Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 22, NUMBER 1 (2019) (Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

SURVEY PAPER

A REVIEW ON VARIABLE-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS: MATHEMATICAL FOUNDATIONS, PHYSICAL MODELS, NUMERICAL METHODS AND APPLICATIONS

HongGuang Sun [1], **Ailian Chang** [1], **Yong Zhang** [2], **Wen Chen** [1]

Abstract

Variable-order (VO) fractional differential equations (FDEs) with a time (t), space (x) or other variables dependent order have been successfully applied to investigate time and/or space dependent dynamics. This study aims to provide a survey of the recent relevant literature and findings in primary definitions, models, numerical methods and their applications. This review first offers an overview over the existing definitions proposed from different physical and application backgrounds, and then reviews several widely used numerical schemes in simulation. Moreover, as a powerful mathematical tool, the VO-FDE models have been remarkably acknowledged as an alternative and precise approach in effectively describing real-world phenomena. Hereby, we also make a brief summary on different physical models and typical applications. This review is expected to help the readers for the selection of appropriate definition, model and numerical method to solve specific physical and engineering problems.

This paper won the "Riemann-Liouville Award: Best FDA Paper (Theory)" at the Conference ICFDA 2018.

MSC 2010: Primary 26A33; Secondary 34A08, 34A34, 34A45, 34K28, 35R11, 60G22, 65M12

Key Words and Phrases: variable-order; fractional calculus; fractional differential equations; numerical methods; applications

---

pp. 27–59 , DOI: 10.1515/fca-2019-0003 DE GRUYTER

1. Introduction

Fractional calculus, containing differentiation and integration (i.e., integration and differentiation of an arbitrary real order), has a history of more than three hundred years [66, 108]. Compared with the integer order calculus, many real-world phenomena can be better described by using fractional operator. In fact, the fractional calculus has been acknowledged as a promising mathematical tool to efficiently characterize the historical memory and global correlation of complex dynamic systems, phenomena or structures. However, various literature indicated that the memory and/or nonlocality of the system may change with time, space or other conditions [66, 106]. The VO fractional operators depending on their non-stationary power-law kernel can describe the memory and hereditary properties of many physical phenomena and processes. Therefore, to accurately characterize complex physical systems and processes, VO fractional calculus was availably employed as a potential candidate to provide an effective mathematical framework [137]. Subsequently, VO-FDEs have attracted more and more attention, ascribing to its suitability in modeling along with a large variety of phenomena, ranging many fields of science and engineering fields, including anomalous diffusion [19, 103, 139], viscoelastic mechanics [30, 36, 50, 96], control system [59], petroleum engineering [80], and many other branches of physics and engineering, just to mention a few [8, 16, 54, 56, 60, 71, 84, 131, 135].

Samko and Ross [89] firstly proposed the concept of VO integral and differential as well as some basic properties in 1993. Lorenzo and Hartley [67] summarized the research results of the VO fractional operators and then investigated the definitions of VO fractional operators in different forms. After that, some new extensions and valuable application potentials of the VO-FDE models have been further explored [30]. It has become a research hotspot and has aroused wide concern in the last ten years. A detailed description will be offered in Section 5.

Extensive investigations have devoted to the physical modeling using VO-FDE models. For examples, Kobelev et al. [58] demonstrated the variable memory problems concerning statistical and dynamical systems, where the fractal dimension changes with time and coordinate. Coimbra et al. [30] investigated the viscoelasticity oscillator via VO fractional operators. Sweilam and Al-Mekhlafi [111] presented a novel multi-strain tuberculosis model using VO fractional derivative as an extension of the nonlinear ordinary differential equation. We also investigated the application potentials of VO-FDE models in characterizing transient diffusion [103].

