A Review on Stress and Deformation Analysis of Curved Beams under Large Deflection

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Abstract. The paper presents a review on large deflection behavior of curved beams, as manifested through the responses under static loading. The term large deflection behavior refers to the inherent nonlinearity present in the analysis of such beam system response. The analysis leads to the field of geometric nonlinearity, in which equation of equilibrium is generally written in deformed configuration. Hence, the domain of large deflection analysis treats beam of any initial configuration as curved beam. The term curved designates the geometry of center line of beam, distinguishing it from the usual straight or circular arc configuration. Different methods adopted by researchers, to analyze large deflection behavior of beam bending, have been taken into consideration. The methods have been categorized based on their application in various formats of problems. The nonlinear response of a beam under static loading is also a function of different parameters of the particular problem. These include boundary condition, loading pattern, initial geometry of the beam, etc. In addition, another class of nonlinearity is commonly encountered in structural analysis, which is associated with nonlinear stress-strain relations and known as material nonlinearity. However the present paper mainly focuses on geometric nonlinear analysis of beam, and analysis associated with nonlinear material behavior is covered briefly as it belongs to another class of study. Research works on bifurcation instability and vibration responses of curved beams under large deflection is also excluded from the scope of the present review paper.

Introduction

Beam is a basic structural element which withstands load, transverse to its longitudinal axis, primarily by offering resistance against bending. Static analysis [1] of a beam entails determination of deflection, slope, curvature, stresses, moments, etc. developed in the beam under the specified condition of loading. Based on the analysis results, one can check whether the beam meets the requirements of offering adequate resistance to prevent failure against the applied loading condition. Traditionally, such static analysis is carried out through linear models [2] in order to simplify the analysis. But linear models cannot capture actual behavior of a structure, since almost all structures behave in some nonlinear manner prior to reaching their limit of resistance. Modern developments in the field of computational mechanics enable researchers to capture nonlinear response of such structures for better system characterization.

Two types of nonlinearities are most commonly encountered in structural analysis and they are termed as geometric and material [3]. Beam response under large deflection reveals nonlinear strain-displacement and curvature-slope relations [4] which necessitates geometric nonlinear analysis. Even for the case of a straight beam with zero initial curvature, the change in curvature in its deformed configuration calls for geometric nonlinear analysis. When the beam undergoing large displacement has initial curvature, solution of the problem becomes much more complicated. Interest in large deflection analysis of beams is ever increasing and research works on this area are numerous. The observations, findings and suggestions put forward by the researchers are classified in five major categories, as indexed below and described in detail in subsequent paragraphs.

(1) Theoretical beam models
    (i)   Classical beam theories
        • Euler-Bernoulli model
        • Winkler-Bach curved beam theory
        • Timoshenko beam theory
    (ii) Non-linear beam theories
        • Combined bending-stretching
        • Large deflection
        • Extra large deflection
    (iii) Geometrically exact beam theories
    (iv) Micro and nano scale beam theories
(2) Analysis methods
    (i)   Analysis without geometry updation
        • Analytical method
          • Approximate analytical method
        • Semi analytical method
        • Numerical method
    (ii) Analysis with geometry updation
(3) Nature of problem parameters
    (i)   Boundary condition
    (ii) Loading condition
        • Snap-through buckling and initial imperfection
        • Thermal load
    (iii) Initial geometry
(4) Type of material
    (i)   Isotropic material
    (ii) Anisotropic material
        • Functionally graded material
    (iii) Material nonlinearity in beam bending
(5) Experimental work and application areas
    (i)   Experimental work
        • Stress measurement technique
        • Displacement measurement technique
    (ii) Application areas

Theoretical beam models: Classical beam theories use continuum mechanics as the foundation and are based on Lame's differential equation of equilibrium $(\lambda + \mu)\nabla(\nabla.u) + \mu\nabla^2 u + \rho b = \rho\ddot{u}$ and Cauchy's stress tensor equation $t_i = \sigma_{ij}n_j$ for isotropic linear materials [5]. The equations relate stress σ_{ij} with the displacements u_i, and the couple stress μ_{ij} associated with the rotations are balanced through equality of cross shears. Furthermore, the equations describe associated boundary conditions in terms of displacement or force, in connection with the problem under study. Continuum mechanics based linear beam theories dominate in the field of structural mechanics over the last few centuries. However, failure of such linear theories in prediction of mechanical behavior of several natural and manmade structures motivates researchers to rethink over the existing theories. To date, several nonlinear beam models have been developed which tends to involve geometrical exactness in analysis [6-8]. However, such nonlinear theories are developed in the framework of continuum mechanics. On the other hand, non-classical beam bending behavior arises if the material is subjected to high strain gradients, e.g. at notches, holes, cracks, etc. Conventional continuum mechanics based theories overlook the microstructure of real materials and hence are insufficient for description of such physical phenomena. The non-classical

beam theories delve deep inside the micro-structure when continuum breaks down at the cellular length scale, and hence are designated by a number of important length scales [9]. Different theoretical beam models are therefore sub-classified under (i) classical, (ii) non-linear, (iii) geometrically exact and (iv) micro and nano scale theories.

Analysis methods: This section reviews different analysis methods of beams following large deflection theory. Change in geometry plays an important role in large deflection analysis and this is usually analyzed through some geometry updation technique [3]. Hence, two major sub-classifications of geometric nonlinear analysis of beam structures are analyses without geometry updation and with geometry updation. Analysis method, whether it considers geometry updation or not, is further categorized in three major categories on the basis of formulation and solution methodologies. These three categories are named as (i) analytical, (ii) semi analytical and (iii) numerical, in the present paper. Analytical method is generally employed for simple problems without much complicating effects and can take up only few classical boundary conditions like simply supported and clamped-free [1, 2]. Whereas, semi analytical method includes a variety of numerical schemes implementing for solving the differential equation derived in variational form for a relatively complicated system [3]. In variational principle, it is implied that problem formulation was made starting from an energy conservation principle, rather than the force balance principle as used in analytical methods. Variational principle can take up a wide variety of boundary conditions and in this connection distinction is made between strong and weak formulations. Numerical methods, as ascribed in the present paper, refer to solution of the system governing equation by means of some domain discretization principle. As the problem domain is discretized, numerical method provides stable solution for complicated boundary value problems [4]. However, in any particular problem the methods can exist in individual or in mixed mode in view of the formulation and solution of the problem.

Nature of problem parameters: This classification is based on the nature of problem under study and has three major sub-categories: (i) boundary condition, (ii) loading condition and (iii) initial geometry. Each of the categories includes several sub-categories, as described in the respective sections.

