Fluctuation and Noise Letters
Vol. 2, No. 2 (2002) R71–R107

A REVIEW OF PARRONDO'S PARADOX

GREGORY P. HARMER and DEREK ABBOTT*
Centre for Biomedical Engineering (CBME) and EEE Department,
University of Adelaide, Adelaide, SA 5005, Australia
dabbott@eleceng.adelaide.edu.au

Received 15 May 2002
Revised 10 June 2002
Accepted 13 June 2002

Inspired by the flashing Brownian ratchet, Parrondo's games present an apparently paradoxical situation. The games can be realized as coin tossing events. Game A uses a single biased coin while game B uses two biased coins and has a state dependent rule based on the player's current capital. Playing each of the games individually causes the player to lose. However, a winning expectation is produced when randomly mixing games A and B. This phenomenon is investigated and mathematically analyzed to give explanations on how such a process is possible.

The games are expanded to become dependent on other properties rather than the capital of the player. Some of the latest developments in Parrondian ratchet or discrete-time ratchet theory are briefly reviewed.

Keywords: Parrondo's paradox; discrete-time Brownian ratchets.

1. Introduction

One of the simplest "games of chance" is the tossing of a biased coin to decide between possible outcomes, usually referred to as 'heads' or 'tails'. Information theorists have long studied these mechanisms [1], which trace back to the work of von Neumann [2]. Using rational biased coins it is possible to mimic any n-sided dice or simulate different biased coins, if n is dyadic only one coin is necessary [3–6]. Furthermore, we can make a fair roulette from biased coins [7], and Durrett *et al.* [8] demonstrate the possibility of winning from fair games.

With Parrondo's games, which can be also played using simple biased coins, we go one step further to generate a positively biased outcome from a combination of two negatively biased processes. This type of behavior is not impossible and exists in other fields. In control theory, the combination of two unstable systems can cause them to become stable [9]. In the theory of granular flow, drift can occur in

*This work was carried out at the University of Adelaide; GPH currently works for Sensor Research & Development Corporation, Orono, ME 04473, USA.

a counterintuitive direction [10, 11]. Furthermore, Pinsky and Schuetzow [12] show that switching between two transient diffusion processes in random media can form a positive recurrent process, which can be thought of as a continuous-time version of Parrondo's games.

The inspiration of Parrondo's games came from the Spanish physicist Juan M. R. Parrondo. He used the games as a pedagogical illustration of the Brownian ratchet. In one of his fields of interest, Brownian motors, he was able to see a link between mathematical games of chance and the Brownian ratchet. Using the equations for detailed balance, analogous to chemical reaction rates [13], enabled Parrondo to devise some working probabilities. Parrondo first devised the games in 1996 and presented them in unpublished form at a workshop in Torino, Italy [14] — then in 1999 the seminal paper appeared [15]. The games were named after their creator as "Parrondo's games," and the results referred to as "Parrondo's paradox."

There are a number of closely related phenomena, where physical processes drift in a counterintuitive direction, e.g. [12, 16–20]. However, Parrondo's games have gained particular attention because: (i) they are the first game-theoretic realization of such processes, (ii) in their original form, they can be directly mapped onto the workings of a flashing Brownian ratchet and (iii) the effect is strikingly counterintuitive and relatively simple to analyze.

Some workers have criticized the use of the term "paradox" — however we use it in the sense of an apparent paradox and this is comparable to existing terminology, such as in "Simpson's paradox" [21, 22], "Braess paradox" [23, 24] and "the renewal paradox" [25].

We first construct Parrondo's original games and consider aspects of fairness, capital distribution and mixed sequences. Using these results we offer several explanations and observations. In Sec. 3, we give an intuitive mathematical analysis using discrete-time Markov-chains. In Sec. 4, we introduce a history-dependent version of the games and show a similar type of analysis can be applied. Section 5 briefly covers further interesting phenomena relating to Parrondo's games.

2. Construction and Characteristics

The games about to be described are remarkably simple to construct. They only involve either a decision, or the result of a random event. In all cases, the random event can be implemented with biased coins [26].

These games are not typically associated with game theory in the von Neumann sense [27] — the difference being we do not have a free choice of what happens or a strategy to play — any choices are dictated by the rules of the game and we are merely observers of a game of chance. On the other hand, this is game theory in the Blackwell sense [28]. However, there is no *a priori* reason why Parrondo's games cannot be extended to allow choices in the von Neumann sense — this is an open area for future research.

The following subsections explain the games and the paradoxical result that occurs when the games are played in certain sequences. This is followed by a description of the characteristics of the games to enable comparisons with Brownian ratchets.

2.1. *Construction of the games*

The games can be formed using elementary probability rules, that is, we win with a probability p or lose with probability $1-p$. Such games or processes are well known and can be likened to going on a biased random walk or tossing a biased coin. Since each step of the games consists of a win-loss decision they can be formed by using a set of biased coins.

Game A is straight forward and consists of a single branch element. The probability of winning and losing is p and $1-p$ respectively.

Game B is a little more complex, as a game rule is first required to chose which of two coins to toss. It is described by the following statement. If the present capital is a multiple of an integer M, then the chance of winning is p_1, if the capital is not a multiple of M, then the chance of winning is p_2. The losing probabilities are $1-p_1$ and $1-p_2$, respectively. Thus, game B uses two coins, the rule of which one to use depends on the value of the current capital. For future reference let the coins B_1 and B_2 have winning probabilities p_1 and p_2 respectively.

The two games are represented diagrammatically in Fig. 1, using branching elements to represent decision conditions or win/loss probabilities. The notation (x, y) at the top of the branch gives the probability (i.e. biasing of the coin) or condition for taking the left and right branch respectively.

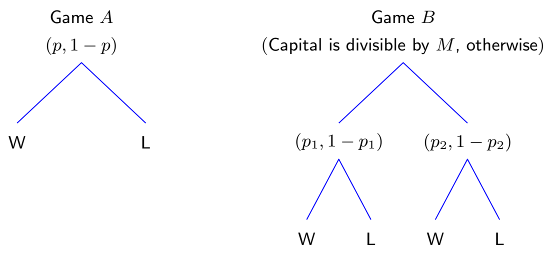

Fig. 1. Construction of Parrondo's games. The games are formed using only simple branching elements. Game A consists of winning (W) or losing (L) with respective probabilities p and $1-p$. Game B involves a decision followed by a coin toss. Both games can be implemented using a total of three biased coins.

If we require control of the three probabilities p, p_1 and p_2 via a single variable, a biasing parameter ϵ can be used to represent a subset of the parameter space with the transformation

$$\begin{aligned} p &= 1/2-\epsilon, \\ p_1 &= 1/10-\epsilon \quad \text{and} \\ p_2 &= 3/4-\epsilon. \end{aligned} \tag{1}$$

This parameterization along with $M = 3$ gives Parrondo's original numbers for the games [15]. For simplicity, most of the simulations and analysis of the games in this paper use $M = 3$, although it is not difficult in most cases to generalize for larger values of M. Where appropriate, details about trends for larger values of M are given.

2.2. *Playing the games*

The games are played between two parties where one wins at the expense of the other. To eliminate confusion we will consider playing against a common opponent, the *house* say. It is also assumed that we can have negative amounts of capital or even play with zero capital — in practice this can be avoided by providing enough initial capital to offset the desired maximum amount of loss to play up to.

The games are traditional gambling games where a bet of one unit is made and we either win a unit (plus our original bet) or lose the unit bet, the net result being either an increase or decrease in our capital by one unit for each round that is played.

It can be deduced via a detailed balance calculation (explained later in Sec. 3.1) and simulations, that both game A and game B lose when ϵ is greater than zero. Consider the scenario if we start switching between the two losing games, play two games of A, two games of B, two of A, and so on for example. The result, which is quite counterintuitive, is that we start winning! That is, we can play the two losing games A and B in such a way as to produce a winning expectation. Furthermore, deciding which game to play next by tossing a fair coin also yields a winning expectation. Figure 2 shows the average progress when playing games A and B individually, switching deterministically and stochastically between them. The switching sequence affects the gain as shown by the different finishing capitals in Fig. 2.

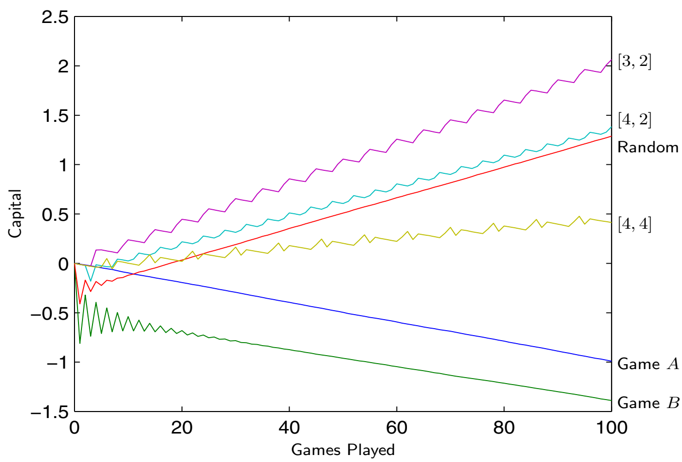

Fig. 2. Progress when playing Parrondo's games deterministically and stochastically. The simulation was performed by playing game A a times, game B b times and so on until 100 games were played, averaged over one million trials. The values of a and b are shown by the vectors $[a, b]$.

This apparent paradoxical result immediately raises questions. Firstly, the reason for the word *apparent* is that this is not a paradox in the narrowest sense; as we shall see later, it is completely explained mathematically. However, the word 'paradox' is also used to describe seemingly contradictory situation that may nonetheless be true, and is the case here. We will discuss a few issues regarding the paradox.

2.3. *Fairness*

So far, we have been careful not to mention a game as being fair, only winning or losing. The reason for this is that the behavior of game B differs from game A as we are likely to win or lose a small amount depending on the starting capital. If the starting capital is a multiple of M then we will lose a little, or conversely gain a little when the starting capital is not a multiple of M. The deviations from different values of starting capital after 30 games are shown in Fig. 3.

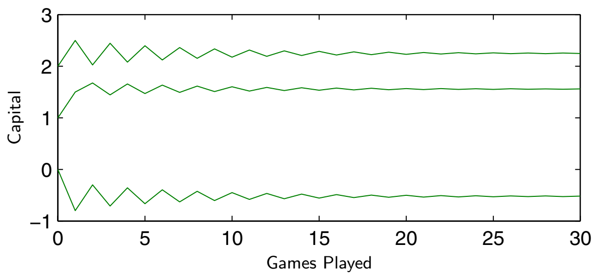

Fig. 3. Transient properties of capital in game B. Depending on the starting capital in game B, a small amount is either lost or gained. This transient effect quickly dies away after playing a small number of games.

A brief discussion of fairness follows, and a more detailed mathematical formulation relating to Parrondo's games is given by [29] and [30]. Consider a gambler repeatedly playing a game. After the nth game the gambler has capital $X(n)$, or X_n for short. Classically, a fair game satisfies [31, p. 299]

$$E[X_{n+1}|X_0, X_1, \ldots, X_n] = X_n \tag{2}$$

for $n \in \mathbb{Z}_+$. That is, the game is a *martingale*, where the expected value of capital after playing a game is the same as the present value.

The difficulty with game B, as shown by Fig. 3, is that when X_0 is a multiple of M, $E[X_1|X_0] < X_0$ and correspondingly when X_0 is not a multiple of M, $E[X_1|X_0] > X_0$. This makes it troublesome to classify game B as either winning, losing or fair [29]. Suffice to say it is argued in [29] that fairness can be defined in terms of drift rates, which were assumed in previous literature [15, 32]. Thus, if the capital tends to drift toward infinity it classifies as winning ($\epsilon < 0$), or if it drifts toward negative infinity it is classified as losing ($\epsilon > 0$). If there is no drift, then the game is fair ($\epsilon = 0$).

Therefore, using the above criterion, both games A and B are fair when ϵ is set to zero in Eq. (1). This is true of game A because the probabilities of moving up and down in capital are equal for all n. It is also true of game B even though the value of starting capital influences the probability of going up and down for small values of n. As n increases, it is clear from Fig. 3 there is no gain (i.e. drift) in capital.

Although there is some concern over whether game B is technically fair, it is not too important in the context of the paradoxical nature of the games as they definitely lose when $\epsilon > 0$. This is satisfactory since the only prerequisite we have for the paradox is that games A and B lose when $\epsilon > 0$. In summary, game B (at $\epsilon = 0$) is fair in the sense that there is zero drift in capital, even though it does not satisfy the narrowest *martingale* definition of fairness.