Whereafter, it is necessary to seek exact solutions or numerical solutions for VO-FDEs. However, it is usually difficult to obtain the analytical solution of VO-FDEs. In general, the numerical methods are employed as efficient developed methods for the numerical approximation of VO-FDEs [27, 90]. For examples, Liu et al. [65] studied the stability and convergence of a new explicit finite-difference approximation for the VO nonlinear fractional diffusion equation. Razminia et al. [86] proved the existence for the solution of VO-FDEs. Zayernouri and Karniadakis [133] developed an exponentially accurate fractional spectral collocation method for solving linear/nonlinear VO-FDEs. Chen et al. [22] introduced a new implicit numerical method to solve the two dimensional (2D) VO fractional percolation equation smoothly. Zhao et al. [138] proposed a second-order approximation formulae for the time VO fractional derivative to describe anomalous diffusion and wave propagation. Afterwards, Cao and Qiu [17] derived a high-order numerical method for VO-FDEs in the light of a second-order numerical approximation. Some properties and inversion formula of the VO operator $d^{\alpha(x)}f(x)/dx^{\alpha(x)}$ using the Riemann-Liouville definition and Fourier transform have been discussed [89]. Furthermore, more details in progress in numerical algorithms for VO-FDEs will be investigated in Section 4.

This paper is organized as follows. Section 2 introduces preliminary definitions of VO fractional calculus. In Section 3, VO fractional integral and derivative models and physical interpretations are investigated. Section 4 provides a review on several widely used numerical methods for VO-FDEs. A wide range of applications concerning VO-FDE models are offered in Section 5. Some conclusions are reported in Section 6.

2. Primary definitions of variable-order fractional calculus

The VO fractional integration and differentiation are increasingly developed and discussed, after the first definition proposed by Samko [88]. Notably, there are three ways to define the mathematical basis for VO fractional integrals and derivatives, including directly extension from fractional operators [89], Laplace-transform [30] and physical-driven [66, 102]. In this section, we offer existing definitions of fractional integral and fractional derivative operators.

2.1. Riemann-Liouville definition.

In the beginning, the Cauchy formula for integer order integration is studied [79]

$$I^n f(x) = \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1} f(y)dy. \tag{2.1}$$

The fractional order integral is then defined as [82, 91]

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad \alpha > 0, \tag{2.2}$$

where $\Gamma(\cdot)$ is the Gamma function as an extension of the factorial function to real numbers [25, 110]

$$\Gamma(t) = \int_0^{\infty} \tau^{t-1} \exp(-\tau) d\tau, \quad t > 0. \tag{2.3}$$

The left-hand and the right-hand Riemann-Liouville fractional derivatives with order α are defined as [69]

$$D_{a+}^{\alpha} f(x) = \frac{d^{\alpha} f(x)}{d_+ x^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(\xi)}{(x-\xi)^{\alpha+1-n}} d\xi, \tag{2.4}$$

and

$$D_{b-}^{\alpha} f(x) = \frac{d^{\alpha} f(x)}{d_- x^{\alpha}} = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{f(\xi)}{(\xi-x)^{\alpha+1-n}} d\xi, \tag{2.5}$$

where n is an integer, $n-1 < \alpha \leq n$, and a and b are the left and right boundary points, respectively. When the fractional order is allowed to vary with time or space, a generalized Riemann-Liouville time integration operator can be written as bellow [65]

$${}_aI_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(\alpha(t))} \int_a^t (t-\tau)^{\alpha(t)-1} f(\tau) d\tau. \tag{2.6}$$

The definitions of left and right VO Riemann-Liouville integrals with hiding memory are then proposed as [5]

$${}_aI_t^{\alpha(t,\tau)} f(t) = \int_a^t \frac{1}{\Gamma(\alpha(t,\tau))} (t-\tau)^{\alpha(t,\tau)-1} f(\tau) d\tau, \tag{2.7}$$

and

$${}_tI_b^{\alpha(t,\tau)} f(t) = \int_t^b \frac{1}{\Gamma(\alpha(t,\tau))} (\tau-t)^{\alpha(t,\tau)-1} f(\tau) d\tau. \tag{2.8}$$

The left-side and right-side VO Riemann-Liouville fractional derivatives are stated as [19]

$${}_a^{RL}D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \int_a^t (t-\xi)^{n-\alpha(t)-1} f(\xi) d\xi, \quad n-1 < \alpha(t) < n, \tag{2.9}$$

and

$$ {}_t^{RL}D_b^{\alpha(t)} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \int_a^t (\xi - t)^{n-\alpha(t)-1} f(\xi) d\xi, \quad n-1 < \alpha(t) < n. \tag{2.10} $$