Type of material: Material, of which a beam is made, is further classified into major two categories based on spatial variation of its mechanical properties and they are (i) isotropic and (ii) anisotropic. However, for both the type of beam material, a further classification may be made depending on its regime of operating stress-strain condition, depicting linear elastic or nonlinear post-elastic behavior.
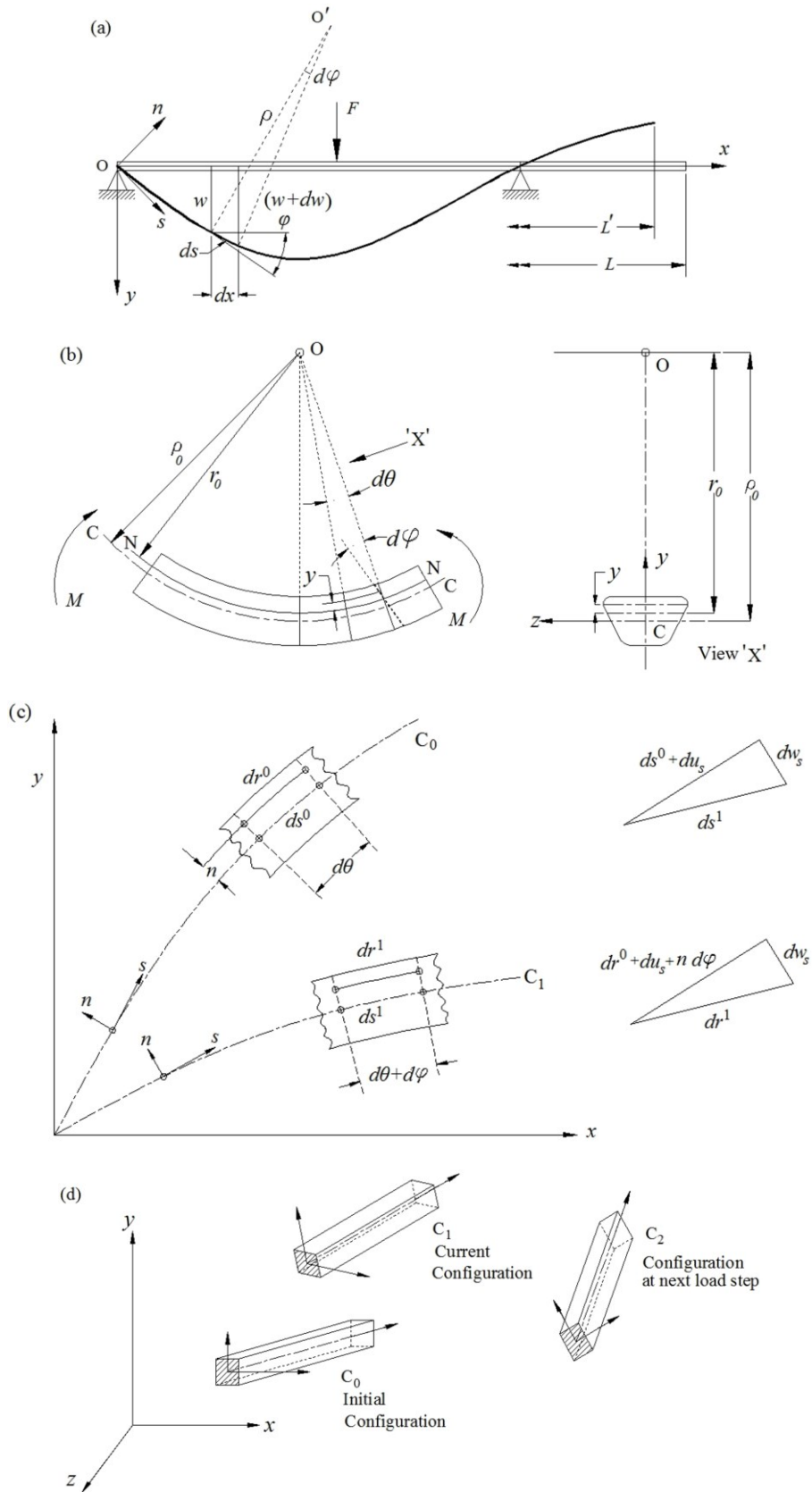
Figure 1. (a) The bend configuration of an initially straight beam, (b) initially curved beam under pure bending, (c) large deformation of a beam element and (d) deformation of beam element in three dimensional space.

Experimental work and application areas: This section investigates the experimental research works to study large deflection behavior of beam like structures under different loading conditions and, some observations and findings from actual engineering application. The section contains two sub-categories: (i) experimental research works and (ii) practical applications of large deflection studies of beams.

The number of research papers available in the subject field is enormous and hence, some most important ones are selected within the boundaries of limited scope. As mentioned earlier, geometric nonlinearities associated with large deformation are focused within the boundary of classical beam theories. The material behavior is mainly restricted to linear elastic and studies of beam-column problems involving extra-large-deflection, where in-plane force takes upper-hand over transverse one, are discussed quite briefly. On the other hand, vibration characteristic of curved beams under large deformation is a vast research area and several theoretical and experimental studies, for example the papers [10, 11], have been published. Hence, dynamic analysis of curved beam incorporating geometrical nonlinearities requires a separate review work and is altogether excluded from the scope of the present paper. From here on, rest of the paper first presents detail description of the above said aspects on geometrical nonlinear analysis of beam. After that the review work is summarized and conclusions are drawn, which are presented in suitable form for ready reference in the subject area.

Theoretical Beam Models

1. Classical beam theories

At the outset, a brief historical overview on the development of research in beam theories and a brief introduction of the subject is provided. Beam bending theory is generally attributed to Bernoulli and Euler, but the study started almost 400 years back. According to the modern studies on historical development of beam theory, Leonardo da Vinci (1493) was the first who correctly identified distribution of stress and strain across a beam section undergoing bending. Later on in 1638, Galileo addressed the problem and derived strength of beam against bending, but his prediction did not match with actual results in terms of breaking loads. However, Daniel Bernoulli together with Leonhard Euler rederived the governing equation of beam bending problem. This Euler-Bernoulli model (1750) is the classical beam theory, extensively used in structural and mechanical engineering applications. The work originated energy-minimizing principle and variational calculus, and in addition, their elastic curves played a major role in developing the theory of elliptic functions. After Bernoulli and Euler, many mathematicians, engineers and scientists provided several models, some of which are described briefly in the following sub-sections [2, 12].

Euler-Bernoulli model: When an initially straight beam is subjected to pure bending moment M, Euler-Bernoulli beam theory states that the bending moment is proportional to the change in curvature (κ) of the beam and may be written mathematically as $1/\rho = M/EI$, where I is the area moment of inertia of the beam cross-section about neutral axis, ρ ($=1/\kappa$) denotes radius of curvature of the neutral surface and E is elasticity modulus of the material [2]. In curvilinear coordinate system (s, n), curvature is given by $1/\rho = d\varphi/ds$, where φ is the change in slope at location s as shown in Fig. 1(a). When the analysis is carried out in Cartesian coordinate system (x, y), the curvature is given by $1/\rho = w''/[1+(w')^2]^{3/2}$ [1]. In classical Euler-Bernoulli beam theory, deflection of beam w is assumed small and thus curvature of the elastic curve is approximated as w''. This assumption results governing equation of beam as $w'' = M/EI$, which is a second order linear differential equation. Solution of this equation defines the shape of the deflection curve or elastic line $w = w(x)$. In case of axially varying transverse load or bending moment, the neutral surface of the bent beam does not necessarily bend in the form of circular arc.

In such non-uniform bending it is generally assumed that the Euler-Bernoulli moment-curvature relation holds at each section of the beam [2].

Winkler-Bach curved beam theory: When a curved beam is subjected to pure bending moment M, the stress developed within the beam is given by $\sigma = -M\,y\,/\,A\,e\left(r_0 - y\right)$. As shown in Fig. 1(b), y is distance of the fibre from neutral surface, A is cross-sectional area of the beam, ρ_0 and r_0 are radii of curvature of centroidal axis and neutral axis and eccentricity $e = \rho_0 - r_0$ [1, 2].

Timoshenko beam theory: Euler-Bernoulli beam theory does not consider the effect of shear deformation and as such its applicability is governed by the slenderness ratio of beam. For beams of shorter lengths, i.e., beams having smaller slenderness ratio, effect of beam thickness becomes important and cannot be neglected. In early twentieth century, Stephen Timoshenko (1921) [12] developed a theory which includes effect of shear deformation in beam bending. According to Timoshenko beam theory, total slope of the center line of beam, subjected to bending, is given by the sum of individual slopes coming from bending and shear deformation. Mathematically speaking, total slope $w' = \psi(x) + \beta(x)$, where $\psi(x)$ and $\beta(x)$ are the rotations of beam element along the center line due to bending and shear deformation effects [4].

2. Non-linear beam theories

Nonlinearity in the theory arises from a nonlinear strain-displacement relation, which may crop in through different sources: i) through combined bending and stretching, ii) when rotation of beam element is large (large deflection) and iii) when large deflection is coupled with axial stretching (extra large deflection).

Combined bending-stretching: Depending on the nature of boundary conditions, sometimes length of beam changes due to large deflection. In small deflection analysis this change of length is generally neglected, but for large deflection of slender beams this produces stretching effect in addition to bending. The combined effects of bending and stretching result nonlinear strain-displacement relation, as given by $\varepsilon_x^b = -y w_x''$ and $\varepsilon_x^m = du_x\,/\,dx + (1/2)(w_x')^2$, where ε_x^b and ε_x^m are strain due to bending and stretching respectively, w_x is transverse displacement of mid-plane of beam and u_x is in-plane displacement of mid-plane of beam [4]. As stretching effect is predominant in slender beams and shear deformation effect gets upper hand in stub beams, their combined effects are not considered in general. However, complexity of such nonlinear analysis is increased when higher beam thickness is considered and in such cases, analysis is carried out in the framework of Timoshenko beam theory [13, 14].