2.3.1. *Game B appears to be winning?*

When investigating game B *prima facie*, it can be mistakenly interpreted as a winning game, thus invalidating the paradoxical result. This is due to taking the wrong path of analysis by considering the games statistically. For example, when $M = 3$ this approach assumes that the chance $X_n \bmod 3$, which is either 0, 1 or 2, occurs with equal probability — a third each. Hence we conclude that coin B_1 is used a third of the time and coin B_2 used the remaining two thirds of the time. Then from the probabilities (1) with $\epsilon = 0$ the winning probability is

$$p_{\text{win}} = \frac{1}{3} \cdot \frac{1}{10} + \frac{2}{3} \cdot \frac{3}{4} = \frac{16}{30}, \tag{3}$$

which is greater than a half. This implies that the game B is winning, which is incorrect — it is actually fair.

The correct analysis involves employing discrete-time Markov chains, which reveals the probability in each state is not a third, but 5/13, 2/13 and 6/13 respectively. Using the correct probabilities for the coins yields the winning probability as

$$p_{\text{win}} = \frac{5}{13} \cdot \frac{1}{10} + \frac{2}{13} \cdot \frac{3}{4} + \frac{6}{13} \cdot \frac{3}{4} = \frac{1}{2}, \tag{4}$$

which correctly dictates the game is fair.

2.3.2. *Trivial paradoxical games*

There have been some claims made that Parrondo's games are not paradoxical, as the Parrondo effect can be easily replicated by simpler games. Such claims artificially contrive games that usually have two points in common:

(i) They are constructed with $M = 2$. This is equivalent to deciding what coin to use in game B based on the capital being odd or even. Thus it is possible to create a version of game B that does not use one of its branches when playing alternatively, but used when played individually, which causes the game to lose.

(ii) They have payoffs that are not simply ± 1. This makes it even easier to construct games when $M = 2$.

Due to the construction of such games, they usually work with only a few different switching combinations. A good test is to play the games randomly or reverse the order. Typically after a short number of tests a sequence is found that breaks the Parrondo effect.

Parrondo's games were physically motivated from the flashing Brownian ratchet — so we must use a *skip free* process in order to properly preserve this ratchet action and hence the constraint of a ± 1 payoff structure is necessary. Using only a single unit payoff it is *impossible* to generate Parrondo's games with $M = 2$. Physically, this creates a symmetric ratchet potential and can never support directed motion. Mathematically, it is possible to show that the constraints for Parrondo's paradox to exist cannot be satisfied, for $M = 2$, and this is shown in Sec. 5.3.

Another erroneous objection against Parrondo's games is that one could chose to trivially redefine the mixture of games A and B, as a new winning game C – and thus there is nothing special about such a winning game. This viewpoint is specious as it ignores the *raison d'etre* for the games and the central ratchet dynamics. As we shall see in Sec. 5.3 games A and B possess interesting dynamics, and the winning expectation from two losing games is a result of a *convex linear combination.* To obscure this interesting behavior by partitioning the problem in a different way, would be akin to replacing the two *Penrose tiles* with one larger single tile!

2.3.3. *Cashing in at the casino?*

An obvious application of the games would be to head to the nearest casino and get rich. Is this possible? The short answer is no.

When the rules of the games are first read, it needs to be realized that the games are not mutually exclusive. This may not be apparent at first inspection. The reason is that the games are linked through the capital. Playing one game changes the capital, which may, or may not affect the probability that is used for the next game. This value of capital can be thought of as memory. The problem is that all the games at the casino are mutually exclusive and definitely do not have any memory — playing one game does not affect other or subsequent games.

An alternative approach is to find three games that model the three coins needed for the probabilities of (1). Finding a game that has slightly less that half a chance of winning is easy, as is finding a game that only wins about 1/10 of the time. However, there are no casino games to our knowledge that win 3/4 of the time as required by one of the coins in game B.

On the other hand, biologists and physical scientists are interested in Parrondo's games because there are many instances in nature where there can be competition between processes of opposite drift — coupling between processes, memory, asymmetry and all the necessary ingredients are available for Parrondian effects to occur.

A full investigation to see if there are subprocesses within existing casino games that contain the necessary ingredients for Parrondo's paradox to occur has not been carried out. For those wishing to carry out such an analysis see Sec. 5.3, which suggests that this would reduce to searching for any signs of *convexity* in the probability parameter spaces of casino games. Until such tests are carried out, we remain agnostic as to whether there is a Parrondian loophole at the casino.

2.4. *Distributions and behavior*

This section looks at the distributions of capital after a number of games have been played. This is to check their behavior does not rapidly diverge or have some other undesirable effect. The distributions also reinforce the ideas of fairness considered in Sec. 2.3.

Several probability density functions (PDFs) have been plotted in Fig. 4. Since we must win or lose at each game, the PDF will consist of only odd or even values depending on how many games have been played and where we started. To compensate for this misleading effect a centered average has been applied denoted by the over-hat and given by

$$\hat{p}(x,n) = \frac{p(x,n-1) + 2p(x,n) + p(x,n+1)}{4}, \tag{5}$$

where x represents the capital and n the number of games played.

The PDFs in Fig. 4 show games A, B and randomized with $\epsilon = -0.1$, 0 and 0.1 after 100 games have been played. It is clear that the drifts of the distributions are dependent on the biasing parameter ϵ. When $\epsilon > 0$ the drift is to the left (the game is losing), when $\epsilon < 0$ the drift is to the right (the game is winning), and when $\epsilon = 0$ the PDF is driftless (the game is fair). As expected, the longer the games are played, the flatter the PDFs become as the standard deviation increases.

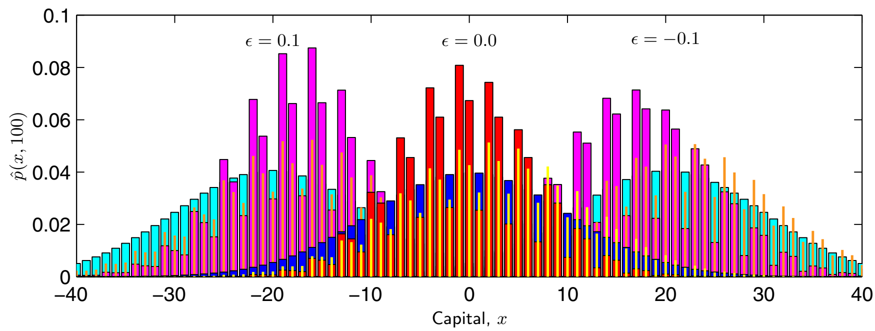

Fig. 4. Probability density functions of the games. The PDFs of the games using the probabilities in (1) with $\epsilon = -0.1$, 0 and 0.1. Games A and B are the bars with the smooth and jagged envelopes respectively, the thick lines are the randomized games.

As game A is well known, we can use its characteristics as a benchmark to judge the other two games. It can be shown that game A follows a normal distribution [15] with the following parameters, $\mathcal{N}(n(p-q), 4npq)$, where $q = 1 - p$ is the losing probability. Using the probabilities of (1), gives a mean of $-2n\epsilon$, in agreement with the PDFs in Fig. 4.

Thus, the standard deviation for game A is $2\sqrt{npq}$, which is proportional to $\sqrt{n}$. Using sample paths from Fig. 2, the standard deviations are plotted in Fig. 5 to show the proportionalities. The linear proportionality with $\sqrt{n}$ for all the games is to be expected since the PDFs resemble normal distributions. The striking result is the standard distribution of game B and the randomized game are smaller (i.e. the distributions are tighter) than that of game A even though the distributions appear more jagged. Thus, one may conclude that game B and the randomized game are as well behaved, if not more, than game A, which is considered reasonably well behaved. As will be discussed later in Sec. 2.6, game A serves to break the "pattern" in the PDF of game B, which explains the proportionalities (i.e. slopes) of the standard deviation.

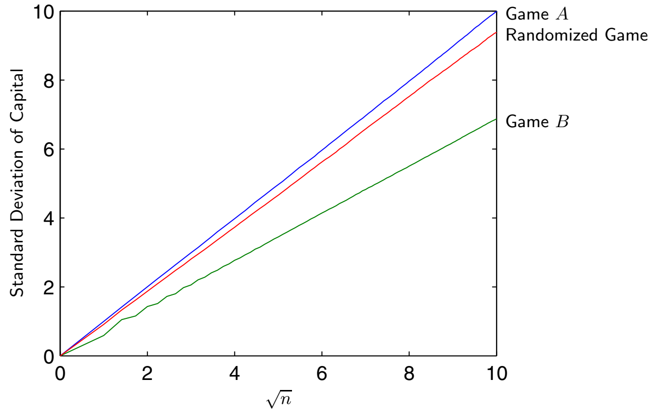

Fig. 5. Standard deviations of the games. Comparing the standard deviations of the games using 100 000 sample paths from Fig. 2. The standard deviations of the games are all proportional to $\sqrt{n}$.

2.5. *Mixing sequences*

This section explores different sequences for mixing games A and B. Firstly, we consider simple deterministic combinations of games A and B, $\{AABAAB\dots\}$ for example. There are more complex deterministic sequences that can reach higher rates of return, such as $\{ABBABABBBAB\dots\}$ for example, they will not be considered. Secondly, different stochastic mixes of games A and B are considered.

Figure 6 shows the results when mixing deterministically according to $[a, b]$. This notation refers to playing game A a times, game B b times and so on. Thus, for example $[2, 3]$ refers to the sequence $\{AABBBAABBB\dots\}$.

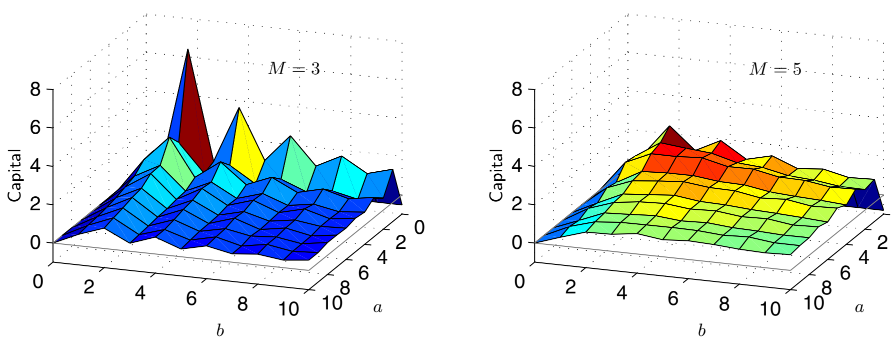

Fig. 6. Using different deterministic sequences between the games. The value of capital after the 100th game was played using the sequence determined by $[a, b]$. The probabilities of (1) with $\epsilon = 0$ were used for $M = 3$. For $M = 5$: $p = 0.5$, $p_1 = 0.1$ and $p_2 = 0.634$.

Clearly from Fig. 6, the larger the switching period the smaller the returns. This is due to playing large stretches of the same game, which as we have seen is bad for the capital. Thus, quickly switching between the games produces the best result — it is the mixing of the games that produces the gain. This is evident by the game $[1, 2]$ having the highest rate of return, followed by other fast switching games.

When $a = 0$ or $b = 0$ we expect zero capital, which is true for $b = 0$ (only playing game A), but not for $a = 0$ (only playing game B). Recall that when only playing game B a small amount is lost or gained. Since we start with zero capital, which is a multiple of M we lose a little, hence the line $a = 0$ is slightly below zero.

When mixing the games stochastically we have chosen which game to play next with fixed probability, namely a half. Let us introduce a *mixing parameter* γ with $0 \le \gamma \le 1$, which is defined as the probability of playing game A. The probability of playing game B is $1 - \gamma$. Using this parameter we can vary the proportions that each game is played; if $\gamma = 0$ then only game B is played and if $\gamma = 1$ then only game A is played. From Fig. 2, both games A and B lose individually ($\gamma = 1$ and 0), but there is a mix that wins, thus there must be a optimal value of γ that gives the best rate of return.

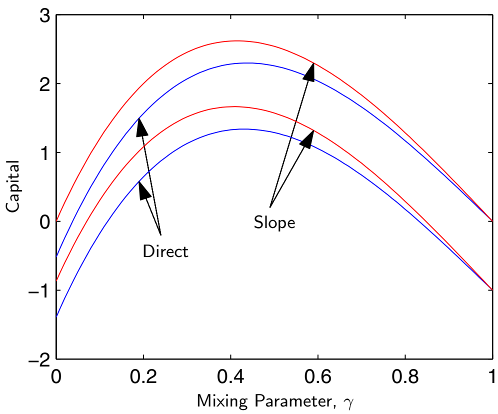

Fig. 7. Varying the mixing parameter of the randomized games. There is an optimal value of γ that gives the highest capital after playing 100 games. The 'direct' curves indicate directly playing the games and the 'slope' curves using the analytically derived slopes. The probabilities of (1) were used with $\epsilon = 0$ (top two curves) and $\epsilon = 0.005$ (bottom two curves).