Meanwhile, the left Riemann-Liouville fractional derivative of order $\alpha(\tau, t)$ is defined as [66, 114]

$$ {}_a^{RL}D_t^{\alpha(\tau,t)} f(t) = \frac{d^n}{dt^n} \left(\frac{1}{\Gamma(n-\alpha(\tau,t))} \int_a^t (t-\tau)^{n-\alpha(\tau,t)-1} f(\tau) d\tau \right). \tag{2.11} $$

The right Riemann-Liouville fractional derivative of order $\alpha(\tau, t)$ is stated as

$$ {}_t^{RL}D_b^{\alpha(\tau,t)} f(t) = \frac{d^n}{dt^n} \left(\frac{(-1)^n}{\Gamma(n-\alpha(\tau,t))} \int_t^b (\tau - t)^{n-\alpha(\tau,t)-1} f(\tau) d\tau \right). \tag{2.12} $$

2.2. **Caputo definition.** Because the initial conditions for the FDEs with the Caputo derivatives are the same as the integer order differential equations, Caputo type definition is extremely useful in many application fields [125]. The Caputo fractional derivative of $f(t)$ is defined as [126]

$$ {}_a^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, n-1 < \alpha < n. \tag{2.13} $$

As a direct extension, the VO fractional derivative is defined as [138]

$$ {}_a^{C}D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha(t)-n+1}} d\tau, n-1 < \alpha(t) < n, \tag{2.14} $$

and

$$ {}_t^{C}D_b^{\alpha(t)} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha(t))} \int_t^b \frac{f^{(n)}(\tau)}{(\tau - t)^{\alpha(t)-n+1}} d\tau, n-1 < \alpha(t) < n. \tag{2.15} $$

Using inverse Laplace transform, Coimbra [30] proposed a formulation of VO differential operator for physical modeling

$$ {}^{C}D^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_{0^+}^t (t-\tau)^{-\alpha(t)} f'(\tau) d\tau + \frac{(f(0^+) - f(0^-)) t^{-\alpha(t)}}{\Gamma(1-\alpha(t))}, \tag{2.16} $$

where $0 \leq \alpha(t) < 1$. Coimbra's definition provided a precise and direct meaning of the VO derivative for a given configuration [81].

If the order $\alpha(t)$ is a constant, then the VO fractional operators are reduced to the corresponding constant-order (CO) derivatives. These two definitions of VO derivatives are not generally equivalent, but they are related by the following relationship

$$ {}_0^{RL}D_t^{\alpha(t)}f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)t^{k-\alpha(t)}}{\Gamma(k+1-\alpha(t))} + {}_0^{C}D_t^{\alpha(t)}f(t). \tag{2.17} $$

When $\alpha(t) \in (0,1)$, the following relationship between (2.9) and (2.13) can be formulated [17]

$$ {}_0^{C}D_t^{\alpha(t)}f(t) = {}_0^{RL}D_t^{\alpha(t)}\left[f(t) - f(0)\right]. \tag{2.18} $$

In addition, the operator ${}_0^{C}D_t^{\alpha(t)}$ satisfies the following property ($1<\alpha(t)<2$) [12]

$$ {}_0^{C}D_t^{\alpha(t)}t^{\gamma} = \begin{cases} 0, & 0, 1, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha(t))}t^{\gamma-\alpha(t)}, & \gamma = 2, 3, ... \end{cases} \tag{2.19} $$

Considering the hiding memory, the left Caputo derivative with order $\alpha(\tau, t)$ is defined as

$$ {}_a^{C}D_t^{\alpha(\tau,t)}f(t) = \int_a^t \frac{1}{\Gamma(n-\alpha(\tau,t))} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha(\tau,t)-n+1}} d\tau, n-1 < \alpha(\tau,t) < n. \tag{2.20} $$

The right Caputo derivative with order $\alpha(\tau, t)$ is written as

$$ {}_t^{C}D_b^{\alpha(\tau,t)}f(t) = \int_t^b \frac{(-1)^n}{\Gamma(n-\alpha(\tau,t))} \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha(\tau,t)-n+1}} d\tau, n-1 < \alpha(\tau,t) < n. \tag{2.21} $$