Large deflection: Several structural members can undergo large displacements without exceeding their specified elastic yield stress limit. For better characterization of such bending problems, analysis is carried out using large deflection theory. Euler-Bernoulli moment curvature relation holds for large deflection analysis of beam, but the square of the first derivative w' cannot be neglected in the slope-curvature formula. Thus moment-curvature relation in Cartesian coordinate system (x, y) becomes $M/EI = w''/[1 + (w')^2]^{3/2}$ [1, 3, 15]. The relation shows that deflection is no longer a linear function of bending moment [3]. However large deflection analysis of beam is generally carried out in curvilinear coordinate system (s, n), in which moment-curvature relationship is given by $M/EI = d\varphi/ds$, and thus reducing the order and degree of the derivative in this form.

Extra large deflection: The situation arises when a large deflection beam problem consist axial stretching, i.e., when axial strain is coupled with the existing nonlinear moment-curvature relation [16]. Solution of this type of problem is known to involve some geometry updation

technique. The phenomenon is shown in Fig. 1(c) for a planar beam containing line elements ds at center line and another element dr at a distance n from neutral layer. The beam displacements in (s,n) coordinate system are w_s and u_s along n and s directions respectively and neglecting higher order terms, it is easy to compute bending and stretching strains as $\varepsilon_s^b = (dr^1 - dr^0)/dr^0 = \{0.5(u_s')^2 + u_s' + 0.5(w_s')^2 + n\varphi'\}/(1 + n\theta')$　　　and　　　$\varepsilon_s^m = (ds^1 - ds^0)/ds^0 = u_s' + 0.5(u_s')^2 + 0.5(w_s')^2$.

3. Geometrically exact beam theories

Beam theories approximate three dimensional structures as one dimensional model as one of their dimensions is much larger than the other two. There exist two different ways of such mathematical abstraction of beam structures. The most classical way is an intrinsic beam formulation, where beam is considered as generalized one dimensional continuum. These theories fulfill essential mechanical principles such as equilibrium of forces and moments, conservation of energy, etc., and postulate material constitutive relation. As intrinsic beam theories are decoupled from three dimensional continuum theory, complicated three dimensional deformation characteristics of beam invalidates the theories [17]. On the other hand, induced beam theories are derived from three dimensional continuum mechanics with one characteristic direction. Such one dimensional continuum theories describe three dimensional deformation behaviors on the basis of proper kinematic, kinetic and constitutive relations [18].

Highly nonlinear mechanical behavior of several beam like structures, for example helicopter blade, fishing rod, non-metallic beam, etc., remind researchers that extensively used Euler-Bernoulli beam model was originally developed in geometrical nonlinear kinematic setting [19]. In the mid nineteenth century, Kirchhoff [18] developed a generalized beam model based on Euler-Bernoulli hypothesis to predict three dimensional deformation characteristics of arbitrarily initially curved beam. Almost after a century, Love extended the Kirchhoff model for extensible beam [18]. Later on Reissner [6, 7] introduced shear deformation in Kirchhoff-Love beam model but unfortunately decreased its exactness. On the other hand, a completely different treatise on geometrically exact nonlinear beam theories was given by Antman [8], in which beams are considered as constrained continuous bodies. In these generalized beam theories, almost any possible configuration of beams is discussed [20]. Using the idea of Antman [8], Simo [21] completed Reissner's model through proper identification of kinematic and kinetic relations in the framework of three dimensional continuum theory, while the constitutive law is postulated. Hence, the theory compromises between intrinsic and induced theories, and known as semi induced theory. This generalized beam theory, popularly known as Simo-Reissner theory, is capable to capture mechanical behavior of a beam like structure undergoing arbitrarily large deformation and strain. In another paper of Simo [22], distortion of beam cross-section was introduced in the existing model which greatly influenced efficiency of the model. Since then, Simo-Reissner beam model is being widely implemented through finite element method for analysis of complicated structural problem.

However, Simo-Reissner beam theory is unable to account for large strain until generalized nonlinear constitutive relation is used. Hence, the theory is still under controversies and sometimes termed as large deformation small strain theory [23]. Reissner beam theory is being extensively used for stress and deformation characteristics of arbitrarily curved beam undergoing large deformation and small strain. Center line of such arbitrarily curved beam in its undeformed configuration is modeled by a space curve in parametric form as $r_0(s) \in R^3$ [23-26]. Among three base vectors, two are directed along principal axes of beam cross section and the other one is along beam center line of the undeformed beam. In deformed configuration, two base vectors are still directed along principal axes but the other one generally does not act along center line for shear deformable beam. Due to high nonlinearity present in such initially curved beam structures, system governing equation is generally derived in weak form rather than in strong form, based on geometrically exact theory. Several numerical schemes are then implemented for solution purpose either considering whole domain or through domain discretization [24-30].

4. Micro and nano scale beam theories

Several micro and nano scale structures, for example Micro Electro-Mechanical system (MEMS), Nano Electro-Mechanical system (NEMS), carbon nanotubes, biosensors, micro actuators, nano probes, etc., are being extensively used nowadays. Mechanical behaviors of such small scale structures are generally predicted by modeling them as beam, having thickness typically in the order of microns [31] or nanometers [32, 33]. The size dependence of deformation behavior of micro and nano scale beams has been observed in various occasions, experimentally [34-36] and through molecular dynamics based simulation [37]. Conventional continuum mechanics based elasticity theories can not predict such size dependent deformation characteristics. Hence, rigorous analysis and design of such small scale structural elements through proper micro and nano scale beam models became an active research area in the field of structural mechanics for the last two decades.

Recently, higher-order continuum theories have been developed to predict size dependences, in which strain gradient or nonlocal terms are involved and additional material length scale parameters are consequently introduced in addition to the classical material constants. All these theories are intermediate between molecular dynamics based and continuum mechanics based modeling and basically they are a mixture of the both concepts. In the present review, only some of the higher order elasticity theories, such as micropolar (Cosserat) elasticity theory [9], couple stress elasticity theory [38, 39], strain gradient elasticity theory [40], modified strain gradient elasticity theory [36] and nonlocal elasticity theory [41, 42], are briefly presented in the following paragraph.

In non-classical beam theories proposed by Cosserat brothers [9], beam cross sections are not restricted to remain plane and rigid. Hence the couple stress, acting across a surface within a material volume or on its boundary, is incorporated in addition to the usual stress coming directly from external force. In addition to translation coming from classical continuum mechanics, local rotation is incorporated in the kinematic model. This higher order elasticity theory of Cosserat brothers [9], also known as micropolar elasticity theory, includes one additional internal length scale parameter together with two Lame's constants μ, λ for isotropic material, to include size effects in the analysis. The classical couple stress elasticity theory, originated by Mindlin and Tiersten [38], contains four material constants (two classical and two additional) for an isotropic elastic material. Based on the elastic theory, Yang et al. [39] proposed a modified couple stress theory by introducing the concept of the representative volume element, in which only symmetric rotation gradient tensor is considered and constitutive equations involve only one additional material length scale parameter. Fleck and Hutchinson [43] extended Mindlin's theory for homogeneous isotropic and incompressible materials, in which the second-order deformation gradient tensor is decomposed into two independent parts: the stretch gradient tensor and the rotation gradient tensor. Lam et al. [36] introduced another strain gradient elasticity theory using three material length scale parameters. These parameters characterize the dilatation gradient tensor, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor. Another extensively used higher order beam theory is the nonlocal continuum theory suggested by Eringen [42], which specifies the stress state at a given point as a function of the strain states at all points in the body. In recent years, several higher order beam models have been developed in the framework of micropolar elasticity theory [34], various strain gradient theories [32, 33, 37, 44-47] and Eringen's nonlocal theory [48-52] to study static [34, 45-47, 51], dynamic [31, 32, 37, 45, 49, 50, 52] and buckling [33, 44, 48] behavior of micro and nano size beam like structures.