Figure 7 shows the rate of return (i.e. capital after 100 games) against the mixing parameter γ. Two methods are used to determine the capital. Firstly, the expected capital when directly playing the games labeled 'Direct', and secondly using the slopes from Fig. 2 (after transients have died away) to extrapolate to 100 games labeled 'Slope'. Methods for finding the slopes are dealt with in Sec. 3.3. The difference being that the former method takes into account the initial transient behavior and is therefore affected by how many games are played, while the latter ignores all the transient effects and is independent of the number of games played. Thus, the difference between the optimal values vanishes as the number of games played approaches infinity. The effect of the biasing parameter is to shift the plots vertically. Note that although the shift is almost linear with ϵ, there are some higher order terms that vary the optimal γ by a small amount.

2.6. *Explanation of the games*

The games were originally formed as an illustration of the Brownian ratchet, so it would be expected that the games could be explained in terms of the Brownian ratchet. This section explains the games intuitively, without the use of mathematics, simply using the characteristics discussed in the previous section.

2.6.1. *Formation of the games*

The aim is to convert the physical system of the Brownian ratchet to a mathematical system. One method is to use simple gambling games. The quantities of the Brownian ratchet must be mapped to analogous mathematical variables. The capital can be used in lieu of particles in the ratchet and the probabilities in lieu of energy potentials. Initially this will be adequate to transform the system.

The easy job is to convert the flat potential, which allows diffusion equally in both directions. This is simply a fair game using a single probability, p say.

The sawtooth potential appears tricky at first, but on closer inspection is not too difficult. The sawtooth shape shown in Fig. 8 can be considered piece-wise linear, consisting of a steep positive slope (segment 1) and a gentle negative slope (segment 2). This suggests that in order to replicate this mathematically two games are required, one for each of the two segments, remembering that this represents a potential and particles will fall downwards. Segment 1 requires a game with a strong losing probability ($1-p_1$, say) and segment 2 requires a game with a weak winning probability (p_2, say). In the Brownian ratchet the particles know what segment to follow due to their spatial location, i.e. there is spatial dependency. This attribute can be mapped to the games by use of an *if* statement. If the capital is within a certain range play the first coin (coin 1), otherwise play the second coin (coin 2). This needs to be periodic, which invites the use of a modulo operator. The capital is an integral amount so the modulo can be formed over a range of capital, say M. Then within a certain range play coin 1, or if in the remaining part of the range play coin 2.

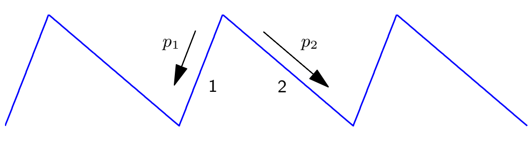

Fig. 8. Sawtooth shape representing the ratchet potential. A piece-wise linear construction consisting of two segments, which can be replicated using probabilities.

To summarize, if the capital is a multiple of M we play coin 1, otherwise we play coin 2. This forms the basis of the games. All that is required is to find values for the probabilities to make it work. A quick way is to use a detailed balance, which is explained later in Sec. 3.1.

2.6.2. *Ratchet potential of the games*

It should be of no surprise that there exists some type of ratchet potential for game B. This is evident in Fig. 4 where the sawtooth shape in the PDFs are modulated by the Gaussian function. The Gaussian shape can be removed by superimposing distributions with different starting values. The resulting distribution for games A and B are shown in Fig. 9. To emphasize the ratchet shape, $M = 7$ was used with probabilities to make the games fair, and the central average $\hat{p}(x, n)$ from (5) was employed.

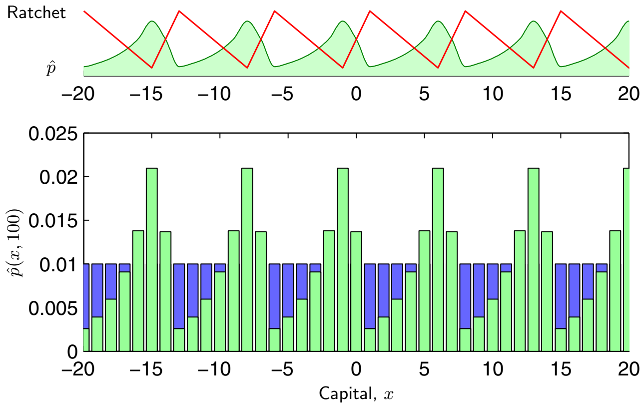

Fig. 9. Ratchet potential formed by game B. The bottom plot is the superposition of PDFs for games A and B, the flat PDF is for game A and the other game B. The top plot shows the distribution of capital in the ratchet formed by game B. Parameters are $p = 0.5$, $p_1 = 0.075$, $p_2 = 0.6032$ and $M = 7$.

From the distributions in Fig. 9, it is possible to deduce the shape of the ratchet. Consider an unknown electrostatic potential — electrodes etched onto a substrate and painted over for example. By placing a solution of charged colloidal particles on the surface, the forces cause the particles diffuse to a point of maximum attraction. The density of particles indicates directly the strength of the potential. Similarly in this system, the distribution of capital indicates directly the ratchet shape of the games, shown in the top part of Fig. 9.

2.6.3. *Distribution localization*

An alternative explanation of the two systems can be given in terms of the localization of particles or capital at system 'ceilings' [33]. Considering game B, the capital tends to localize between the $Mn-1$ and Mn states for an integer n that represents the subsystems. This is due to the chosen probabilities of p_1 and p_2. At $Mn-1$, there is a high probability (p_2) the capital will increase to Mn and at that state there is an even higher probability $(1-p_1)$ the capital will be pushed back down to $Mn-1$. Due to the oscillatory behavior between these two states and the tendency for capital to drift upwards when in states $(M(n-1)+1)$ to $(Mn-1)$, there is a localization of capital at the Mn ceilings. In the same way, the particles in the ratchet teeth are localized to the pits, a small movement is met with an opposing long gentle or short steep edge. Thus, in each subsystem an extra 'kick' is required to move the capital or particles backwards or forwards one period.

Adding game A to the playing sequence improves the chance of moving up to the next subsystem because most of the capital is localized at these ceilings. Switching to an approximately fair game allows almost half of the capital at these ceilings to move up to the next subsystem (i.e. from Mn to $M(n+1)$), while the other half moves down a few steps only to be brought back to the Mn ceiling when game B is played again. This is exactly what happens when the ratchet teeth are made to disappear in the flashing Brownian ratchet — about half of the particles can easily move over the steep edge into the next pit while the remaining particles fall back into the same pit via the gentle edge when the ratchet teeth appear again.

2.7. *Observations*

2.7.1. *Breaking the equilibrium distribution*

We deduce from the previous discussions that in order for game B to function properly, it is dependent on the ratchet shape. We will consider the games in terms of coins. For game B we have a *bad* coin B_1, and a *good* coin B_2. Even though the state-dependency allows the good coin B_1 to be played more often (i.e. when not in state Mn), the bad coin B_2 is sufficiently bad to cause the game as a whole to be losing [26].

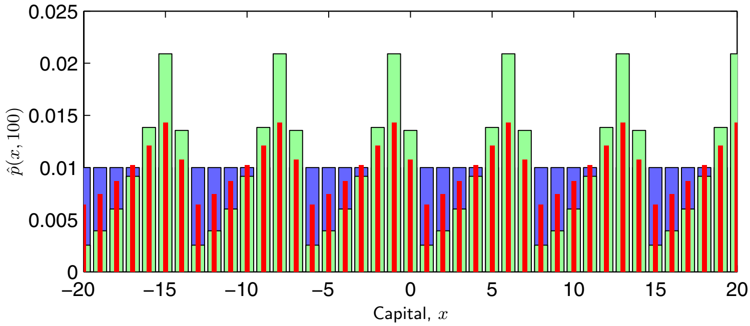

Fig. 10. The breaking of the individual PDFs by the randomized game. The randomized game is shown by the thick lines. Adding game A to B causes degradation of the ratchet shape, which increases the probability of winning.

To make this into a winning game, the distribution needs to be perturbed to alter the proportions of how many times each coin is played, that is, we want more of B_1 and less of B_2. This is accomplished by adding in game A, which creates the new PDF as shown by the thick lines in Fig. 10. This breaks the distributions formed by game B alone, and allows coin B_1 to be used more often, thus creating an overall winning game [30]. This new game needs to win sufficiently to offset the slight hindrance caused by the introduction of game A.

2.7.2. *Analogous quantities*

Observing that Parrondo's games and the flashing Brownian ratchet use the same transport mechanism, we can form analogies between the variables for each of the systems. A summary of some of the analogies is shown in Table 1.

The source of the potential in Brownian ratchets can be provided by a variety of means, but typically an electrostatic potential is used. For the games, the potential source is generated by the game rules, which define the shape of the potential. The

Table 1. The relationship between quantities in Parrondo's games and the Brownian ratchet.

Quantity	Brownian Ratchet	Parrondo's Paradox
Source of Potential	Electrostatic, Gravity	Rules of games
Switching	U_{on} and U_{off} applied	Games A and B played
Switching Durations	for τ_{on} and τ_{off}	a and b
Duration	Time	Number of games played
Biasing	Macroscopic field gradient	Parameter ϵ
Transport Quantity	Brownian particles	Capital
Measurable Output	Displacement x	Capital amount X_n
External Energy	Switching U_{on} and U_{off}	None
Potential Shape	Depends on α	Probabilities p_1, p_2 and M
Mode of Analysis	Fokker-Planck equation	Discrete-time Markov chain

capital can be thought of as the transport quantity analogous to the particles in the Brownian ratchet.

Further depth between these two analogies is found by considering the dynamics of the two systems. The Brownian ratchet is continuous in time and space. The particles can exist at any real displacement along the potential, which is flashed on and off at any real time. This is in contrast to Parrondo's games that are discrete in both the analogous time and space. The capital of the games must be an integral number of units and each game is an indivisible operation. This is highlighted by the mode of analysis — the Brownian ratchet is analyzed via continuous variables in the Fokker-Planck equation, where as Parrondo's ratchet is via discrete-time Markov chain analysis.

When we consider the ratchet and pawl machine, directed motion is only achieved when energy is added to the system, like in a heat engine. Similarly for a flashing Brownian ratchet, energy is taken up by switching between two states to produce directed motion of Brownian particles. From the simulations and mathematical analysis of Parrondo's games, the two losing games can yield a winning expectation, without any apparent cost. This creates a further paradox, "money for free." Where is the *energy* coming from in Parrondo's games? Of course, the money itself is conserved since the winnings of the player are at the expense of the losing opponent.

In stock market models, for example, switching energy could be thought of as the buying and selling transaction cost. For the flashing ratchet, the work done by the system is less than the external energy used to flash the potentials (i.e. efficiency is less than unity). This flashing is costly in the physical system as dictated by the laws of thermodynamics.

The equivalent 'law' for the Parrondian ratchet is that winning rate of the mixed A and B games can never be greater than the 'good' coin B_2 played on its own. This idea shows that the concept of 'efficiency' still applies, but the analogy breaks down in that there is no 'external energy' as such. An expanded discussion on this issue is given in [33].

3. Analysis of the Games

As has been hinted throughout the previous sections, the mode of analysis for the games is via discrete-time Markov chains (DTMCs). Each value of capital is represented by a state, and the transition probabilities between the states are determined by the rules of the games. Note though, the analysis presented here is not a rigorous one and uses only basic Markov chain theory. More in depth mathematical approaches can be found in [25, 34, 35].

For those not familiar with DTMCs an excellent introductory level textbook by Yates and Goodman [36] is recommended. More in depth information can be found in [37].

3.1. *Modeling the games as DTMCs*

The DTMC representing game A is shown in Fig. 11, where the states represent the value of capital. Since the range of capital can extend to $\pm\infty$ it is referred to as *doubly infinite*. We move up a state with probability p and down with $1-p$.

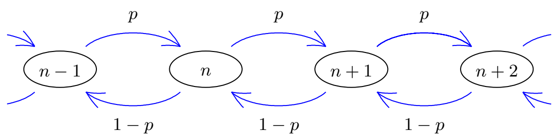

Fig. 11. Discrete-time Markov chain representing game A.

