=90 \mathrm{deg},
\end{array} \quad \begin{array}{ll}
M^{\prime}=\frac{4}{3} \pi \rho a^{3}, & k=\frac{1}{2}\left(1+\frac{3}{16} \frac{a^{3}}{h^{3}}+\ldots .\right) .
\end{array}
$$

Sphere moving past center of fixed spherical shell:


$$
\begin{aligned}
& T=\frac{\pi}{3} \rho a^{3} \frac{b^{3}+2 a^{3}}{b^{3}-a^{3}} U^{2}, \quad \text { as in Equation [129e], } \\
& M^{\prime}=\frac{4}{3} \pi \rho a^{3}, \quad k=\frac{1}{2} \frac{b^{3}+2 a^{3}}{b^{3}-a^{3}} .
\end{aligned}
$$

## TABLE IV (continued)

Prolate spheroid (or ovary ellipsoid), $a>b ;$ see Section 137:
Let $e=$ eccentricity of sections through axis of symmetry,

$$
\begin{aligned}
& \alpha_{0}=\frac{1-e^{2}}{e^{3}}\left(\ln \frac{1+e}{1-e}-2 e\right), \\
& \beta_{0}=\frac{1-e^{2}}{e^{3}}\left(\frac{e}{1-e^{2}}-\frac{1}{2} \ln \frac{1+e}{1-e}\right) .
\end{aligned}
$$

(1) Translation "end on'":


$$
\begin{aligned}
& T=\frac{2}{3} \rho \pi a b^{2} U^{2} \frac{\alpha_{0}}{2-\alpha_{0}}, \\
& M=\frac{4}{3} \rho \pi a b^{2}, \quad k=k_{1}=\frac{\alpha_{0}}{2-\alpha_{0}} .
\end{aligned}
$$

(2) Translation "broadside on'":


$$
\begin{aligned}
& T=\frac{2}{3} \rho \pi a b^{2} U^{2} \frac{\beta_{0}}{2-\beta_{0}}, \\
& M=\frac{4}{3} \rho \pi a b^{2}, \quad k=k_{2}=\frac{\beta_{0}}{2-\beta_{0}} .
\end{aligned}
$$

(3) Rotation about an axis perpendicular to axis of symmetry:


$$
\begin{aligned}
& T=\frac{1}{2} k^{\prime} l \omega^{2}, \quad l=\frac{4}{15} \rho \pi a b^{2}\left(a^{2}+b^{2}\right), \\
& k=k^{\prime}=\frac{\left(a^{2}-b^{2}\right)^{2}\left(\beta_{0}-\alpha_{0}\right)}{\left(a^{2}+b^{2}\right)\left[2\left(a^{2}-\beta^{2}\right)-\left(a^{2}+b^{2}\right)\left(\beta_{0}-\alpha_{0}\right)\right]}
\end{aligned}
$$

## See Table A

TABLE IV (continued)

TABLE A
Coefficients of Inertia for Prolate Spheroid

| $a / b$ | $k_{1}$ <br> Translation <br> "end on"' | $k_{2}$ <br> Translation <br> "broadside on" | $k^{\prime}$ <br> Rotation about <br> Minor Axis |
| :---: | :---: | :---: | :---: |
| 1.00 | 0.500 | 0.500 | 0 |
| 1.50 | 0.305 | 0.621 | 0.094 |
| 2.00 | 0.209 | 0.702 | 0.240 |
| 2.51 | 0.156 | 0.763 | 0.367 |
| 2.99 | 0.122 | 0.803 | 0.465 |
| 3.99 | 0.082 | 0.860 | 0.608 |
| 4.99 | 0.059 | 0.895 | 0.701 |
| 6.01 | 0.045 | 0.918 | 0.764 |
| 6.97 | 0.036 | 0.933 | 0.805 |
| 3.01 | 0.029 | 0.945 | 0.840 |
| 9.02 | 0.024 | 0.954 | 0.865 |
| 9.97 | 0.021 | 0.960 | 0.883 |
| $\infty$ | 0 | 1.000 | 1.000 |

Oblate spheroid (or planetary ellipsoid), $a<b$, see Section 138, where $b=c$ :
Let $e=$ eccentricity of sections through axis of symmetry,

$$
\begin{gathered}
\alpha_{0}=\frac{2}{e^{3}}\left(e-\sqrt{1-e^{2}} \sin ^{-1} e\right), \\
\beta_{0}=\frac{1}{e^{3}}\left[\sqrt{1-e^{2}} \sin ^{-1} e-e\left(1-e^{2}\right)\right]
\end{gathered}
$$

(1) Translation "broadside on" or parallel to axis:


$$
\begin{aligned}
& T=\frac{2}{3} \rho \pi a b^{2} U^{2} \frac{\alpha_{0}}{2-\alpha_{0}}, \\
& M=\frac{4}{8} \rho \pi a b^{2}, \quad k=k_{1}=\frac{\alpha_{0}}{2-\omega_{0}} .
\end{aligned}
$$