Material behaves differently at free surface than bulk of the material. Classical continuum mechanics based structural analysis neglects such surface effect. However, in case of micro and nano sized structural elements surface energy plays significant role in total energy measure of the system, due to considerable surface to volume ratio. The most widely used mathematical framework which incorporates surface elasticity in mechanical analysis of beam was proposed by Gurtin and Murdoch [53]. In this surface elasticity theory, surface is modeled as two dimensional layer having similar displacement conditions as bulk of the material near the surface. However, kinetic behavior of surface and bulk material is completely different. Since then, several researchers [54-57] have

used the surface elasticity theory in combination with higher order continuum mechanics based beam theories for more accurate prediction of mechanical behavior of micro and nano sized beam like structures. Beam modeling based on higher order continuum theories, incorporating intrinsic size effects and surface effects, are amenable to numerical computations but their application is limited by the difficulties in calibration of the internal length scale and formulating the boundary conditions [47, 51, 54].

Analysis Methods

1. Analysis without geometry updation

In many text books on elementary strength of materials beam bending theory is included for undergraduate studies [1, 2]. However, the discussion is limited to the consideration of small deflection and in such cases a simple analytical solution is possible. However for geometric nonlinear problems simple analytical solution is impossible [15], and when initially curved beams [58] are considered, the complexity of the problems becomes much greater. In such cases approximate methods are generally applied, which over the ages has changed dramatically due to improvement in the field of computational mechanics. At present various nonlinear mathematical models and solution algorithms are employed to arrive at realistic predictions regarding the behavior of mechanical systems. These techniques, which were earlier accessible to researchers only, are well documented in some graduate level textbooks [3, 4].

The present paper mainly deals with static displacement and stress distribution in beams through geometric nonlinear model. Three major classes of analysis methods are identified in the present review work. In the first type, governing equation of beam is derived based on classical mechanics and solved mathematically by analytical equations or using some indirect transcendental approximate techniques. These approximate techniques result closed form solution in absence of nonlinear terms. In case of second type, governing equation of the system is derived by using force balance or energy balance methods, based on continuum mechanics. The nonlinear system governing equations are expressed in variational or some other special forms, which are then solved by employing simple numerical techniques. The third class of analysis method discretizes the problem domain into finite number of segments. These three classes of analysis methods are named as analytical, semi analytical and numerical methods respectively.

Analytical method: An analytical solution to any physical problem is always desirable because it provides complete information about the system behavior at any point within the solution domain. Unlike linear systems, nonlinear systems do not lend themselves to closed form solution because of the presence of nonlinear terms in governing equation. An exact solution is only possible for a few nonlinear beam problems [14, 15, 59, 60]. One of the classical analytical methods for large deflection analysis of beam is elliptic integral approach [15, 59]. In this approach, load-deflection behavior is expressed in terms of complete and incomplete elliptic integrals. These elliptic integrals are solved by expressing them either as power series or in terms of elliptic functions [60]. The difficulty associated with elliptic integral method is that it is not readily applicable for beams having geometry variation and for beams under combined loading.

When exact or closed form solution is not possible, nonlinear governing equation is solved by expressing the desired variable in the form of an expansion in integer powers of a small parameter. One of these approximate methods is perturbation method. Fundamental perturbation technique provides solution of desired unknown in form of power series in a small parameter. However, for complicated structural problem, multi-parameter perturbation [61] method is used, in which the involved parameters take care of multiple characteristics of structure. Solution of a nonlinear equation through fundamental perturbation method is possible if nonlinearity present in equation is small compared to the linear terms, i.e., the system governing equation is weakly nonlinear [61]. Geometric nonlinear analysis of non-uniform beam under complicated loading condition results strongly nonlinear governing equation. In such cases homotopy analysis method [62] is employed,

which does not depend on the scale of physical parameter and capable for better convergence of series solution. Another series approximation method, widely used in nonlinear structural mechanics, is Adomian decomposition method [63]. This approach does not simply linearize the system governing equation, but provides solution convergence for nonlinear portion of the governing equation employing Adomian polynomials. Taylor expansion technique [64, 65] is also used for large deflection analysis of beam, which provides solution for unknown displacement field in terms of unknown higher order derivatives evaluated at a suitable point along the beam axis. Large deflection analysis of beam results boundary value problem and solution of such a problem is constructed through Eigen function expansion technique [66].

Approximate analytical method: Approximate solution to geometric nonlinear beam bending problem is generally displacement based and statically admissible stress field is derived from the displacement field [67, 68]. However, stress based formulation [69, 70] is also used for analysis of large deflection behavior of beam. In stress based approach, general solution of equilibrium equation is obtained in terms of some stress functions. For two dimensional elasticity solution of beam bending problem, stresses are expressed in terms of Airy's stress function [69, 70]. When three dimensional analysis is carried out, other stress functions such as Maxwell's stress functions are used.

Use of force balance to formulate large deflection behavior of beam is well established in classical mechanics, in which compatibility condition and constitutive law are considered. Governing equation of large deflection beam problem is generally derived in the framework of Euler-Bernoulli beam theory [71-76]. Mathematical manipulation transforms the nonlinear governing equation in some special forms. In case of elliptic integral [74-76], instead of using solution in terms of elliptic functions, the complete and incomplete elliptic integrals are evaluated by using some iterative scheme [75, 76]. For example, as implemented through computational platform of Mathematica [74], the analytical elliptic integral approach becomes approximate analytical. In another approach, two point boundary value problem is converted to an initial value problem by estimating slope of beam at particular position as one of the required initial condition for a particular load step. The initial value problem is then solved using several iterative shooting techniques [77, 78]. The problem associated with such iterative process is convergence of the solution. Presence of multiple solutions for similar boundary value problem is another frequently encountered problem in nonlinear analysis of beam structures [79-81]. However, most widely used numerical methods [82-86] for solution of nonlinear governing equations of beam problems are Newton iterative method [82], Runge-Kutta-Fehlberg method [83], Runge-Kutta method [77, 78, 85, 86], etc.

Semi analytical method: Application of variational principle to governing equation [87, 88] is the basis of semi analytical method. This method expresses governing equation in variational form and solution of such formulation is obtained by minimization of residual error [89-91] encountered in the analysis of system response. In this method the unknown, generally displacement field in beam bending problem, is approximated by polynomial [89, 90] or combination of coordinate functions [91]. Substitution of this approximate field into system governing equation provides residual error, which is then minimized through some iterative scheme. This leads to approximate solution of desired unknown. Variational formulations of beam problems are of two types. In the first type, governing equation is derived by using force or moment balance, which is then given a variational form through some mathematical manipulations. The second variational approach uses some energy measure of a structure to formulate its governing equation. As energy is scalar quantity, the approach reduces complexity of a structural problem. Thus energy based method is extensively used in nonlinear analysis of structural members like beam. Two most widely used energy methods, in structural analysis under static loading, are virtual displacement method and minimization of total potential energy method.

According to virtual displacement method, work done by external force distribution due to virtual displacement is equal to the integration of product of stress and virtual strain over the body. Mathematically virtual displacement principle is given by $\iiint\limits_{V} B\,\delta u\,dv + \oiint\limits_{S} T\,\delta u\,dA = \iiint\limits_{V} \sum_i \sum_j \tau_{ij}\,\delta\varepsilon_{ij}\,dv$ [4], where B is body force distribution, T is surface force distribution and δu is virtual displacement. Advantage of this approach is that it can undertake nonlinear curvature-slope and nonlinear stress-strain relations easily [92, 93].

Principle of minimization of total potential energy [94-98] is a generalization for deformable elastic bodies of the total potential energy of particles and rigid bodies. Mathematically the principle is represented as $\delta(\pi) = 0$, where δ is the variational operator and π is the total potential energy of the system. π consists of two parts, strain energy and potential energy and the final variational form is given as $\iiint\limits_{V} \sum_i \sum_j \tau_{ij}\,\delta\varepsilon_{ij}\,dv - \iiint\limits_{V} B\,\delta u\,dv - \oiint\limits_{S} T\,\delta u\,dA = 0$ [4].

It is clear from the above energy expressions that energy based formulation yields governing equation directly in variational form. Solution of such a formulation follows some stepwise procedure. Firstly, unknown displacement field is assumed satisfying boundary conditions through several approximate methods such as Ritz method, Galerkin method, etc. Then the assumed displacement field is substituted in energy equation, resulting in residual error of the system, minimization of which provides desired solution [94, 97, 98].