This has the transition matrix $\mathbb{P}_A$, which has zeros everywhere except for the two diagonals that are offset from the leading diagonal by ± 1.

$$\mathbb{P}_A = \begin{bmatrix} 0 & 1-p & & & (p) \\ p & 0 & 1-p & & \\ & \ddots & \ddots & \ddots & \\ & & p & 0 & 1-p \\ (1-p) & & & p & 0 \end{bmatrix}. \tag{6}$$

The DTMC in Fig. 11 extends to $\pm\infty$, hence so do the dimensions of $\mathbb{P}_A$. However, in practice the size of $\mathbb{P}_A$ can be restricted to the range of interest. For example, if we are only playing N games in a row then a $2N+1$ square matrix will suffice as the capital will not exceed the bounds of the matrix. Hence, the bracketed terms that cater for the boundary conditions are not required in practice, and are shown for completeness.

The DTMC shown in Fig. 12 represents game B. This consists of n periodic subsystems of length M, which is taken as three. This representation clearly shows the state dependency that is exhibited by game B — the probabilities leaving the $M i$th state are p_1 and $1-p_1$, while the probabilities leaving all the other states are p_2 and $1-p_2$.

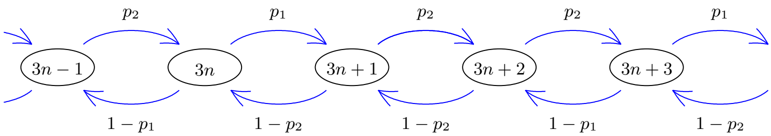

Fig. 12. Discrete-time Markov chain representing game B.

This has the transition matrix $\mathbb{P}_B$ of

$$\mathbb{P}_B = \begin{bmatrix} 0 & 1-p_2 & & & & & (p_2) \\ p_1 & 0 & \ddots & & & & \\ & p_2 & \ddots & 1-p_2 & & & \\ & & \ddots & 0 & 1-p_1 & & \\ & & & p_2 & 0 & 1-p_2 & \\ & & & & p_1 & 0 & \ddots \\ (1-p_1) & & & & & \ddots & \ddots \end{bmatrix}, \qquad (7)$$

where the losing and winning probabilities in every Mth column are $1-p_1$ and p_1 respectively. The bracketed terms are included for completeness.

By extracting the periodic subsystem from the DTMC representation in Fig. 12, the dynamics of the games can be more easily studied. The subsystem is be defined by

$$Y_n \equiv X_n \bmod M. \qquad (8)$$

Though this representation does not reveal the absolute value of capital, meaningful trends can be easily calculated. The DTMC defined by Y_n has the states $\{0, \ldots, M-1\}$, and is cyclic. That is, if we win at the highest state $M-1$ we go back to state 0 and vice versa from state 0 to $M-1$. The corresponding DTMC to Y_n is shown in Fig. 13.

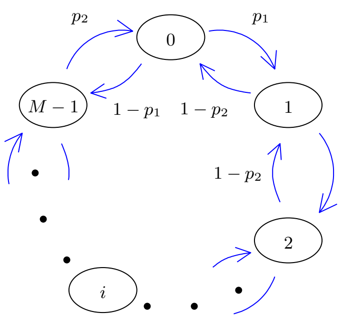

Fig. 13. Discrete-time Markov chain corresponding to the modulo game B. The cyclic DTMC defined by (8) represents one of the subsystems of the DTMC representing game B in Fig. 12.

Y_n can now also be used for game A by letting $p_1 = p_2 = p$. Having the games represented by non-infinite DTMCs allows useful calculations to be performed, as shown in subsequent sections. For example, from Onsager's Nobel prize winning work on reversible chemical reactions, we can use a detailed balance to quickly determine which direction has the greatest drift: clockwise or counter-clockwise [13, 38, 39]. The detailed balance simply entails finding the product of probabilities in one direction, and comparing this to the product of the probabilities in the opposite direction. For the DTMC in Fig. 13, clockwise is the winning direction — thus from the detailed balance game B is winning if

$$p_1 p_2^{M-1} > (1-p_1)(1-p_2)^{M-1}. \tag{9}$$

Although mathematicians may feel hesitant about this "back of the envelope" type of calculation, it does give the same result as a more formal analysis [32, 40] and is more physically intuitive.

The transition matrix representing the modulo game B in Fig. 13 is

$$\mathbb{P}_B = \begin{bmatrix} 0 & 1-p_2 & & & p_2 \\ p_1 & 0 & \ddots & & \\ & p_2 & \ddots & 1-p_2 & \\ & & \ddots & 0 & 1-p_2 \\ 1-p_1 & & & p_2 & 0 \end{bmatrix}. \tag{10}$$

This is restricted to $M \times M$ in size and the previously bracketed terms now become important as they enforce the modulo rule.

3.1.1. *The randomized game*

Dealing with the randomized game is not as difficult as it first appears. Recall that the mixing parameter γ (from Sec. 2.5) gives the relative probability of playing game A, where we have assumed the value of a half. Therefore, when the capital is a multiple of M the probability of winning is

$$q_1 = \gamma p + (1-\gamma)p_1. \tag{11a}$$

This is the chance of playing game A, multiplied by the chance of winning, added to the chance of playing game B, multiplied by the chance of winning it. Alternatively, when the capital is not a multiple of M, the probability of winning is

$$q_2 = \gamma p + (1-\gamma)p_2. \tag{11b}$$

The respective losing probabilities are $1 - q_1$ and $1 - q_2$. Using these probabilities we can treat the randomized game exactly the same as game B, except replacing the p_is with q_is. This does not affect the DTMC analysis as a combination of two DTMCs simply form another DTMC, which accordingly also obeys Markov chain theory.

3.1.2. *Playing the games analytically*

Having determined the transition matrices representing the games and using the distribution vector giving the capital in each of the states, playing the games n times gives

$$\boldsymbol{\pi}_n = \mathbb{P}^n \boldsymbol{\pi}_0, \tag{12}$$

where $\boldsymbol{\pi}_n$ is short for $\boldsymbol{\pi}(n)$, not to be confused with π_j.

If we wish to determine the distribution of capital we need to use the transition matrices that represent the doubly infinite chains. As mentioned earlier, the size can be reduced to a $2N + 1$ square matrix. Starting with zero capital gives $\boldsymbol{\pi}_0 = [\ldots, 0, 1, 0, \ldots]^T$. To play either game A or B, $\mathbb{P}_A$ or $\mathbb{P}_B$ is accordingly substituted into (12). To play a mix of the games, the appropriate transition matrix to be substituted is dependent on n. Thus, we could have $\boldsymbol{\pi}_n^{[a,b]} = \mathbb{P}_X^n \boldsymbol{\pi}_0$, using the notation $[a, b]$ described earlier and where

$$\mathbb{P}_X = \begin{cases} \mathbb{P}_A & \text{if } ((n-1) \bmod (a+b)) < a \\ \mathbb{P}_B & \text{otherwise,} \end{cases}$$

where $n = 1, 2, \ldots$.

For the random mix of games, we have $\mathbb{P}_R = \gamma \mathbb{P}_A + (1 - \gamma)\mathbb{P}_B$, which can be substituted in (12) to easily replicate the randomized games when averaged over an infinite number of trials. That is, the expected result according to the central limit theorem.

Using the technique of (12) we can determine statistical properties of the games, namely the mean μ, and standard deviation σ. We define the vector $\boldsymbol{x} = [-N, \ldots, N]$, which contains all the values of capital possible (i.e. states) when playing N games. The mean is then given by $\mu_n = \boldsymbol{x}\boldsymbol{\pi}_n$, where the matrix multiplication provides the summing. Similarly, the standard deviation is given by $\sigma_n = \sqrt{(\boldsymbol{x} - \mu_n)^2 \boldsymbol{\pi}_n}$, where the square is an element-wise operation. These results agree with the sample paths shown in Fig. 2 and standard deviations in Fig. 5.

3.1.3. *Equilibrium distribution*

The equilibrium (or stationary) distribution of a DTMC occurs when the distribution of capital in the states does not change from one game to the next. This means the distribution is invariant under the action of $\mathbb{P}$, that is, $\boldsymbol{\pi}(n+1) = \mathbb{P}\boldsymbol{\pi}(n)$. Given that at equilibrium we have $\lim_{n\to\infty} \boldsymbol{\pi}(n) = \boldsymbol{\pi}$, we need to solve $(\mathbb{I} - \mathbb{P})\boldsymbol{\pi} = 0$. Although there are many ways to solve this typical eigenvalue problem, a systematic method states that the stationary distribution is proportional to the diagonal cofactors of $\mathbb{I} - \mathbb{P}$ [34]. That is

$$\boldsymbol{\pi} = \frac{1}{D} \mathrm{diag}(\mathrm{cofac}(\mathbb{I} - \mathbb{P})), \tag{13}$$

where D is the normalization constant. The function 'diag' returns the main diagonal of a matrix and 'cofac' gives the cofactors [41, p. 373] of a matrix.

Alternatively one can use the global balance equations with $\mathbb{P}$ [35]. By either method, the stationary distribution for $M = 3$ is

$$\boldsymbol{\pi}^B = \frac{1}{D}\begin{bmatrix} 1 - p_2 + p_2^2 \\ 1 - p_2 + p_1 p_2 \\ 1 - p_1 + p_1 p_2 \end{bmatrix}, \tag{14}$$

where $D = 3 - p_1 - 2p_2 + 2p_1p_2 + p_2^2$. If we let $p_1 = p_2 = p$ to represent game A, then the stationary distribution simplifies to

$$\boldsymbol{\pi}^A = \frac{1}{3}[1, 1, 1]^T \tag{15}$$

as expected for a three state chain with identical transition probabilities.

Using the probabilities of (1) with $\epsilon = 0$, the stationary distribution for game B is found to be

$$\boldsymbol{\pi}^B = \frac{1}{13}[5, 2, 6]^T, \tag{16}$$

which are the quantities used in (4). For the randomized game with $\gamma = 1/2$, we have

$$\boldsymbol{\pi}^R = \frac{1}{709}\begin{bmatrix} 245 \\ 180 \\ 284 \end{bmatrix} = \begin{bmatrix} 0.346 \\ 0.256 \\ 0.401 \end{bmatrix}.$$

3.2. *Constraints of Parrondo's games*

It would be desirable, given a set of parameters, to find constraints to determine if Parrondo's paradox would be exhibited. An intuitive approach is finding the probability of winning using the stationary distribution, given by

$$p_{\text{win}} = \sum_{j=0}^{M-1} \pi_j p_j, \tag{17}$$

where p_j is the winning probability in state π_j. The games are winning, losing or fair when p_{win} is greater, less or equal to a half, which implies that $\langle X_n \rangle$ is a decreasing, increasing or constant with respect to n respectively.

From (17) we require $p < 1/2$ for game A, or alternatively

$$\frac{1-p}{p} > 1. \tag{18}$$

For game B, the winning probability of (17) becomes

$$\begin{aligned} p_{\text{win}}^B &= \pi_0 p_1 + \pi_1 p_2 + \cdots + \pi_{M-1} p_2 \\ &= \pi_0 p_1 + (1 - \pi_0) p_2. \end{aligned} \tag{19}$$

Using $M = 3$ to simplify the algebra and using the stationary distribution (14) with $p_{\text{win}}^B < 1/2$ yields

$$\frac{(1-p_1)(1-p_2)^2}{p_1 p_2^2} > 1. \tag{20}$$

This is the condition that needs to be satisfied for game B to be losing.

For the randomized game we use the expression for game B except replacing the p_is with q_is from (11) and setting $p^R_{\text{win}} > 1/2$ we get

$$\frac{(1-q_1)(1-q_2)^2}{q_1 q_2^2} < 1. \tag{21}$$

This is the condition for the randomized game to win. Therefore, in order for Parrondo's paradox to be exhibited we require probabilities and parameters to satisfy (18), (20) (i.e. to make game A and B lose) and (21) (i.e. make the randomized game win).

This type of analysis becomes tedious as M becomes larger due to the necessity of finding the equilibrium distribution. A more formal analysis considers the conditions of recurrence of the corresponding DTMC for the games and is given in [32, 40].