Furthermore, with consideration of the initial condition, the left Caputo derivative with order $\alpha(t) \in (0, 1)$ is established as [114]

$$ {}_a^{C}\mathbb{D}_t^{\alpha(t)}f(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha(t)} [f(\tau) - f(a)] d\tau, \tag{2.22} $$

and the right Caputo derivative with order $\alpha(t)$ can be written as

$$ {}_t^{C}\mathbb{D}_b^{\alpha(t)}f(t) = \frac{-1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_t^b (\tau-t)^{-\alpha(t)} [f(\tau) - f(b)] d\tau. \tag{2.23} $$

More details on the relationship between two types of Caputo VO fractional derivatives can be found in related reference [114].

The relationship between Caputo (2.20) and Riemanm-Liouville (2.9) VO fractional derivatives ($\alpha(t) \in (0,1)$) is stated as follows [114]

$$\begin{aligned} {}_{a}^{RL}D_{t}^{\alpha(t)} f(t) &= {}_{a}^{C}\mathbb{D}_{t}^{\alpha(t)} f(t) + \frac{f(a)}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_{a}^{t} (t-\tau)^{-\alpha(t)} d\tau \\ &= {}_{a}^{C}D_{t}^{\alpha(t)} f(t) + \frac{f(a)}{\Gamma(1-\alpha(t))} (t-a)^{-\alpha(t)} \\ &+ \frac{f(a)\alpha'(t)}{\Gamma(2-\alpha(t))} (t-a)^{1-\alpha(t)} \left[\frac{1}{1-\alpha(t)} - \ln(t-a) \right], \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} {}_{t}^{RL}D_{b}^{\alpha(t)} f(t) &= {}_{t}^{C}\mathbb{D}_{b}^{\alpha(t)} f(t) + \frac{f(b)}{\Gamma(1-\alpha(t))} (b-t)^{-\alpha(t)} \\ &+ \frac{f(b)\alpha'(t)}{\Gamma(2-\alpha(t))} (b-t)^{1-\alpha(t)} \left[\frac{1}{1-\alpha(t)} - \ln(b-t) \right]. \end{aligned} \tag{2.25}$$

2.3. **Grünwald-Letnikov definition.** Gründwald-Letnikov fractional derivative is defined as follows [15, 69, 94]

$${}_{a}^{G}D_{t}^{\alpha} f(t) = \lim_{h\to 0^{+}} \frac{1}{h^{\alpha}} \sum_{r=0}^{\frac{t-a}{h}} (-1)^{r} \begin{pmatrix} \alpha \\ r \end{pmatrix} f(t-rh), \ \alpha > 0, \tag{2.26}$$

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} = \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}. \tag{2.27}$$

Using a direct extension, the Grünwald-Letnikov VO fractional integration is proposed as [118]

$${}_{a}^{G}I_{t}^{\alpha(t)} f(t) = \lim_{h\to 0^{+}} \frac{1}{h^{\alpha(t)}} \sum_{r=0}^{\frac{t-a}{h}} (-1)^{r} \begin{pmatrix} -\alpha(t) \\ r \end{pmatrix} f(t-rh), \ \alpha(t) > 0. \tag{2.28}$$

The left fractional VO derivative of Grünwald-Letnikov type is formulated as [67]

$${}_{a}^{G}D_{t}^{\alpha(t)} f(t) = \lim_{h\to 0^{+}} \frac{1}{h^{\alpha(t)}} \sum_{r=0}^{\frac{t-a}{h}} (-1)^{r} \begin{pmatrix} \alpha(t) \\ r \end{pmatrix} f(t-rh), \ \alpha(t) > 0, \tag{2.29}$$

and the right VO fractional derivative is stated as

$${}_{\mathrm{t}}^{G}D_{b}^{\alpha(t)} f(t) = \lim_{h\to 0^{+}} \frac{1}{h^{\alpha(t)}} \sum_{r=0}^{\frac{b-t}{h}} (-1)^{r} \begin{pmatrix} \alpha(t) \\ r \end{pmatrix} f(t+rh), \ \alpha(t) > 0, \tag{2.30}$$

where the order $\alpha(t)$ also can be replaced by $\alpha(rh)$ or $\alpha(t-rh)$, which has the same meaning as the $\alpha(\tau)$ in the Riemann-Liouville and Caputo derivatives.