Numerical method: Closed form solution to beam bending problem, following large deflection theory, is not straight forward. Initially curved and non-prismatic beam adds complexity in analytical approach. In such complicated situations, numerical scheme is the only approach available. The different methods included in this category are finite difference, boundary element, finite element, nonlinear finite element and analysis with geometry updation. Finite difference method (FDM) is a technique for solving nonlinear differential equations through approximation of derivatives by finite difference [99]. Whereas in boundary element method (BEM), solution to nonlinear beam bending problems is approximated within the problem domain using solutions at boundaries [100].

Another very popular method widely used in structural analysis is finite element method (FEM), in which the whole domain of the structure is divided into finite number of elements of required shape. Whole domain of the structure is modeled by element wise simple equations. Energy based variational principle, such as principle of virtual displacement, minimization of total potential energy, etc. [58, 101-104], is mostly used to derive such element wise equation. However, force based formulation is also used [105]. Finite element analysis then assembles these element wise equations to model the whole problem.

Improvement in the modern day computers enables realistic simulation of structural behavior at loaded condition. Engineers and scientists generally use finite element package to find potential of a structure in actual working environment. For this purpose a structure is modeled in virtual environment using nonlinear finite element (NFEM) analysis software. For design of beam like structures, several commercial finite element packages such as MSC/NASTRAN [90], ANSYS [106, 107], ABAQUS [108], etc. are being extensively used in industry nowadays.

Applicability of the above mentioned methods fail in case of large deformation, when change in geometry of beam structure need to be considered. Detail on this geometry updation technique is presented in the next sub-section.

2. Analysis with geometry updation

A basic characteristics associated with geometric nonlinear analysis of beam is that stiffness of beam is a function of displacement. In other words, change in geometry cannot be neglected in large deflection analysis of beam. Hence, to obtain the final deformed configuration of a body under a given state of loading, the intermediate deformed states are captured and it is considered as the initial configuration for the next state of deformation coming from the corresponding incremental

loading. This method is commonly known as incremental Lagrangian formulation [109-117]. The method is also used in applications, where the complicated body geometry calls for a nonlinear 'strain-displacement' relation. Such nonlinearity may arise from large rotational and translational displacements of the body, without including much strain [3, 5, 109, 115].

It is well known that in Lagrangian coordinate system, the initial configuration of a body is considered to measure finite displacement, whereas Eulerian coordinate system is body fitted. Based on the configuration to which the reference is transformed, Lagrangian approach is classified into two categories and they are total Lagrangian [109-116] and updated Lagrangian [115, 117] formulations. In total Lagrangian approach, all parameters involved in energy expressions are referred to the initial configuration (C_0). Fig. 1(d) shows configurations of a beam element of length ds (refer Fig. 1(c)), undergoing large deformation, in three dimensional space with respect to global Cartesian coordinate system (x, y, z). Green's strain tensor and its energy conjugate, the second Piola Kirchhoff stress tensor are used to define the internal energy of the body [115]. Energy expression for total Lagrangian formulation is given by

$$\int_{0_V} (_0D_{ijrs})(_0\varepsilon_{rs})\delta(_0\varepsilon_{ij})\,^0dv + \int_{0_V} (_0^1S_{ij})\delta(_0\eta_{ij})\,^0dv = {}^2R - \int_{0_V} (_0^1S_{ij})\delta(_0e_{ij})\,^0dv \text{ [115, 116]}.$$

In the above equation, $_0D_{ijrs}$, $_0\varepsilon_{rs}$, $_0^1S_{ij}$ and ^2R are incremental constitutive tensor, Green strain increment tensor, second Piola-Kirchhoff stress tensor and total external virtual work respectively with respect to C_0 configuration. In addition, $_0e_{ij}$ and $_0\eta_{ij}$ are linear and nonlinear components of incremental strain tensor $_0\varepsilon_{ij}$. On the other hand in updated Lagrangian approach, parameters involved in energy balance are expressed with respect to the last calculated configuration (C_1) as shown in Fig. 1(d). Hence, mathematical expression of virtual work statement for updated Lagrangian formulation involves Cauchy stress tensor $^1\tau_{ij}$ in place of $_0^1S_{ij}$ with other similar parameters defined in C_1 configuration as given by

$$\int_{1_V} (_1D_{ijrs})(_1\varepsilon_{rs})\delta(_1\varepsilon_{ij})\,^1dv + \int_{1_V} (^1\tau_{ij})\delta(_1\eta_{ij})\,^1dv = {}^2R - \int_{1_V} (^1\tau_{ij})\delta(_1e_{ij})\,^1dv \text{ [115, 117]}.$$

It is important to note the importance and significance of body fitted Eulerian coordinate system in implementing Lagrangian method. The geometry of the body in every state of deformation is tracked through a number of material points, usually defined in its initial state. The kinematic requirement of compatibility conditions need to be satisfied by the body fitted material points, in addition to the overall validation of equilibrium conditions [3]. A large displacement analysis problem may be implemented through various means. The expression, derived through Lagrangian approach, is highly nonlinear in incremental displacement and closed form solution based on continuum mechanics is rarely available in literature [112, 113]. Generally, Lagrangian formulation is used as the basis of finite element analysis of nonlinear beam bending problem, which results element wise nonlinear governing equation [109-111, 114-117]. These nonlinear equations are generally linearized using small incremental strain steps and solved using some iterative method [109-111, 113, 115, 116] to obtain element stiffness matrix. Several computer programs like ADINA [115], NACS [117], etc. are also used to implement finite element formulation based on Lagrangian approach.

Nature of Problem Parameters

Development of stresses and deflections in beam structures highly depends on the nature of problem parameters. Three most important parameters of a structural problem are boundary condition, loading condition and initial geometry. Several aspects of these three types of problem parameters are described in the following sub-sections.

1. Boundary condition

Characteristics of deflection and stress field developed in beam under different loading conditions depend on their boundary conditions. Generally, displacement kinematics is prescribed at boundaries, whereas force kinetics [93, 95] is prescribed sometimes and mixed boundary condition arises rarely in some complicated beam bending problems [66, 69, 105, 118]. Two classical displacement based boundary conditions, used for stress and deflection analysis of beam, are fixed-free and simply supported. Many other boundary conditions, such as fixed-simply supported [61], clamped-clamped [59, 91, 96, 98, 110, 119], hinged-clamped [92], simply supported-elastically restrained [16, 76], hinged-hinged [103], stiffened lateral ends [68], are also prescribed for deflection analysis of beam structures. These different types of classical and non-conventional boundary conditions of beams are shown in Fig. 2. When beam is subjected to combined bending and stretching stress field, membrane boundary condition [70] is also prescribed in addition to displacement boundary conditions.
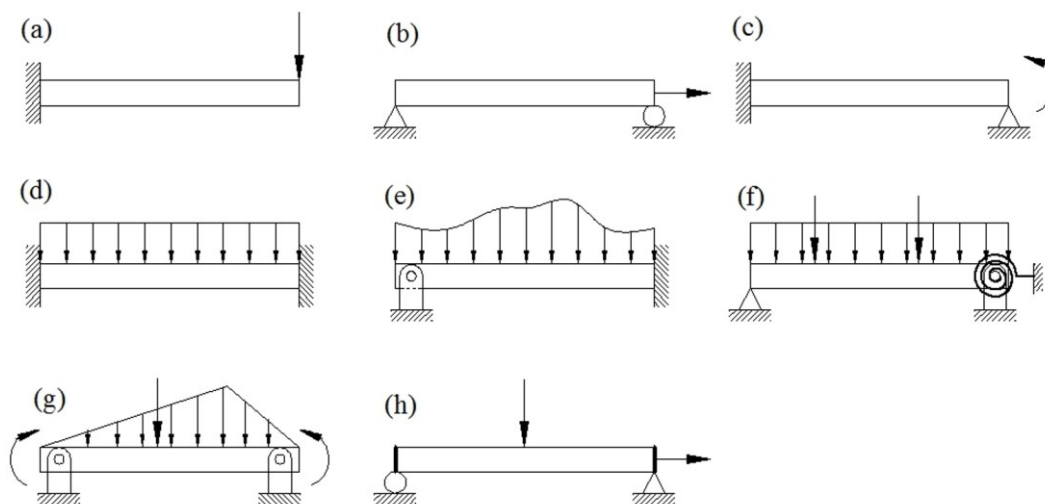

Figure 2. Different types of boundary and loading condition of beam: (a) fixed-free under concentrated transverse load, (b) simply-roller supported under concentrated in-plane load, (c) fixed-simply supported under pure bending moment, (d) clamped-clamped under uniformly distributed load, (e) hinged-clamped under non-uniformly distributed load, (f) simply supported-elastically restrained under combined concentrated and distributed transverse load, (g) hinged-hinged under transverse concentrated, distributed load and bending moment, (h) roller-simply supported with stiffened lateral ends under combined bending and in-plane load.