However, also notice that the numerator is the product of all the losing probabilities over one period and the denominator is the product of all the winning probabilities. This is the reverse drift over the forward drift, as also found using Onsager's detailed balance in Sec. 3.1. This approach allows us to generalize the results by replacing the exponent of 2 with $M-1$ in (20) and (21). Thus, for Parrondo's paradox to be exhibited in the general modulo M game the following inequalities need to be satisfied,

$$\begin{aligned} \frac{1-p}{p} &> 1, && \text{(22a)} \\ \frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}} &> 1 \quad \text{and} && \text{(22b)} \\ \frac{(1-q_1)(1-q_2)^{M-1}}{q_1 q_2^{M-1}} &< 1. && \text{(22c)} \end{aligned}$$

3.2.1. *Range of biasing parameter*

In Sec. 2.3, the biasing parameter was shown to control whether a game was winning or losing. In terms of Parrondo's paradox, we have shown that randomizing the games improves the performance. However, if ϵ is too large then all the games lose, albeit the randomized game does not lose by as much. Conversely if ϵ is too small (negative), then all the games win. Thus, ϵ needs to be chosen such that it biases games A and B to lose, but the improvement gained by mixing is greater than the offset made by ϵ.

By substituting the probabilities of (1) into the equations (22) we deduce a range of ϵ for which Parrondo's paradox exist. The equations (22) are respectively,

$$\begin{aligned} \epsilon &> 0, && \text{(23a)} \\ \epsilon(80\epsilon^2 - 8\epsilon + 49) &> 0 \quad \text{and} && \text{(23b)} \\ 320\epsilon^3 - 16\epsilon^2 + 229\epsilon - 3 &< 0. && \text{(23c)} \end{aligned}$$

For the quadratic part of (23b), $\triangle = b^2 - 4ac < 0$, so the roots are imaginary, meaning that $80\epsilon^2 - 8\epsilon + 49 > 0$ for all ϵ, which leaves $\epsilon > 0$. For (23c) we can numerically find the roots or use Cardan's method for cubic polynomials to deduce that there is one real and two imaginary roots. Either way the real root is $\epsilon_{\max} \approx$

0.0131, which gives the possible range of the biasing parameter as $0 < \epsilon < 0.0131$. To approach the upper limit of this range $\epsilon_{\max}$, n needs to be large to offset the initial transient behavior.

3.2.2. *Probability space*

The probability space refers to the reachable space in $\mathbb{R}^3$ by the point formed by $(p,\ p_1,\ p_2)$. Using the constraints of (22), the regions of probability space where Parrondo's paradox exists are defined.

Game A only depends on p, the winning and losing regions are separated by the plane $p = 1/2$.

Game B depends on p_1 and p_2 for a given M and the curves constituting fair games can be derived from the equality (22b). This is shown for various values of M in Fig. 14. The straight line shows the region of space that is used with the parameterization given in (1). From the trends related to ϵ discussed in Sec. 2.3, we can deduce that the region above the curves is winning, and the region below losing.

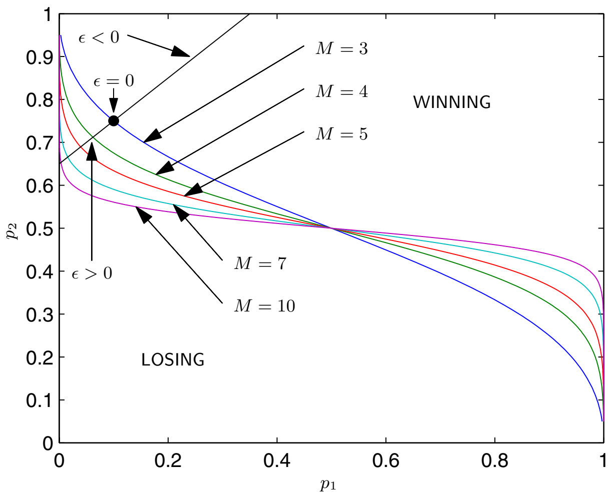

Fig. 14. Probability space of game B. The curves for various values of M divide the winning and losing regions of probability space. The straight line is the parameterization given in (1).

The only 'fair' version of game B considered so far is the one represented by the large dot in Fig. 14, but there are a continuous range of probabilities (p_1, p_2). At the extreme values of p_1 we lose or win by the greatest amount. This is due to the increasing asymmetry of the teeth in the ratchet potential, the magnitude of the slope increases at the limits of the probability space.

In Fig. 15 the surface Π_R separating the winning and losing volumes for the randomized game has been plotted, as well as the surfaces Π_A and Π_B for games A and B respectively. The arrows indicate the side of the surface where a point is required to be to satisfy (22). The point needs to be below Π_A and to the right of Π_B for games A and B to lose, but above Π_R for the randomized games to win.

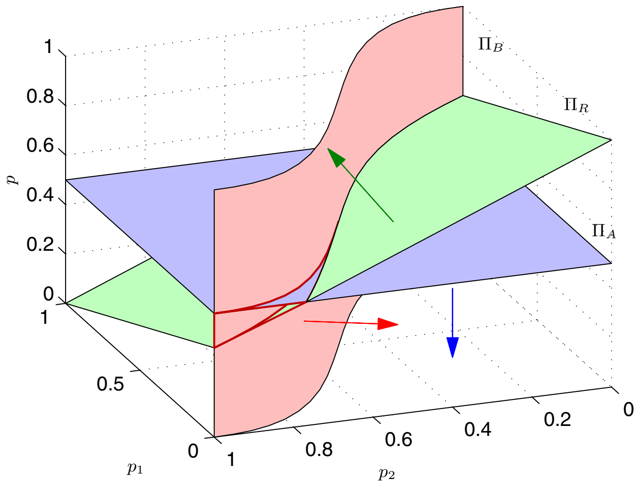

Fig. 15. Probability space of all the games showing the paradoxical region. The surfaces Π_A, Π_B and Π_R represent the boundaries between winning and losing games. The small volume also bounded by the $p_1 = 0$ plane at the front left of the plot is where Parrondo's paradoxical games exist. Parameters are $M = 3$ and $\gamma = 1/2$.

A small volume that satisfies all these constraints exists in the front left side of the plot — any point in this region will give rise to Parrondo's paradox being exhibited.

There is also a region at the opposite side of the volume that gives rise to the opposite of Parrondo's paradox. Two winning games can combine to form a losing game — this is the anti-Parrondo case. In a practical sense this is like changing the observer's perspective of the games — i.e. whether from the player's or the bank's point of view. The region in probability space that exists for Parrondo's paradox is not large, only a mere 0.032% of the whole probability space.

3.3. *Rate of winning*

From the distributions of the games it is possible to find the rate of winning as a function of the number of games played, $r(n)$. The method given here simply uses stationary distributions, whereas a more formal approach is given in [33].

Using the stationary distributions we can find the rate of winning by subtracting the probability of losing from the probability of winning. Thus, we have

$$\frac{\mathrm{d}\langle X_n \rangle}{\mathrm{d}n} \equiv r = \sum_{i=0}^{M-1} 2\pi_i p_i - 1. \tag{24}$$

For game A, $\pi_i = 1/M$ and $p_i = p$ for all i and the expression (24) reduces to $r_A = 2p - 1$, which is the expected result.

For game B the rate of winning is $r_B = 2p_2 - 1 + 2\pi_0(p_1 - p_2)$, which is valid for all M, though one still needs to find the stationary distribution to obtain π_0. For the probabilities in (1) with the stationary distribution for $M = 3$ given in (14), the slope of game B is $r_B = -1.740\epsilon + 0.119\epsilon^2 + O(\epsilon^3)$, using a Taylor series expansion with center $\epsilon_0 = 0$. This is negative for small $\epsilon > 0$, which indicates game B is losing. As suspected earlier in Fig. 7, this contains higher order nonlinearities with respect to ϵ.

The same calculation holds for the randomized game by appropriately changing the probabilities to give $0.0254 - 1.937\epsilon + 0.0136\epsilon^2 + O(\epsilon^3)$, which is positive for small ϵ, but becomes negative for larger ϵ.

When comparing simulations with this analysis the initial transient game needs to be ignored by determining the slope from the 100th to 600th game for example.

4. History Dependent Games

It has now been shown that two losing capital dependent games can win, but are there any other types of games that have this characteristic? Although state dependent games are applicable in some areas, it may be desirable to have a version of the games independent of capital. It turns out that we can answer the aforementioned question in the affirmative, in the form of history dependent games. Such games were also devised by Parrondo in [42], although other implementations are possible [29].

4.1. *Construction*

The probability of winning and losing each game depends on the result of the previous two games, as shown diagrammatically in Fig. 16.

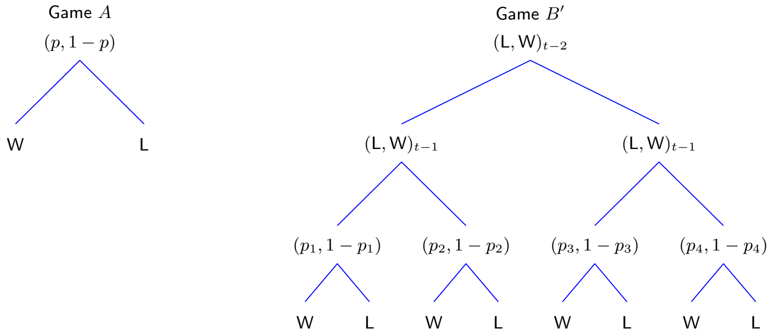

Fig. 16. Construction of Parrondo's history dependent games. The decisions are based on previous results, either winning (W) or losing (L). The games can be implemented as biased coins.

Game A is identical to the state dependent games, hence the same name. Again, game B', the counterpart to game B, is a little more complex and the probabilities depend on the two previous results. The subscript $t-1$ refers to the previous game and $t-2$ to the game prior to that. If we had previously lost then won, the next game would be played with a winning probability of p_2 according to Fig. 16.

These probabilities can be parameterized using the following transformation,

$$\begin{aligned} p &= 1/2 - \epsilon, \\ p_1 &= 9/10 - \epsilon, \\ p_2 &= p_3 = 1/4 - \epsilon \quad \text{and} \\ p_4 &= 7/10 - \epsilon. \end{aligned} \tag{25}$$

This parameterization gives Parrondo's original numbers for the history dependent games [42], which behave very similarly to the parameterization of the capital dependent games in (1). That is, the games can be considered fair when $\epsilon = 0$, losing when $\epsilon > 0$ and winning when $\epsilon < 0$. The method of analysis for the games will follow closely to that of the original capital dependent games.

4.2. *Results*

The same counterintuitive result occurs when playing games A and B', that is, when playing the games individually they lose, but switching between them creates a winning expectation. The switching can be either stochastic or deterministic as shown by the selected games plotted in Fig. 17. Similarly, there are some initial starting transients; the magnitude and shape depend on the initial conditions used, i.e. LL, LW, WL or WW. The sequences shown in Fig. 17 are averaged from each of the four possible starting conditions, thus eliminating much of the transient behavior.

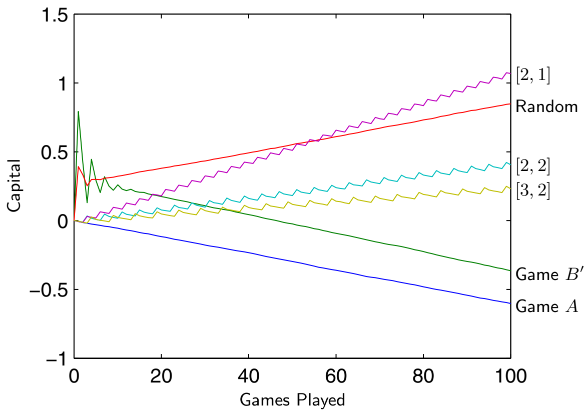

Fig. 17. Progress when playing Parrondo's history-dependent games. Simulations were performed using the probabilities in (25) with $\epsilon = 0.003$. A total of one million sample paths were averaged using each of the four initial conditions a quarter of the time.

4.3. *Analysis of the games*

To analyze the history dependent games we need to manipulate their representation into a workable DTMC. Upon doing so the same method can be employed as was used for the capital dependent games. An alternative approach using a quasi-birth-and-death process to represent the games is given in [29].