2.4. **Riesz definition.** The Riesz type of VO time fractional integration is defined as [45]

$$ {}^{R}I_t^{\alpha(x,t)} f(x,t) = \frac{1}{2\cos(\pi\alpha(x,t)/2)\Gamma(\alpha(x,t))} \int_{-\infty}^{+\infty} \frac{f(x,\eta)}{|t-\eta|^{1-\alpha(x,t)}} d\eta. \quad (2.31) $$

More specifically, the Riesz type VO time-dependent fractional derivative of the function u is expressed as [87]

$$ \frac{\partial^{\alpha(t)} u(x,t)}{\partial |x|^{\alpha(t)}} = -(-\Delta)^{\alpha(t)/2} u(x,t) = -\mathcal{F}^{-}|\xi|^{\alpha(t)} \mathcal{F} u(\xi,t). \quad (2.32) $$

Then the Riesz type VO space-dependent fractional derivative is obtained in the spatial domain $x \in [a,b]$ [139]

$$ \frac{\partial^{\alpha(x)} u(x,t)}{\partial |x|^{\alpha(x)}} = -\frac{1}{2\cos(\pi\alpha(x)/2)} \left[{}_aD_x^{\alpha(x)} u(x,t) + {}_xD_b^{\alpha(x)} u(x,t) \right], \quad (2.33) $$

and the alternative expression is

$$ \begin{aligned} -(-\Delta)^{\alpha(x)/2} f(x) = -\tfrac{1}{2\cos(\pi\alpha(x)/2)} \Big[& \tfrac{1}{\Gamma(n-\alpha(x))} \int_a^x \tfrac{f^n(\eta)d\eta}{(x-\eta)^{\alpha(x)-n+1}} \\ & + \tfrac{(-1)^n}{\Gamma(n-\alpha(x))} \int_x^b \tfrac{f^n(\eta)d\eta}{(\eta-x)^{\alpha(x)-n+1}} \Big]. \end{aligned} \quad (2.34) $$

It is worth stressing that the Riesz fractional derivative is recognized as a powerful tool to describe some nonconservative models, but it is not suitable for all kinds of variational problems.

2.5. **Other definitions.** To the best of our knowledge, an extension to Hadamard fractional operator was presented by Almeida and Torres [3]. The left and right Hadamard VO fractional integrals are defined respectively as ($\alpha(t)>0$)

$$ {}_aI_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(\alpha(t))} \int_a^t (\ln \frac{t}{\tau})^{\alpha(t)-1} \frac{x(\tau)}{\tau} d\tau, \quad (2.35) $$

and

$$ {}_tI_b^{\alpha(t)} x(t) = \frac{1}{\Gamma(\alpha(t))} \int_t^b (\ln \frac{\tau}{t})^{\alpha(t)-1} \frac{x(\tau)}{\tau} d\tau. \quad (2.36) $$

While the left and right Hadamard VO fractional derivatives are respectively given by [3]

$$ {}_aD_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_a^t (\ln \frac{t}{\tau})^{-\alpha(t)} \frac{x(\tau)}{\tau} d\tau, \quad (2.37) $$

and

$$ {}_tD_b^{\alpha(t)}x(t) = \frac{-t}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_t^b (\ln\frac{\tau}{t})^{-\alpha(t)}\frac{x(\tau)}{\tau}d\tau. \tag{2.38} $$

Furthermore, many different definitions of the VO fractional integral and derivative have been introduced. Ultimately, for more information about definitions, we suggest that readers check the book [2]. In addition, the VO fractional operator has also been used in other non-local definitions with non-singular kernels, which is beyond the scope of this review [99, 127].

3. Physical discussion on VO fractional integral and derivative models

3.1. VO fractional operators in system memory characterization. VO fractional integral and derivative are non-local operators to characterize the memory property of systems. It offers a new approach to investigate the complex dynamics, hereditary effects and self-similarity from physical viewpoints [83, 85, 106, 111]. Therefore, increasing attentions have been focused on theoretical analysis and physical modeling by using VO fractional derivative and integral.