Large deflection analysis of cantilever beam assumes constant beam length at any configuration [15]. Condition of constant beam length is generally used as convergence criteria in iterative solution to obtain deflection profile [71]. When a cantilever beam is subjected to transverse follower type loading, constancy of beam length holds good. But for the case of large deflection under non-follower type loading, length changes appreciably due to stretching effect. This stretching effect, in addition to bending, results nonlinear displacement field.

In case of beam undergoing large deflection under three point bending, contact locations between beam and supports are shifted from original position to attain equilibrium [13, 14, 120]. When point load is applied at the mid-span of beam, horizontal components of reaction forces balance each other. But asymmetric loading yields non-equal horizontal components at supports [118]. Presence of friction between beam and simple supports [60] produces stretching under large

deflection, irrespective of symmetric and asymmetric loading. Moreover, local deformation at contact points affects deflection behavior of beam under three point bending. Effects of local deformations at supports and load application point are generally captured by modeling them through spring elements [121].

2. Loading condition

Structural member like beam is generally subjected to mechanical loads such as force, moments, etc. Different types of mechanical loading, applied to beam, are also shown in Fig. 2 along with the boundary conditions. Generally beam bending analysis under conservative loading considers no change in direction of load during the course of large deformation (refer Fig. 3(a)). Three major categories of loading are concentrated, uniformly or non-uniformly distributed and combined distributed and concentrated loads. In addition, a beam can also be subjected to couple or moment [58, 63, 73, 87, 92, 111].

Another broad category of mechanical load applied to beam, whose angle of inclination remains fixed with respect to the deformed axis of beam, is called follower type load (refer Fig. 3(a)) [77-81, 86]. Under such follower loading, the beam system becomes non-conservative. Non-conservative beam bending problem becomes complicated due to static instability [81].

In large deflection analysis, when geometry of beam changes appreciably, the intensity of distributed load never remains constant along the undeformed axis of the beam. Total external load remains constant but the load intensity changes along the axis of the deformed beam (refer Fig. 3(b)) [62, 71, 75, 90].
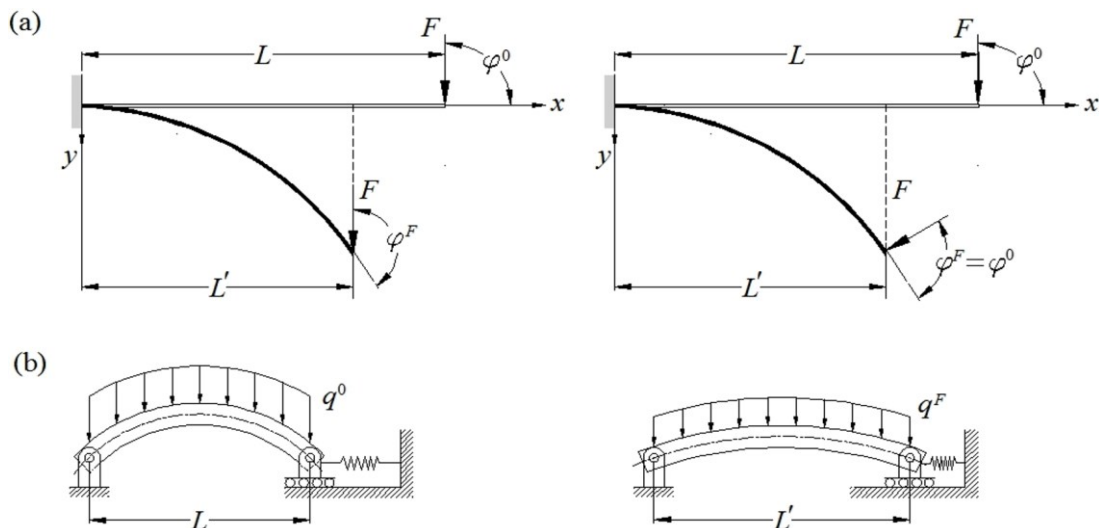

Figure 3. Loading diagrams: (a) showing change in direction of load for conservative and non-conservative loading and (b) showing change of uniformly distributed load intensity of a hinged-roller supported curved beam.

Snap-through buckling and initial imperfection: The load-deflection behavior of an initially curved beam in the form of a shallow arch is shown in Fig. 4, under transverse [59] and axial [119] loading. In snap through buckling the transverse load "snaps through" when load reaches at a critical value. Bifurcation buckling under such loading is also possible if the thickness of beam is much higher than the width, but such cases are rarely encountered in real life applications. For deeper arches multiple snap through modes [122] with anti-symmetric behavior is possible and sometimes this is prevented by using two beams in parallel [119] where any possible friction force within the layer has been prevented.

Thermal load: Thermal stress poses a great challenge in the design of many engineering structures as it give rise to premature failure or yielding. Thermal expansion of solids leads to the subsequent thermal stresses when the expansion of the body is restrained in one or more directions [91, 98, 99]. Hence axial stress is imposed in a beam structure at constrained boundaries (refer Fig.

4(b)) due to temperature variation. Stress developed in a beam structure, subjected to thermal loading, is a function of imposed temperature distribution across its surface or cross-section [98] or along axial direction [123]. Temperature field applied in a beam structure causes variation of its mechanical properties, which is modeled similar to the variation of properties for a functionally graded material [91, 98].
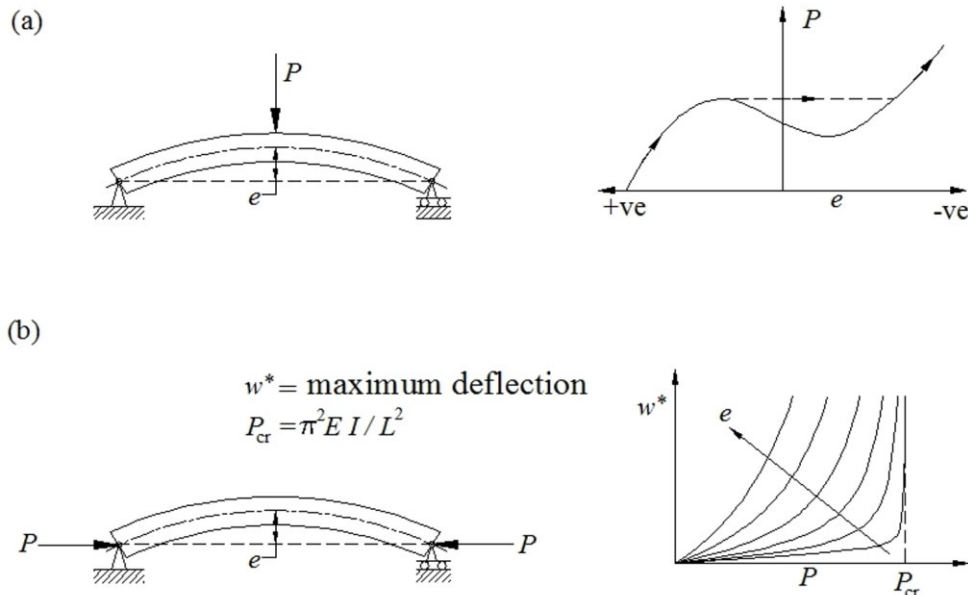

Figure 4. Representation of shallow arch and the corresponding load-deflection behavior under: (a) transverse and (b) axial loading.

3. Initial geometry

Development of stresses and deflections in beam undergoing large deflection greatly depends on initial geometry of beam, as shown in Fig. 5. Initial geometry of beams can be broadly divided into two major categories and they are straight and curved. Initially curved beams are defined as beams whose axes are not straight and are curved in the elevation. However, there is another class of initially curved beam whose centroidal axis has curvature in-plan [114].