The discrete-time chain for game B' is shown in Fig. 18. The states are periodic, that is, each column-wise slice is identical. The rows represent the history of the previous two games, which are $\{\text{LL}, \text{LW}, \text{WL}, \text{WW}\}$. It is the role of the rows to keep track of the history for 'each capital' so that it knows what probability (i.e. coin) to use for the next game. The amount of capital we possess is indicated by the column

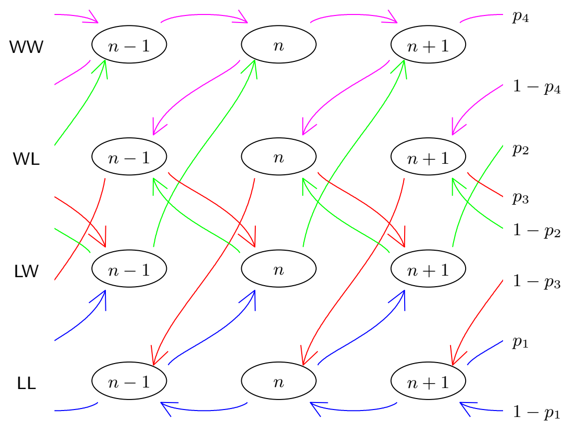

Fig. 18. Discrete-time chain for game B'. The rows encode the previous results (LL, LW, WL and WW) and the column indices hold the current value of capital X_n.

index, which simply needs to be summed. Therefore, for every game that is played we move back or forth one column and move to the appropriate row depending on the outcome of the game.

However, the chain shown in Fig. 18 is not a standard Markovian process since the next state depends not only on the previous states (i.e. the columns), but the rows as well. This is remedied by taking the periodic components of the chain in Fig. 18. This only records the past histories and not the value of capital. We can define this using the previous two states, giving the vector notation

$$Y'(n) = [X(n-1) - X(n-2), X(n) - X(n-1)] \,. \tag{26}$$

This gives four states as $[-1,-1]$, $[-1,+1]$, $[+1,-1]$ and $[+1,+1]$, where $+1$ represents a win and -1 a loss. This representation forms a discrete-time Markov chain as shown in Fig. 19.

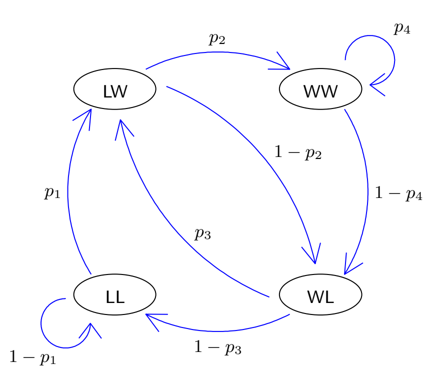

Fig. 19. Discrete-time Markov chain for game B' using previous histories. The DTMC is formed by $Y(n)$ defined in (26).

The corresponding transition matrix for the chain in Fig. 19 is

$$\mathbb{P}_{B'} = \begin{bmatrix} 1-p_1 & 0 & 1-p_3 & 0 \\ p_1 & 0 & p_3 & 0 \\ 0 & 1-p_2 & 0 & 1-p_4 \\ 0 & p_2 & 0 & p_4 \end{bmatrix}, \tag{27}$$

with the rows and columns representing the four states LL, LW, WL and WW, labeling from the top left corner. This matrix is always 4×4 since we only ever record the results of the previous two games.

When randomly mixing the games, the probabilities are given by

$$q_i = \gamma p + (1-\gamma)p_i \tag{28}$$

for $i = 1, \ldots, 4$ and γ is the mixing parameter.

4.3.1. *Equilibrium distribution*

Having represented the history dependent games as a DTMC and formed the corresponding transition matrix, the standard DTMC analysis can be performed.

The equilibrium distribution is found using (13), which gives

$$\boldsymbol{\pi}^{B'} = \frac{1}{D'} \begin{bmatrix} (1-p_3)(1-p_4) \\ (1-p_4)p_1 \\ (1-p_4)p_1 \\ p_1 p_2 \end{bmatrix}, \tag{29}$$

where the normalization constant $D' = p_1p_2 + (1+2p_1-p_3)(1-p_4)$. Using Parrondo's original probabilities of (25) with $\epsilon = 0$ gives $\boldsymbol{\pi}^{B'} = \frac{1}{22}[5, 6, 6, 5]$.

If one naively assumed that each state was occupied for a quarter of the time, the probability of winning appears to be

$$p_{\mathrm{win}} = \frac{1}{4}\left(\frac{9}{10} + \frac{1}{4} + \frac{1}{4} + \frac{7}{10}\right) = \frac{21}{40} > \frac{1}{2},$$

which is winning — like the similar analysis in the capital dependent games, this is also incorrect. The correct probability of winning using $\boldsymbol{\pi}^{B'}$ is

$$p_{\mathrm{win}} = \frac{5}{22} \cdot \frac{9}{10} + 2\left(\frac{3}{22} \cdot \frac{1}{4}\right) + \frac{5}{22} \cdot \frac{7}{10} = \frac{1}{2},$$

which is exactly fair.

4.3.2. *Paradoxical games test*

Since the construction of the games only involves using the past two results, the calculations remain tractable. Thus, we can simply use the probability of winning to find constraints for the paradox to exist.

Using

$$p_{\mathrm{win}} = \sum_{i=1}^{4} \pi_i p_i \tag{30}$$

with the stationary probabilities of game B' in (29) we get

$$p_{\text{win}} = \frac{p_1(1+p_2-p_4)}{p_1p_2+(1-p_4)(1+2p_1-p_3)}. \tag{31}$$

Subjecting this to the constraint $p_{\text{win}} > 1/2$ for a winning game or $p_{\text{win}} < 1/2$ for a losing game, we have the following conditions,

$$\begin{aligned} \frac{1-p}{p} &> 1, && \text{(32a)} \\ \frac{(1-p_3)(1-p_4)}{p_1p_2} &> 1 \quad \text{and} && \text{(32b)} \\ \frac{(1-q_3)(1-q_4)}{q_1q_2} &< 1 && \text{(32c)} \end{aligned}$$

for game A and B' to lose and the randomized game to win. Using the probabilities in (25), the range of biasing parameters possible for the paradox to exist are $0 < \epsilon < 1/168$.

4.4. *Probability space*

Similarly to the capital dependent games there exists a probability space defined by the inequalities in (32). Unfortunately these relations depend on five variables, which makes it difficult to visualize. To combat this we fix $p_2 = p_3$ and $p = 1/2$ to leave a visualizable probability space, plotted in Fig. 20. The surfaces $\Pi_{B'}$ and $\Pi_{R'}$ represent the boundaries between winning and losing for game B' and the randomized game respectively. The four bounded volumes that are created by the surfaces are denoted Q_i for $i = 1, 2, 3, 4$.

In Fig. 20, points below the surfaces are losing, and above are winning. Hence, the volumes denoted by Q_1 and Q_3 represent regions where the games are paradoxical — two losing games combine to form a winning game. The effect of making game A losing (rather than fair as shown $p = 1/2$) is to cause $\Pi_{R'}$ to lift, while $\Pi_{B'}$ is unaffected. This reduces the volumes of Q_1 and Q_3, as expected. Note, there is almost always a region in the top right corner of the probability space where the paradox appears to always exist. This is an exception as it is an extreme case where the probabilities are very near 0 or 1.

Conversely, in the other regions, Q_2 and Q_4, the anti-Parrondo case occurs — two winning games combine to form a losing game. Adjusting game A to win slightly causes $\Pi_{R'}$ to fall, and the volumes of Q_2 and Q_4 decrease as expected.

Comparing Fig. 20 with the probability space of the capital dependent games (Fig. 15 on page 22), a striking difference is its lack of symmetry. Although there are regions that have opposite properties, they are not geometrically related to each other. However, there does exist some symmetry along the boundaries of $\Pi_{B'}$ and $\Pi_{R'}$. The dashed lines at the bottom of the plot are the projected lines of interception between $\Pi_{B'}$ and $\Pi_{R'}$, although this symmetry breaks down when $p \neq 1/2$.

The other noticeable departure between the two probability spaces is the volume occupied by the paradoxical regions. Using a numerical approach, approximately

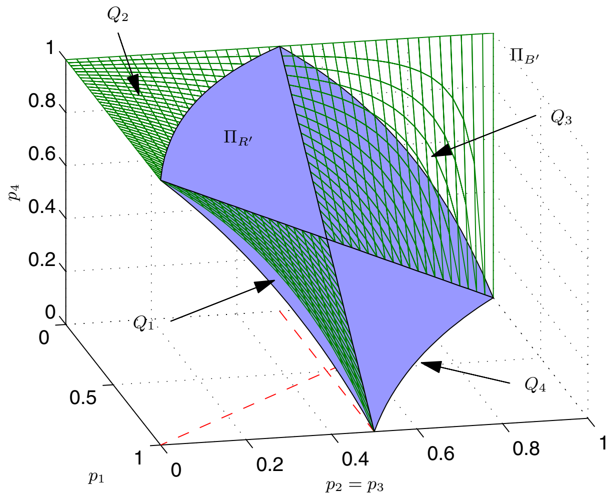

Fig. 20. Probability space of the history-dependent games. $\Pi_{B'}$ and $\Pi_{R'}$ are the surfaces separating winning and losing regions for game B' and the randomized game respectively. The paradox occurs in the volumes marked Q_1 and Q_3 and the reverse effect occurs in Q_2 and Q_4. Parameters are $\gamma = 1/2$ and $p = 1/2$.

0.032% of the probability space of the original games is occupied by the paradoxical region. This compares to a much larger 1.76% probability space for the history dependent games. The volumes are also dependent on other parameters (γ and M for example) and conditions (setting $p_2 = p_3$), but they do indicate about a 50 times difference. Caution must be exercised when using the comparisons as the entities are not that similar, i.e. $\mathbb{R}^3$ and $\mathbb{R}^5$. The rate of return, is larger for the original games (approximately 1.25 in Fig. 2 on page 4) than the history dependent games (approximately 0.8 in Fig. 17). Thus the increased volume of the paradoxical regions is a trade-off against the smaller rate of return. This is also evident in the deterministic outcomes. See Fig. 6 for example.

5. Other Phenomena

The following sections briefly cover other selected phenomena inspired by Parrondo's games.

5.1. *Co-operative Parrondo's games*

We have described games that are state dependent and history dependent. Inspired by this work, cooperative Parrondo games have been formed by Toral [43]. These games rely on the state of a player's neighbors. We refer to this state as whether a player has either lost or won the previous game.

A group of N players are arranged in a circle so each that player has two neighbors. Game A remains unchanged, thus does not have any dependencies. Game B depends on the state of the neighbors to the left and right of a player. This gives the possible states as $\{\mathrm{LL}, \mathrm{LW}, \mathrm{WL}, \mathrm{WW}\}$, where each pair is the previous state of the left and right neighbor respectively. The games are classified by

$$C_n = \sum_i C_n(i),$$

where $C_n(i)$ is the capital of the ith player after the nth game.

The results shown in [43] demonstrate that these games indeed exhibit the properties of Parrondo's paradox. Toral also considers the variance to determine if the average winning is due to a few players becoming excessively wealthy at the expense of all the other players. This is not the case as the variance grows with n as expected. Moreover, the trends in the variance shown by the cooperative games closely resemble the standard deviation of the capital dependent games in Fig. 5. Namely, the variance of the mixed games lies between that of the individual games.

5.2. *Fractal properties of Parrondo's games*

Recent work by Allison *et al.* [44] has shown that Parrondo's games exhibit fractal patterns in their state space.

Parrondo's games with $M = 3$ are considered so the state space is restricted to $\mathbb{R}^3$. The state space is defined by the equilibrium distribution $\boldsymbol{\pi}$. This is visualized in Cartesian space by letting π_0, π_1 and π_2 represent the three orthogonal axes, c.f. x, y and z. From the total law of probability we have

$$\pi_0 + \pi_1 + \pi_2 = 1, \tag{33}$$

which is the equation of a plane. The portion of the plane that satisfies the other constraint of $0 \leq \pi_i \leq 1$ is shown in Fig. 21. This shows all the possible state vectors that are allowed by the games.

Two sample paths are shown on the state space at the top of Fig. 21. Since we have used the starting distribution of $\boldsymbol{\pi} = [1, 0, 0]$ there are only two initial paths that can be taken as opposed to the usual eight. These are both shown in Fig. 21. Even after the small sequence of 40 games that were used, both trajectories reduce to similar paths. The average of these similar paths reduces to a point — the stationary distribution.

Since the state space shown in Fig. 21 is planar, it is possible to easily transform it to more appropriate axes. This is desirable to improve the visualization of the state space.

The transforms are achieved using elementary rotations and translations. The steps are listed below and shown in Fig. 21.

(i) The aim is to place the state space in the xy plane, that is, to have no z dependency. Thus, we need to place the centroid of the state space, which is the *foot of the normal* in this case, at the origin of the axes. The foot is a distance of $1/\sqrt{3}$ from the origin, so we require a translation of $-1/3$ in each direction.

(ii) Next, it is rotated about the z-axis so the base of the state space (as shown in Fig. 21) is perpendicular to the y-axis. To maintain a reference to direction, it is rotated so the 'top' of the state space is in the positive y direction. Simple geometry reveals a $-3\pi/4$ rotation is required.