## TABLE IV (continued)

(2) Translation "edge on" or perpendicular to axis:


$$
\begin{aligned}
& T=\frac{2}{3} \rho \pi a b^{2} U^{2} \frac{\beta_{0}}{2-\beta_{0}}, \\
& M=\frac{4}{3} \rho \pi a b^{2}, \quad k=k_{2}=\frac{\beta_{0}}{2-\beta_{0}} .
\end{aligned}
$$

(3) Rotation about axis perpendicular to axis of symmetry:


$$
\begin{aligned}
& T=\frac{1}{2} k^{\prime} l \omega^{2}, \quad I=\frac{4}{15} \rho \pi a b^{2}\left(a^{2}+b^{2}\right) \\
& k=k^{\prime}=\frac{\left(b^{2}-a^{2}\right)^{2}\left(\alpha_{0}-\beta_{0}\right)}{\left(a^{2}+b^{2}\right)\left[2\left(b^{2}-a^{2}\right)-\left(a^{2}+b^{2}\right)\left(\alpha_{0}-\beta_{0}\right)\right]}
\end{aligned}
$$

See Table 3

TABLE $B$
Coufficients of Inertia for Oblate Spheroid

| $b . a$ | $k_{2}$ <br> Translation <br> "edge on" | $k_{1}$ <br> Translation <br> "broadside on" | $k \prime$ <br> Rotation about <br> Equatorial Axis |
| :---: | :---: | :---: | :---: |
| 1.00 | 0.500 | 0.500 | 0 |
| 1.50 | 0.384 | 0.803 | 0.115 |
| 2.00 | 0.310 | 1.118 | 0.337 |
| 2.50 | 0.260 | 1.428 | 0.587 |
| 3.00 | 0.223 | 1.742 | 0.840 |
| 4.00 | 0.174 | 2.379 | 1.330 |
| 5.00 | 0.140 | 3.000 | 1.978 |
| 6.00 | 0.121 | 3.642 | 2.259 |
| 7.00 | 0.105 | 4.279 | 2.697 |
| 8.00 | 0092 | 4.915 | 3.150 |
| 9.00 | 0.084 | 5.549 | 3.697 |
| 10.00 | 0.075 | 6.183 | 4.019 |
| $\infty$ | 0.000 | $\infty$ | $\infty$ |

TABLE IV (continued)
Circular disk in translation perpendicular to its faces:

$T=\frac{4}{3} \rho a^{3} U^{2}, \quad$ as in Equation $\left[1380^{\circ}\right] ;$
$\frac{\text { (apparent increase in mass) }}{(\text { spherical mass of fluid of radius } a \text { ) }}=\frac{2}{\pi}$.
Circular disk rotating about a diameter; see Section 138:

Elliptic disk of ellipticity $e$ in translation perpendicular to its faces, $a>b$; References (240) and (235):


$$
T=\frac{2 \pi}{3 E} \rho u^{2} b U^{2}, \quad e=\frac{1}{a} \sqrt{a^{2}-b^{2}}
$$

$$
\frac{\text { (apparent increase in mass) }}{\left(\frac{4}{3} \rho \pi a^{2} b=\begin{array}{c}
\text { ellipsoidal mass of fluid with } \\
\text { axes } a, a, b)
\end{array}\right.}=k^{\prime \prime}=\frac{1}{E},
$$



| $a / b=1$ | 1.25 | 1.5 | 1.75 | 2 | 2.5 | 3 | 4 | 6 | 9 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $k^{\prime \prime}=0.637$ | 0.705 | 0.756 | 0.795 | 0.826 | 0.869 | 0.898 | 0.932 | 0.964 | 0.981 |

Ellipsoid, any ratio of the axes $a, b, c$; see Section 141:

$$
\text { Let } \begin{aligned}
\alpha_{0} & =a b c \int_{0}^{\infty} \frac{d \lambda}{\left(a^{2}+\lambda\right)^{3 / 2}\left(b^{2}+\lambda\right)^{1 / 2}\left(c^{2}+\lambda\right)^{1 / 2}}, \\
\beta_{0} & =a b c \int_{0}^{\infty} \frac{d \lambda}{\left(a^{2}+\lambda\right)^{1 / 2}\left(b^{2}+\lambda\right)^{3 / 2}\left(c^{2}+\lambda\right)^{1 / 2}},
\end{aligned}
$$

$$
\begin{aligned}
& T=\frac{8}{45} \rho a^{5} \omega^{2}, \\
& \frac{\text { (apparent increase in moment of inertia) }}{\text { (moment of inertia of sphere of fluid of }}=\frac{2}{3} \text {. } \\
& \text { radius } a \text { or } 8 \rho \pi a^{5} / 15 \text { ) }
\end{aligned}
$$