There are two types of definitions regarding the memory systems which are discussed by the VO fractional integral/derivative. That is to say, one is to describe the system memory changes with time and spatial coordinates [58]. Another one is associated with the history memory of orders; namely, VO is influenced by previous values of the differentiation orders and has a special feature of memory [92, 118]. The difference of these two definitions in characterizing the system memory property has been investigated by considering VO fractional relaxation-type differential equation in [106]. In addition, Zhang and Liu [136] investigated the influence of time-dependent memory and variable spatial correlation of medium heterogeneity on tracer dynamics. Dabiri et al. [34] proposed an optimization method, VO fractional proportional-integral-derivative, to obtain the optimal control parameters and assess bounded closed-loop response for linear dynamical systems under distinct initial conditions.

3.2. Dynamic-order fractional dynamic system. In recent years, the usage of fractional order dynamical systems play a vital role in real-world applications, including for examples, the affine cipher using date of birth fuzzy [77], fractional integral sliding mode control [10], fractional order modified Duffing systems [39], fractional order King Cobra chaotic system [75], digital cryptography [74], and authenticated encryption scheme [76]. Hence, we investigate the fractional order dynamical systems which have great application potentials in real-world engineering fields.

In some VO fractional dynamic systems, the VO is a function of certain variables, such as temperature, concentration, density and so on. For instance, Glöckle and Nonnenmacher [41] have found that the differential order of protein relaxation is a function of temperature. He and Luo [44] extensively investigated the dynamic behavior which is the chaotic property of fractional order Duffing systems. Wang and Wu [120] proposed the fractional order 5D hyperchaotic system based on the hyperchaotic Lorenz system. Nowadays, the nonlinear dynamic systems of fractional order and the synchronization of VO fractional chaotic systems have become the focus in scientific research.

The dynamic-order fractional dynamic system has been employed to explore the physical mechanism of VO fractional dynamic system and further provides the determination method. Furthermore, the multi-system interaction and multi-field coupling from the VO fractional derivative modeling approach have been increasingly investigated. Especially, the behavior of a dynamic system may change with the VO in multi-system physical processes. In our previous work, a dynamic-order fractional dynamic system has been investigated to explain the multi-system physical processes, which can be written as below [107]

$$\begin{cases} \dfrac{d^{\alpha(y(t))}x(t)}{dt^{\alpha(y(t))}} = Ax(t) + B, \\ \dfrac{d^{\alpha(x(t))}x(t)}{dt^{\alpha(x(t))}} = Cy(t) + D, \end{cases} \tag{3.1}$$

in which $\alpha(x(t))$ and $\alpha(y(t))$ are VO fractional derivative orders, and A, B, C and D are system parameters. Then a generalized form of dynamic-order fractional dynamic system has been pointed as a powerful tool to tackle the real-world complex phenomena and problems [57]. In addition, the VO fractional derivative can be well approximated through a fuzzy system; hereby this physical system can be recognized as a fuzzy-order dynamic system [108].

3.3. **Random-order fractional dynamic system.** Recently, the random-order fractional derivative model is generally used to describe the system relaxation, attenuation and diffusion phenomena. In the real-world applications, the physical systems usually suffer from some noises including fluctuations of the external pressure field in anomalous diffusion system, or unstable temperature field in the energy dissipation [102]. These noises inevitably cause the fluctuations of the whole system. In this case, the random-order fractional derivative model is a preferential choice to depict this type of fluctuation process.

In consequence, to better describe the influence of system noise on the dynamic behavior of physical system, the random-order FDE models have been effectively developed. For the models, the fractional derivative order include a constant and a random term, in which the constant term characterizes the average memory rate of the system and the random term represents the fluctuations caused by the random noises. The definition of random-order fractional integral can be stated as follows

$$I_{0+}^{\alpha_0+\varepsilon_t} f(t) = \frac{1}{\Gamma(\alpha_0+\varepsilon_t)} \int_0^t (t-\tau)^{(\alpha_0+\varepsilon_t-1)} f(\tau) d\tau, \ \ \alpha_0+\varepsilon_t > 0. \quad (3.2)$$