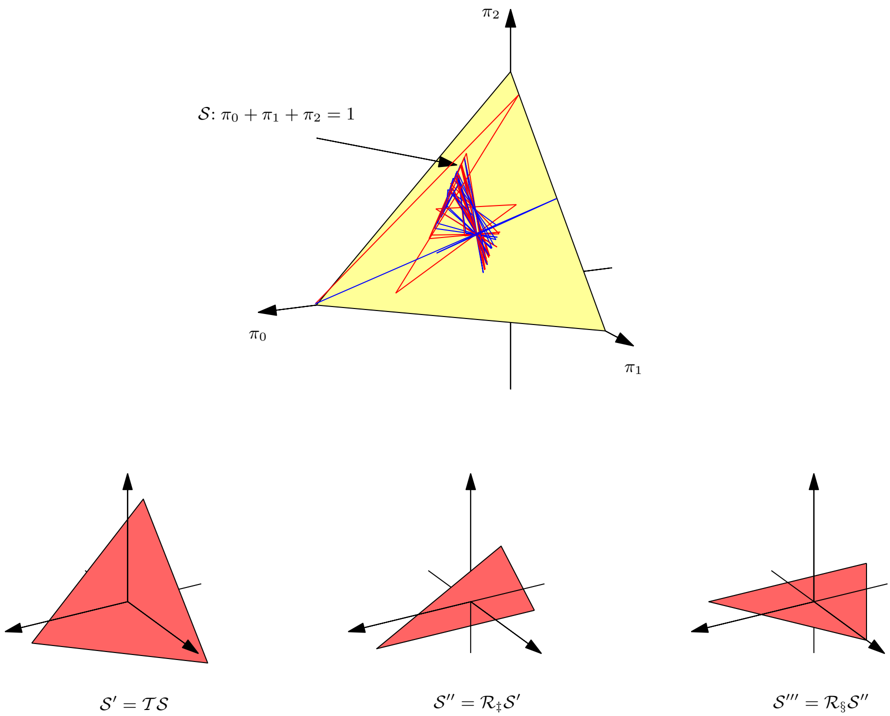

Fig. 21. State space and transformation of Parrondo's games. The shaded area in the top plot is the valid state space $\mathcal{S}$ for the games. Two sample paths of the randomized game are shown by the lines on $\mathcal{S}$. The three bottom plots are the required transformations to put $\mathcal{S}$ in the xy plane. From left to right they are a translation, a rotation and another rotation.

(iii) The last step flattens the state space into the xy plane via a rotation about the x-axis. Making a rotation of $-(\arctan\sqrt{2})$ leaves the state space oriented upwards (with respect to the y axis), with its centroid at the origin as desired.

The steps are summarized by the following transformation,

$$\begin{aligned} \tilde{\boldsymbol{\pi}} &= R_x R_z T \boldsymbol{\pi} = R\boldsymbol{\pi} \\ &= \frac{1}{6}\begin{bmatrix} -3\sqrt{2} & 3\sqrt{2} & 0 & 0 \\ -\sqrt{6} & -\sqrt{6} & 2\sqrt{6} & 4\sqrt{6}/3 \\ 2\sqrt{3} & 2\sqrt{3} & 2\sqrt{3} & 2/\sqrt{3} \\ 0 & 0 & 0 & 6 \end{bmatrix}\boldsymbol{\pi}, \end{aligned} \tag{34}$$

using the points $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, 1)$, which takes into account the translation. R_x and R_z are rotation matrices about the x and z axes respectively and T is the translation matrix. $\tilde{\boldsymbol{\pi}}$ is the transformed state space, which has the property that $\tilde{\pi}_2 = 0$ for all π. Alternatively, the top left 3×3 submatrix can be used, and the translation carried out separately.

Using the transformation of (34), the fractals for Parrondo's games can be viewed, as shown in Fig. 22. The zoomed sections demonstrate the self-similarity

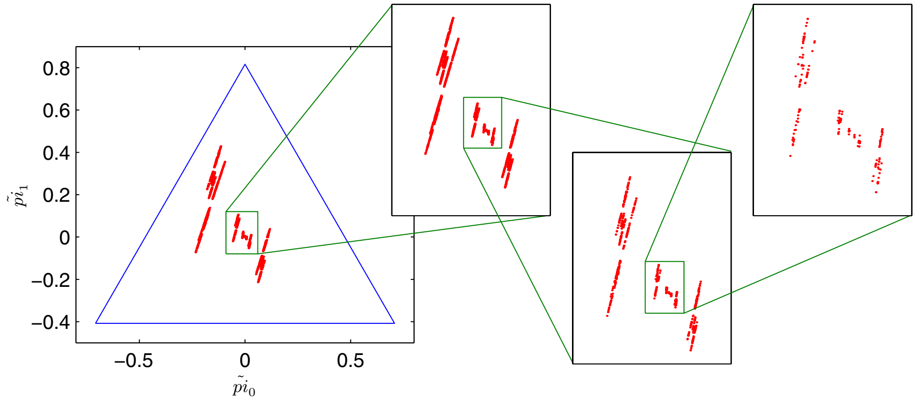

Fig. 22. Fractals in Parrondo's games. The left plot shows the transformed triangular state space with points from the randomized game using (1) with $\epsilon = 0.005$. The smaller plots to the right show zoomed sections to demonstrate the self-similarity of the fractal.

property of the fractal. As we magnify the fractal, the resolution diminishes. This is due to the limited number of games used (i.e. 10 000), and can be improved by playing more games to increase the number of sample points.

It is interesting to note that changing the biasing parameter ϵ does not vary the fractal greatly; only slightly translating and rotating it. Also for much larger values of ϵ, so the games are not paradoxical, rotations in the zoomed sections are clearly noticeable. However, changing the probability values of p_1 and p_2 produces completely new fractal patterns.

5.3. *Parrondo's games are ubiquitous*

A number of authors have used the fact that the randomized game is a linear combination of the other two games to explain the paradox [29, 34, 45]. In a generalized form the games are described by

$$G(\rho_1, \rho_2, M).$$

This describes game A by setting $\rho_1 = \rho_2 = p$, game B by setting $\rho_1 = p_1$ and $\rho_2 = p_2$ and the randomized game with $\rho_1 = \gamma p + (1-\gamma)p_1$ and $\rho_2 = \gamma p + (1-\gamma)p_2$. The parameter γ varies the relative strengths of game A and B. Using this notation we can plot the parameter space as shown in Fig. 23.

Game A only exists on the straight line indicated, whereas game B exists everywhere in the plot area. The curved line represents the manifold that separates the winning and losing games for B when $M = 3$.

If we select game A and B as the points P_1 and P_2, the randomized game can exist anywhere along the straight connecting line by varying γ. It is not hard to see that any randomized game in the highlighted section of the line will produce Parrondo's paradox. Furthermore, we can choose any two games of B and as long as the connecting line crosses the manifold twice, Parrondo's paradox will be exhibited. Thus, we can clearly see how two winning games can form a losing game.

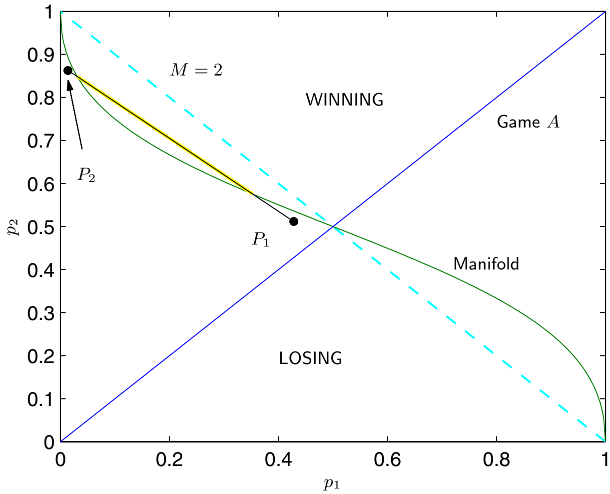

Fig. 23. The probability space of Parrondo's games in p_1 and p_2. The manifold shown is with $M = 3$ and the line segment joining P_1 and P_2 are randomized games formed by varying γ. Since the manifold is convex Parrondo's paradox exists. The manifold when $M = 2$ and the possibilities for game A are also shown.

The reason why this occurs is because the randomized game is formed by a *convex linear combination* of two other games. It is also pointed out by [29] that any game where the manifold is not a hyperplane, which include all but the simplest games, has the potential to exhibit Parrondo's paradox.

This is also the reason that games with $M = 2$, previously discussed in Sec. 2.3.2, cannot exhibit the paradox. The manifold for $M = 2$ is shown by the dashed line in Fig. 23, which is a hyperplane in $\mathbb{R}^2$.

5.4. *Related works*

There have been brief reviews of Parrondo's games in the literature, such as [20, 46], as well as some general articles [47–49] — this paper, however, represents the first extensive review. For a summary of a number of open problems in the area, refer to Ref. [33]. For a fuller mathematical treatment refer to Refs. [25, 30, 34]. The fractal properties of the games have been investigated [44] and also the optimal mixing sequences for games have been considered [50]. Arena *et al.* [51] play the games using chaotic switching sequences and they mix the capital and history dependent games together. The games have also been considered from an information entropy viewpoint [35, 52] and in terms of signal-to-noise ratio [53].

After the seminal publication of Parrondo's games in the literature, a number of papers appeared that related the games to other disciplines. These are listed as follows.

Van den Broeck *et al.* [39] use the games to give an example of a discrete "pulsating ratchet." They describe playing the games against Maxwell's demon to highlight their apparent paradoxical nature.

The games have be related to lattice gas automata [54], spin models [45], random walks and diffusions [34], biogenesis [55], molecular transport [56, 57], stochastic resonance [58], stochastic control [9, 59] and stock market modeling [60]. The switching dynamics of the games has also inspired an extension into the area of randomly induced spatial patterns [61, 62] and new types games that also exhibit counterintuitive drift [63].

There is now great interest in recasting Parrondo's games into the domain of quantum game theory. In the same way that classical games can be used to explore classical information theory, quantum games are a useful tool for investigating the relatively new field of quantum information theory. The seminal paper on the quantum Parrondo game was by Meyer and Blummer [54], where a quantum particle performs an unbiased random walk within a lattice gas.

Parrondo's history dependent game has been directly transformed into a quantum game by performing $SU(2)$ operations on qubits, which can be thought of as 'quantum coins' [64]. Lee and Johnson [65] perform a quantum Parrondo based on operations directly on the state-vector within the Bloch sphere. These investigations may have important implications for the control of decoherence and development of new quantum algorithms [65].

Finally, there have been many diverse (but equivalent) ways of explaining the operation of Parrondo's games. The physicist's choice is to think in terms of a Brownian ratchet, the mathematician tends to view it either as a convex combination or a perturbation of equilibrium probabilities, the engineer may prefer to use the Boston Interpretation that sees noise as breaking up the state-dependence within the games [66] and the physical chemist sees a noise-induced breaking of detailed balance in winning rates (*a la* Onsager).

What is remarkable is how all these valid viewpoints all compress into one simple toy model — this is one of the reasons why Parrondo's paradox has has generated such interest across so many different fields.

6. Summary

We have given a detailed explanation of Parrondo's games and the circumstances that surround their apparent paradoxical nature. The mechanism behind the games was first described heuristically in terms of the Brownian ratchet. In the same way the ratchet directs the motion of random particles, Parrondo's games use the randomized switching of the games to direct flow of capital. This analogy between the games and the ratchet allows us to compare the respective role of the variables in each of the systems. The question is, what information can the games provide about discrete time and space ratchets that cannot be extrapolated from the standard continuous ratchet? Moreover, can the games infer anything back to the continuous ratchet?

The ratchet explanation is based on the underlying physical structure. However, as one may expect, the apparent paradoxical nature of the games can be fully explained mathematically with the use of discrete-time Markov chains. This requires no knowledge of the physical origins of the games. The mathematical analysis allows us to generalize the games and determine trends that can characterize them.

The original capital dependent games were then modified to be history dependent. That is, the rules depend on the results of previous games. This type of rule maybe more appropriate for gambling games as the results are public information whereas the capital of a player is not.

Other aspects of the games were also raised, as well as some physical systems where the games can be applied. As with any toy model, what is important is the application of the principles illustrated by the model, rather than the model *per se*. One key principle is that noise or randomness can become useful when in the presence of a nonlinearity or an asymmetry. Furthermore, a random mixture of game A with game B could be thought of as a noisy process that breaks up the state dependence in game B causing it to favor the good coin. Too little of game A in the mixture or too much game A is detrimental. However, there is an optimum amount of game A that is mixed with B to maximize winnings. This points us towards the study of stochastic resonance.