TABLE IV (continued)

$$
\gamma_{0}=a b c \int_{0}^{\infty} \frac{d \lambda}{\left(a^{2}+\lambda\right)^{1 / 2}\left(b^{2}+\lambda\right)^{1 / 2}\left(c^{2}+\lambda\right)^{3 / 2}} .
$$

(1) Translation parallel to the a-axis:


$$
\begin{aligned}
& T=\frac{2}{3} \rho \pi a b c \frac{\alpha_{0}}{2-\alpha_{0}} U^{2}, \\
& M^{\prime}=\frac{4}{3} \rho \pi a b c, \quad k=\frac{\alpha_{0}}{2-\alpha_{0}}
\end{aligned}
$$

(2) Rotation about the $a$-axis:


$$
\begin{aligned}
& T=\frac{2}{15} \rho \pi a b c \omega^{2} \frac{\left(b^{2}-c^{2}\right)^{2}\left(\gamma_{0}-\beta_{0}\right)}{2\left(b^{2}-c^{2}\right)+\left(b^{2}+c^{2}\right)\left(\beta_{0}-\gamma_{0}\right)} \\
& I=\frac{4}{15} \rho \pi a b c\left(b^{2}+c^{2}\right), \quad k^{\prime}=\frac{\left(b^{2}-c^{2}\right)^{2}\left(\gamma_{0}-\beta_{0}\right)}{2\left(b^{4}-c^{4}\right)+\left(b^{2}+c^{2}\right)^{2}\left(\beta_{0}-\gamma_{0}\right)} .
\end{aligned}
$$

For the expression of $\alpha_{0}, \beta_{0}, \gamma_{0}$ in terms of elliptic integrals, see N.A.C.A. Report 210 by Tuckerman (235) or Volume I of Duran's Aerodynamic Theory (3). Some values of $k$ and of $k^{\prime}$, distinguished by a subscript to denote the axis of the motion, were given by Zahm (174).

Fluid inside ellipsoidal shell rotating about its $a$-axis, any relative magnitudes of $a, b, c$ (see last figure):

$$
\begin{aligned}
& T=\frac{2}{15} \rho \pi a b c \frac{\left(b^{2}-c^{2}\right)^{2}}{b^{2}+c^{2}} \omega^{2} \quad \text { as in Equation [140f], } \\
& I^{\prime}=\frac{4}{15} \rho \pi a b c\left(b^{2}+c^{2}\right), \quad k=\left(\frac{b^{2}-c^{2}}{b^{2}+c^{2}}\right)^{2} .
\end{aligned}
$$

Solid of revolution formed by revolving about its axis of symmetry the limason defined by $r=b(s+\cos \theta) /\left(s^{2}-1\right)$ where $b$ and $s$ are constants. The curve for $s=1$ is a cardioid. A few values of $k$ are given by Bateman in Reference (240):


TABLE $V$
(From Reference 9)
NOTATION FOR TABLES II AND IV : SEE ALSO REFERENCE 9

| $a, b, c$ | Radius of a circle or semiaxis of an ellipse or ellipsoid, or half-width or width of a lamina |
| :---: | :---: |
| $e$ | Ellipticity |
| $k$ | Coefficient of inertia, a dimensionless constant |
| In translation, | $k=\frac{\text { apparent increase in mass }}{\text { mass of displaced fluid }} \text {; }$ |
|  | $k=\frac{2 T}{M^{\prime} U^{2}} \text { or } \frac{2 T_{1}}{M_{1}^{\prime} U^{2}}$ |
| In rotation, | $k=\frac{\text { apparent increase in moment of inertia }}{\text { moment of inertia of displaced fluid }}$ |
|  | $k=\frac{2 T}{I^{\prime} \omega^{2}} \text { or } \quad \frac{2 T_{1}}{I_{1} \omega^{2}}$ |
| 10 | Moment of inertia of displaced fluid rotating as a rigid body about the assumed axis of rotation |
| 11 | See under $T_{1}$ |
| $M^{\prime}$ | Mass of fluid displaced by body |
| $M_{i}^{\prime}$ | See under $T_{1}$ |
| $T$ | Kinetic energy of fluid |
| $T_{1}, I_{1}, M_{i}:$ | Values of $T, 1 ; M^{\prime}$ for fluid between two planes parallel to the motion and unit distance apart, in cases of twodimentional motion |
| $U$ | Velocity of translation of body |
| $\theta$ | An angle in radians |
| $\rho$ | Density of the Ruid, in dynamical units |
| $\omega$ | Angular velocity of rotation of a body, in radians per second. |

The nuid is assumed to surround the body and to be of infinite extent and at rest at infinity, except where other conditions are indicated. In regard to units, see Sections 18, 147.

$$
\mid
$$