Many of beam like structures, like leaf springs [102, 107], arches [92, 95, 103, 110, 112, 122], etc., are not initially straight and undergo large amplitude transverse displacement during operation. They are generally analyzed by modeling them as initially curved beam. In case of initially straight beam, moment-curvature relation gives curvature of deformed configuration, whereas, for initially curved beam moment-curvature relation is modified to incorporate the initial curvature [89]. For simplification of analysis, initial configuration of curved beam is generally considered as part of circle. For such constant curvature initially curved beam, analytical solution is possible. On the other hand, analysis is not so simple for initially curved beam having non-uniform curvature [58, 61, 75, 118]. Large displacement of initially curved beam induces a stretching effect and provides additional stiffening to the structures. The beam curvature leads to a coupling between the displacement fields coming from bending and stretching which leads the analysis of system response in the field of geometric nonlinearity [97, 118]. Moreover, non-prismatic beam geometry [71, 80, 90, 91] adds complexity in the analysis. Applications of varying cross-section beams are known from our historical observations, like arches of an aqua-duct or profile of a gear tooth. In addition, presence of any geometric irregularity in the body of beam produces stress concentration effect [118, 124]. As a result, actual stress value increases near the geometric discontinuity above the calculated stress level based on continuum mechanics approach.
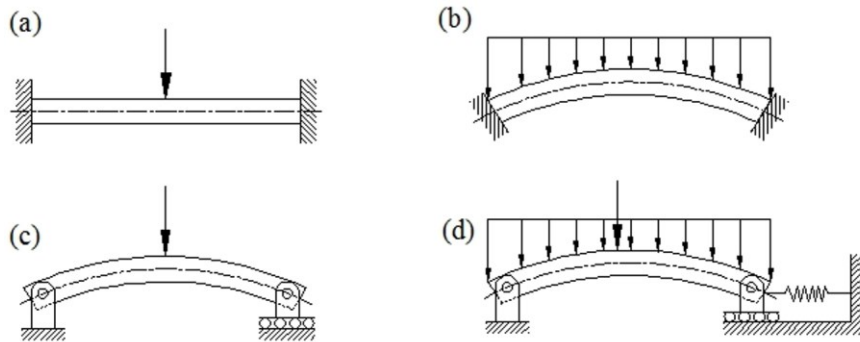
Figure 5. Different types of initial geometry: (a) straight beam with clamped ends under concentrated transverse load, (b) curved beam with clamped ends under distributed transverse load, (c) curved beam with hinged-roller supported ends under concentrated transverse load, (d) curved beam with hinged-roller supported ends under transverse and in-plane load.

Type of Material

As the present review work mainly focuses on beam deformation up to yield limit, the material behavior remains linear elastic. Material is broadly classified into two major categories, namely isotropic and anisotropic both of which show nonlinear stress-strain behavior beyond their elastic limit. Material nonlinearity associated with post elastic deformation is mainly dealt with in the theory of plasticity, which is a vast subject in itself, involving various types of material modeling and a number of theories predicting stress-strain relations. A brief description of this type of material nonlinearity is presented in a separate sub-section under the present section.

1. Isotropic material

Structural steel is traditionally used for construction of beams and such other structural elements. They exhibit linear relation between stress and strain within its elastic limit. However, several metals undergo work hardening and show nonlinearity in stress-strain relation within elastic limit [72, 85]. Stress-strain relation for this Ludwick type nonlinear material is generally modeled by the relation $\sigma = E\varepsilon^{1/n}$, as presented graphically in Fig. 6(a) [85]. Common engineering materials also show some nonlinearity within elastic domain, during transition from elastic to plastic regime. This type of nonlinear stress-strain relation is generally modeled by Ramberg-Osgood material modeling, which is given by $\varepsilon = (\sigma/E) + K(\sigma/E)^n$ and shown graphically in Fig. 6(b) [125]. σ and ε represents stress and strain respectively and E, n are material dependent constants.

2. Anisotropic material

Limited resource and minimum weight criteria without sacrificing strength make material selection a critical issue in design of several beam like structures nowadays. To meet increasing expectation from structural material, positive properties of several materials are mixed in solid state that develops anisotropic material popularly known as composite material. Such anisotropic materials enhance bending resistance and are realized by varying material property along a particular direction within the beam. The most widely used composite materials for several machine elements are carbon fiber reinforced epoxy [126], glass fiber reinforced epoxy [126] and glass and carbon fibers reinforced epoxy [126], glass fiber reinforced plastic [107], graphite fiber reinforced epoxy [110], etc. Use of laminated glass [96] as beam material is also reported in literature, in connection with civil structures. In addition, beam made of piezoelectric material [101] is also used as machine element that generates AC voltage when subjected to stress during operation. Beam bending under thermal loading is another example of beam anisotropy [91, 98, 99]. Major problem in use of composite laminated material for beam structures is delamination. This failure occurs due

to development of shear stress between the layers and generally seen at free edges. Reddy's layer-wise theory is most commonly used for static analysis of interlaminar stresses and free edge effects in laminated composite beams [93]. However, demerits of composite laminated materials coming from discontinuous variation of properties at transition zone between two solids lead towards development of a more advanced anisotropic material known as functionally graded material (FGM). A brief description of FGM as beam material is presented in the following sub-section.

Functionally graded material: Functionally graded materials are a class of inhomogeneous materials in which the compositions are engineered to change material properties smoothly over volume. FGM are usually composed of two different materials, such as metal and ceramic and they take advantage of the improved property from each constituent [127]. The variation in properties of straight beam made of FGM is generally controlled in one direction, either along beam center line [128] or across beam thickness [91, 98, 129]. On the other hand, in case of FGM curved beams bidirectional gradation of properties is required [130, 131]. If the variations of material properties are along three mutually orthogonal twofold axes of rotational symmetry, the material is called orthotropic material [68, 69]. Smooth variations of mechanical properties along particular direction of FGM beam are commonly modeled by exponential functions [91], power law [98], etc. Many of the functionally graded composite materials generally exhibit different elastic response in tension and compression [83]. In addition, if material exhibits viscoelastic property, its time dependent response is generally captured through quasi-linear viscoelastic model [99]. Due to smooth variation of material properties of FGM as per the requirement, the FGM structures are widely used in various engineering field and as such their application in beam structures are evolving. Recently several research papers, for example [55, 129], have been published in which FGM material modeling is coupled with emerging structural research problems in the framework of geometrically exact beam models [129], higher order beam models [55], etc.

3. Material nonlinearity in beam bending

In the course of large deformation in post elastic state of material behavior, the 'stress-strain' relation governing material parameters become a function of the current deformation state. Generally Von Mises' [82] and Tresca's [132] yield criteria are used to define limit of elasticity for a material under any combination of stresses. Most of the engineering materials strengthen by plastic deformation commonly known as strain or work hardening. Thus material nonlinearity is mainly manifested beyond yielding, although there are some nonlinear elastic materials as discussed earlier (refer Fig. 6(b)). However, trivial solution does not consider strain hardening in analysis [133], stress-strain curve of such elastic perfectly plastic material is shown in Fig. 6(c(i)). Practical engineering application generally considers linear strain hardening model [134] for structural analysis beyond elastic limit. However, more rigorous constitutive models for beam bending analysis include nonlinear strain hardening models [82, 135]. These different types of stress-strain curves for material with linear elastic behavior are shown in Figs. 6(c(ii)) and (c(iii)). However, material nonlinearity is directly related to large strain and is not generally considered in large deflection analysis mainly with large rotations [125].
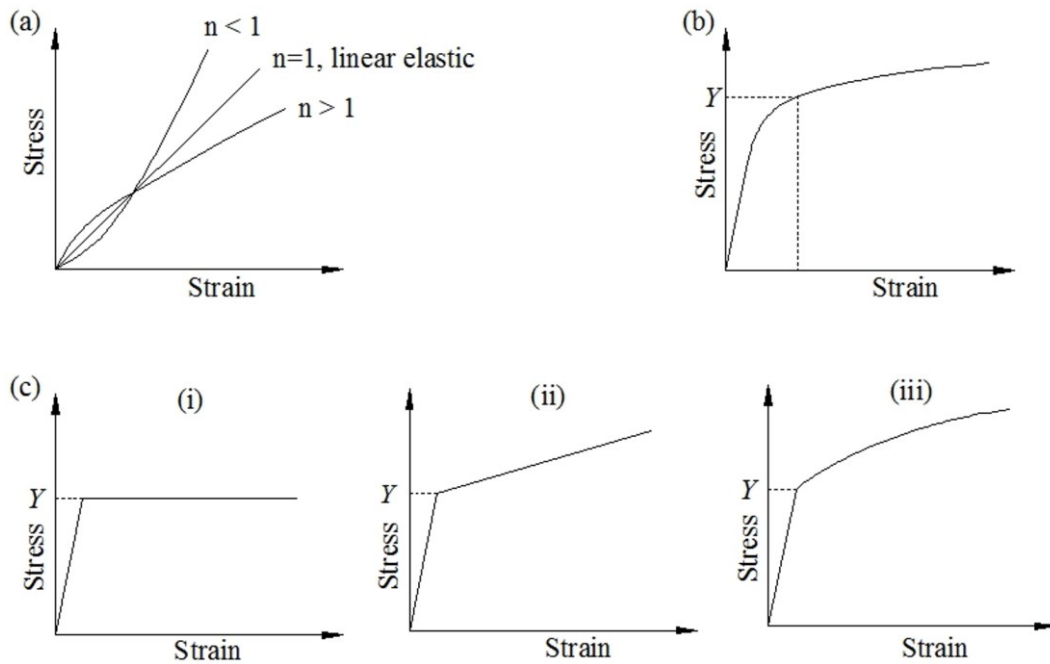
Figure 6. Stress-strain relations for (a) linear elastic and Ludwick type material, (b) Ramberg-Osgood type material and (c) material behavior beyond elastic limit: (i) perfectly plastic, (ii) linear hardening and (iii) nonlinear hardening.