Acknowledgements

Special thanks are due to A. Allison (University of Adelaide), C. Doering (University of Michigan), A.P. Flitney (University of Adelaide), J.M.R. Parrondo (Universidad Complutense de Madrid), C.E.M. Pearce (University of Adelaide), R. Pyke (University of Washington, Seattle), P.G. Taylor (University of Melbourne) and R. Toral (Universitat de les Iles Balears) for useful communications. Funding from GTECH Australasia, Sir Ross and Sir Keith Smith Fund and the Australian Research Council is gratefully acknowledged.

References

[1] J. R. Roche, *Efficient generation of random variables from biased coins*, *Tech. rep.*, Bell Tech. Rep., AT&T Lab., File Case 20878 (1992).

[2] J. von Neumann, *Various techniques used in connection with random digits*, *Applied Math Ser.* **12** (1951) 36–38.

[3] D. Feldman, R. Impagliazzo, M. Naor, N. Nisan, S. Rudich and A. Shamir, *On dice and coins: Models of computation for random generation*, *Information and Computation* **104** (1993) 159–174.

[4] T. Itoh, *Simulating fair dice with biased coins*, *Information and Computation* **126** (1996) 78–82.

[5] L. Gargamo and U. Vaccaro, *Efficient generation of fair dice with few biased coins*, *IEEE Transactions on Information Theory* **45** (1999) 1600–1606.

[6] W. Hoeffding and G. Simons, *Unbaised coin tossing with a biased coin*, *Ann. Math. Statist.* **41** (1970) 341–352.

[7] E. W. Dijkstra, *Making a fair roulette from a possibly biased coin*, *Information Processing Letters* **36** (1990) 193.

[8] R. Durrett, H. Kesten and G. Lawler, *Making money from fair games*, in Random Walks, Brownian motion, and Interacting particle systems, eds. R. Durrett and H. Kesten, Birkhäuser: Boston, vol. 28 (1991) 255–267.

[9] A. Allison and D. Abbott, *Control systems with stochastic feedback*, *Chaos* **11** (2001) 715–724.

[10] A. Rosato, K. J. Strandburg, F. Prinz and R. H. Swendsen, *Why the Brazil nuts are on top: Size segregation of particulate matter by shaking*, *Physical Review Letters* **58** (1987) 1038–1040.

[11] D. Kestenbaum, *Sand castles and cocktail nuts*, *New Scientist* **154** (1997) 25–28.

[12] R. Pinsky and M. Scheutzow, *Some remarks and examples concerning the transient and recurrence of random diffusions*, *Annales de l'Institut Henri Poincaré — Probabilités et Statistiques* **28** (1992) 519–536.

[13] L. Onsager, *Reciprocal relations in irreversible processes I*, *Physical Review* **37** (1931) 405–426.

[14] J. M. R. Parrondo, *How to cheat a bad mathematician*, in EEC HC&M Network on Complexity and Chaos (#ERBCHRX-CT940546), ISI, Torino, Italy (1996), Unpublished.

[15] G. P. Harmer and D. Abbott, *Parrondo's paradox*, *Statistical Science* **14** (1999) 206–213.

[16] A. Adjari and J. Prost, *Drift induced by a periodic potential of low symmetry: Pulsed dielectrophoresis*, *C. R. Acadamy of Science Paris, Série II* **315** (1993) 1635–1639.

[17] S. Maslov and Y. Zhang, *Optimal investment strategy for risky assests*, *Int. J. of Th. and Appl. Finance* **1** (1998) 377–387.

[18] H. V. Westerhoff, T. Y. Tsong, P. B. Chock, Y. Chen and R. D. Astumian, *How enzymes can capture and transmit free energy contained in an oscillating electric field*, *Proc. Natl. Acad. Sci.* **83** (1986) 4734–4738.

[19] E. S. Key, *Computable examples of the maximal Lyapunov exponent*, *Probab. Th. Rel. Fields* **75** (1987) 97–107.

[20] D. Abbott, *Overview: Unsolved problems of noise and fluctuations*, *Chaos* **11** (2001) 526–538.

[21] M. P. Klay and D. J. Foulis, *Maximum likelihood estimation on generalized sample spaces: An alternative resolution of Simpson's paradox*, *Foundations of Physics* **20** (1990) 777–799.

[22] E. Neufeld, *Simpson's paradox in artificial intelligence and in real life*, *Computational Intelligence* **11** (1995) 1–10.

[23] H. Kameda, E. Altman, T. Kozwaw and Y. Hosokawa, *Braess-like paradoxes in distributed computer systems*, *IEEE Transactions on Automatic Control* **45** (2000) 1687–1691.

[24] Y. A. Korilis, A. A. Lazar and A. Orda, *Avoiding the Braess paradox in non-cooperative networks*, *Journal of Applied Probability* **36** (1999) 211–222.

[25] C. E. M. Pearce, *On Parrondo paradoxical games*, in Second International Conference on Unsolved Problems of Noise and Fluctuations, Adelaide, Australia, eds. D. Abbott and L. B. Kish, American Institute of Physics (2000), 201–206.

[26] G. P. Harmer and D. Abbott, *Parrondo's paradox: Losing strategies cooperate to win*, *Nature* **402** (1999) 864.

[27] J. V. Neumann, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, USA (1944).

[28] D. Blackwell and M. A. Girshick, *Theory of Games and Statistical Decisions*, John Wiley & Sons, New York, NY, USA (1954).

[29] A. Costa, M. Fackrell and P. G. Taylor, *Two issues surrounding Parrondo's paradox*, in Annals of the International Society on Dynamic Games, ed. A. Nowack (2002), Birkhäuser. Submitted.

[30] E. S. Key, M. M. Kłosek and D. Abbott, *On Parrondo's paradox: How to construct unfair games by composing fair games* Preprint math.PR/0206151, `http://xxx.lanl.gov`.

[31] J. L. Doob, *Stochastic Processes*, John Wiley & Sons, Inc., New York (1953).

[32] G. P. Harmer, D. Abbott and P. G. Taylor, *The paradox of Parrondo's games*, *Proceedings of the Royal Society A* **456** (2000) 247–260.

[33] G. P. Harmer, D. Abbott, P. G. Taylor and J. M. R. Parrondo, *Parrondo's games and Brownian ratchets*, *Chaos* **11** (2001) 705–714.

[34] R. Pyke, *On random walks related to Parrondo's games* Preprint math.PR/0206150, `http://xxx.lanl.gov`.

[35] C. E. M. Pearce, *Entropy, Markov information sources and Parrondo games*, in Second International Conference on Unsolved Problems of Noise and Fluctuations, eds. D. Abbott and L. B. Kish, American Institute of Physics, Adelaide, Australia (2000), 207–212.

[36] R. D. Yates and D. J. Goodman, *Probability and Stochastic Processes: A Friendly Introduction for Electrical and Computer Engineers*, John Wiley & Sons (1999).

[37] J. R. Norris, *Markov Chains*, Cambridge University Press (1997).

[38] L. Onsager, *Reciprocal relations in irreversible processes II*, *Physical Review* **38** (1931) 2265.

[39] C. Van den Broeck, P. Reimann, R. Kawai and P. Hänggi, *Coupled Brownian motors*, in Lecture Notes in Physics: Statistical Mechanics of Biocomplexity, eds. D. Reguera, M. Rubí and J. M. G. Vilar, Springer-Verlag: Berlin, Heidelberg, New York (1999), vol. 527, 93–111.

[40] G. P. Harmer, D. Abbott, P. G. Taylor and J. M. R. Parrondo, *Parrondo's paradoxical games and the discrete Brownian ratchet*, in Second International Conference on Unsolved Problems of Noise and Fluctuations, Adelaide, Australia, eds. D. Abbott and L. B. Kish, American Institute of Physics (2000), vol. 511, 189–200.

[41] E. Kreyszig, *Advanced Mathematics for Engineers*, John Wiley & Sons, Inc., New York (1953).

[42] J. M. R. Parrondo, G. P. Harmer and D. Abbott, *New paradoxical games based on Brownian ratchets*, *Physical Review Letters* **85** (2000) 5226–5229.

[43] R. Toral, *Cooperative Parrondo's games*, *Fluctuation and Noise Letters* **1** (2001) L7–L12.

[44] A. Allison, C. E. M. Pearce and D. Abbott, *State-space visualisation and fractal properties of Parrondo's games* Preprint cond-mat/0205536, `http://xxx.lanl.gov`.

[45] H. Moraal, *Counterintuitive behaviour in games based on spin models*, *Journal of Physics A* **33** (2000) L203–L206.

[46] P. Reimann, *Brownian motors: Noisy transport far from equilibrium*, *Phys. Rep.* **361** (2002) 57–265.

[47] R. D. Astumian, *Making molecules into motors*, *Scientific American* **285** (2001) 57–64.

[48] E. Klarreich, *Playing both sides*, *The Sciences* **41** (2001) 25–29.

[49] J. L. Casti, *Losing games for your winning plays*, *Complexity* **6** (2001) 11–14.

[50] D. Velleman and S. Wagon, *Parrondo's paradox*, *Mathematica in Education and Research* **9** (2000) 85–89.

[51] P. Arena, S. Fazzino, L. Fortuna and P. Maniscalco, *Non linear dynamics and the Parrondo paradox* (2001). XIV Italian Meeting on Game Theory and Applications, Ischia, Italy. Unpublished.

[52] G. P. Harmer, D. Abbott, P. G. Taylor, C. E. M. Pearce and J. M. R. Parrondo, *Information entropy and Parrondo's discrete-time ratchet*, in Stochastic and Chaotic Dynamics in the Lakes, eds. D. S. Broomhead, E. Luchinskaya, P. McClintock and T. Mullin, American Institute of Physics, Melville, NY, USA (2000), vol. 502, 544–549.

[53] C. Fuh and Y. Yeh, *Random perturbation in games of chance*, *Studies in Applied Mathematics* **107** (2001) 207–215.

[54] D. A. Meyer and H. Blumer, *Parrondo games as a lattice gas automata*, *J. Stat. Phys.* **107** (2002) 225–239.

[55] P. C. W. Davies, *Physics and life: The Abdus Salam Memorial Lecture*, in Sixth Trieste Conference on Chemical Evolution, Trieste, Italy, eds. J. Chela-Flores, T. Tobias and F. Raulin, Kluwer Academic Publishers (2001), 13–20.

[56] D. Kinderlehrer and M. Kowalczyk, *Diffusion-mediated transport and the flashing ratchet*, *Arch. Ration. Mech. An.* **161** (2002) 299–309.

[57] D. Heath, D. Kinderlehrer and M. Kowalczyk, *Discrete and continuous ratchets: From coin toss to molecular motor*, *Discrete and Continuous Dynamical Systems-Series B* **2** (2002) 153–167.

[58] A. Allison and D. Abbott, *Stochastic resonance on a Brownian ratchet*, *Fluctuation and Noise Letters* **1** (2001) L239–L244.

[59] L. Kocarev and Z. Tasev, *Lyapunov exponents, noise-induced synchronization and Parrondo's paradox*, *Phys. Rev. E.* **65** (2002) 046215.

[60] M. Boman, S. J. Johansson and D. Lyback, *Parrondo strategies for artificial traders* Preprint cs.CE/0204051, `http://xxx.lanl.gov`.

[61] J. Buceta, K. Lindenberg and J. M. R. Parrondo, *Spatial patterns induced by random switching*, *Fluctuation and Noise Letters* **2** (2002) L21–L29.

[62] J. Buceta, K. Lindenberg and J. M. R. Parrondo, *Stationary and oscillatory spatial patterns induced by global periodic switching*, *Phys. Rev. Lett.* **88** (2002) 024103.

[63] B. Cleuren and C. V. den Broeck, *Random walks with absolute negative mobility*, *Phys. Rev. E* **65** (2002) 030101.

[64] A. P. Flitney, J. Ng and D. Abbott, *Quantum Parrondo's games* Preprint quant-ph/0201037, `http://xxx.lanl.gov`.

[65] C. F. Lee and N. Johnson, *Parrondo games and quantum algorithms* Preprint quant-ph/0203043, `http://xxx.lanl.gov`.

[66] Y. Lee, A. Allison, D. Abbott and H. E. Stanley, *Minimal Brownian ratchet: an exactly solvable model* Preprint cond-mat/0205302, `http://xxx.lanl.gov`.