Experimental Work and Application Areas

Ample research works on theoretical stress and deflection analysis of beams following large deflection theory are available, but on the other hand experimental work is very few. Several stress and displacement measurement techniques, used by the researchers are described first and the next sub-section describes application of large deflection study of curved beams in practical engineering field.

1. Experimental work

Experiments in industry are generally carried out with the goal of recreating the conditions of actual working environment of a structural component, in order to predict its behavior under such conditions [124, 126, 136, 137]. On the other hand most of the laboratory experimental works are done to validate theoretical simulation models. Experimental research works are carried out on various types of straight [106, 108, 109, 121] and curved [65, 73, 96, 109] beams and their equivalent models such as leaf spring [75, 84, 107, 118, 126], arch [122], rectangular plate [136, 137], and beam and plate with geometric discontinuity in form of hole [124]. These structures involve various boundary conditions, like simply supported-simply supported [96, 107, 121, 126], clamped-free [73, 75, 106, 109], clamped-simply supported [84], clamped-clamped [122], etc. and various loading conditions such as three point bending [96, 118, 121], compression [126], tension [126, 136], torsion [109, 126], cyclic loading [126], pure bending [124], concentrated load [65, 73, 75, 107, 109], distributed load [109], combined action of concentrated and distributed loads [106], end moment [73, 109] and combined bending-stretching [118], etc. Many experiments are carried out on specially designed rig [65, 73, 75, 84, 106, 109, 118] and universal testing machine [96, 121, 126, 136] whereas some other researchers carried out experiments on standard leaf spring testing rig [107] and electronic post buckling testing platform [59]. Measurement techniques for stress and deformation analysis of beam structures, as implemented by the researchers, are described in the following sub-sections.

Stress measurement technique: Most widely used method for structural stress measurement is strain gauge [96, 108, 118, 136, 137] which are generally mounted on beam along longitudinal direction. Sometimes strain gauges are also mounted in transverse direction to account for Poisson's ratio effect. However, strain gauge is capable to measure micro scale strain only. Several other strain measurement techniques are also used other than strain gauge and these include speckle shearing interferometry [137], strain measurement extensometer [136], crosshead motion of electromechanical testing machine [136], mechanical deflectometer [137], etc. Stress distribution around geometric discontinuity in structure is measured by photo-elasticity technique using polariscope set-up [124]. The method is based on the property of birefringence, as exhibited by some transparent material. For materials which do not show photoelastic behavior, models made of transparent material are tested having geometry similar to the real structure under investigation. Moreover, strain gauge technique is also used to obtain the effect of stress concentration, by mounting them at locations in the vicinity of the discontinuity [118].

Displacement measurement technique: Deflection is the degree to which a structural element is displaced under a load and it may refer to an angle or a distance. Usually deflection is measured with respect to its initial undeformed configuration. Conventional displacement measurement techniques include tensile testing machine [126], Hopkinson pressure bar [108], contact type [65, 73, 97] and non-contact type [84] displacement sensors, vernier height gauge [118], rulers [106, 118], vernier caliper and dial caliper [109], electronic device like LVDT [59], etc. Several displacement measurement methods based on image processing technique are being extensively used nowadays [75, 118].

2. Application areas

Some applications of curved beam structures, as found in nature, in ancient structures, in engineering applications and their large deflections are shown respectively in Fig. 7(a-d). Various flexural members in mechanical and civil structures and machine elements are generally analyzed and designed by modeling them as beam. Most of the research works are performed on stress and deformation of beam like structures following geometric nonlinear theory for betterment of their design. Many researchers have reported these applications in automobile industry [102, 107, 126], aerospace application [104], designs of large displacement hinges in planar compliant mechanisms [73], arch [92, 95, 103, 110, 112, 122] and adjustable stiffness spring [84], etc. One particular research paper reported stress analysis for simulation of sheet metal forming process [135]. Application of higher order beam theories in design of several Micro Electro-Mechanical system (MEMS), Nano Electro-Mechanical system (NEMS), carbon nanotubes, biosensors, micro actuators, nano probes, etc., have also been reported in some recent papers [31, 33, 34, 37, 46-52, 54-57, 59]. Several non-metallic materials offer high strength and low weight and hence modern trend in structural mechanics is to replace metallic structural elements by non-metallic ones, for example leaf springs [107, 126], arches [103], micro and nano devices [31, 55], etc., made of composite and functionally graded materials. Large deformation behavior of such non-metallic structures [67-70, 83, 91, 93, 96, 98, 103, 107, 114, 126] within elastic limits call for analysis in geometrical nonlinear setting.
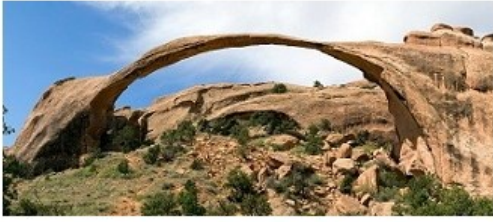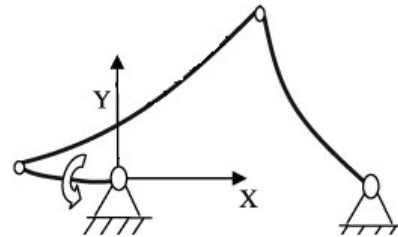
Figure 7. Application of curved beams: (a) in nature, (b) in ancient structures, (c) in engineering appliance and (d) in an example of large deflection (figures collected from open domain).

Conclusions

Stress and deformation behavior of curved beam undergoing large deflection is thoroughly investigated in the present review paper. Various methods used for geometric nonlinear analysis of beam, various system parameters and complicating effects are identified from the reviewed research papers. Several aspects of their effects in beam bending are discussed subsequently, and the bottom lines are summarized below for ready reference.

- Most of the theoretical works resort to numerical solutions because of the presence of strong nonlinearity in governing equation.
- Few classical analytical methods along with several approximate techniques are also reported in this regards.
- In spite of the widespread theoretical nonlinear studies available in literature, there is lack of experimental fidelity.
- From material point of view it is observed that various non-metallic materials attracted attention of researchers in recent times due to their high strength to weight ratio.
- Engineered materials like FGMs, composites, plastics, etc., are making inroads involving nonlinear elastic, viscoelastic and other types of material modeling.
- Practical engineering application of large deflection studies is also significant, especially in MEMS and NEMS.

Research works regarding static analysis beyond elastic limit and dynamic analysis in both elastic and post elastic regimes for curved beam are not included in the present scope of review. However, the observations reported in this review work on large deflection static analysis of curved beam will facilitate researchers to identify various complicating effects and challenges involved in analyzing some practical structural problems and would pave way towards future challenges.

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