The expression of random-order fractional derivative is then developed as

$$D_{0+}^{\alpha_0+\varepsilon_t} f(t) = \frac{d^n}{dt^n} (I_{0+}^{n-\alpha_0-\varepsilon_t} f(t)), \ \ n-1 < \alpha_0+\varepsilon_t < n. \quad (3.3)$$

In the real-world engineering problem analysis, the random-order fractional derivative model can be quantitatively employed to evaluate and describe the fluctuation of the system. It is pointed that some possible applications include environmental pollution prediction, engineering risk estimates, system stability analysis, etc. [105]. In addition, the random-order fractional derivative can be extended into random-order FDE model to display anomalous diffusion on discrete finite domains [122].

4. **Numerical methods for VO-FDEs**

It is notable to mention that the derivation of an analytical solution for VO-FDE is still in its infancy due to the definitions of VO fractional operators. Hence, many numerical approximation methods and computational techniques have been suggested to investigate the VO-FDE models. For this purpose, many numerical methods have been studied in literature, such as finite difference methods (FDMs), spectral methods, matrix methods and spline interpolation methods, etc. [7, 23, 138]. Especially, more and more mathematical physical equations have been solved by using computationally efficient numerical methods [73, 95, 124].

4.1. **Numerical methods for time FDEs.** It is well-known that the finite difference schemes for VO time and/or space FDEs have been widely studied. There are several discretization schemes regarding different definitions.

The discretization of the Caputo-type VO time fractional derivative can be stated as follows

$$\begin{aligned}\frac{\partial^{\alpha_i^{k+1}} u(x_i,t_{k+1})}{\partial t^{\alpha_i^{k+1}}} &= \frac{1}{\Gamma(1-\alpha(x_i,t_{k+1}))}\int_0^{(k+1)\tau}\frac{\frac{\partial u(x_i,\tau)}{\partial \tau}}{(t_{k+1}-\tau)^{\alpha(x_i,t_{k+1})}}d\tau \\ &= \frac{1}{\Gamma(1-\alpha(x_i,t_{k+1}))}\sum_{j=0}^{k}\frac{u(x_i,t_{j+1})-u(x_i,t_j)}{\tau}\int_{j\tau}^{(j+1)\tau}\frac{d\tau}{(t_{k+1}-\tau)^{\alpha(x_i,t_{k+1})}} \\ &= \frac{\tau^{-\alpha_i^{k+1}}}{\Gamma(2-\alpha_i^{k+1})}\Big\{u(x_i,t_{k+1})-u(x_i,t_k)+\sum_{j=1}^{k}[u(x_i,t_{k+1-j})-u(x_i,t_{k-j})] \\ &\times[(j+1)^{1-\alpha_i^{k+1}}-j^{1-\alpha_i^{k+1}}]\Big\}+O(\tau).\end{aligned} \tag{4.1}$$

Cao and Qiu [17] introduced the following shifted Grünwald approximation concerning VO Riemann-Liouville derivative

$$A_{\tau,p}^{\alpha(t)}y(t)=\frac{1}{\tau^{\alpha(t)}}\sum_{k=0}^{\infty}g_k^{\alpha(t)}y(t-(k-p)\tau). \tag{4.2}$$

Moreover, Lorenzo [66] proposed the derivation of the Laplace transform of the VO integral as follows

$$\mathcal{L}\left\{{}_0D_t^{-\alpha(t)}f(t)\right\}=\int_0^{\infty}e^{-st}\left(\int_0^t\frac{(t-\tau)^{\alpha(t-\tau)-1}}{\Gamma(\alpha(t-\tau))}f(\tau)d\tau\right)dt,\ \alpha(t)>0,t>0. \tag{4.3}$$

In our previous investigations, we have examined three finite difference schemes including the explicit scheme, the implicit scheme and the Crank-Nicholson scheme for VO time FDEs. The accuracy, stability and convergence of these three schemes are tested and summarized [104]. The Crank-Nicholson scheme for the VO time fractional derivative can be formulated as follows

$$\begin{aligned}\frac{\partial^{\alpha_i^{k+1}}u(x_i,t_{k+1})}{\partial t^{\alpha_i^{k+1}}} &= \frac{\tau^{-\alpha_i^{k+1}}}{\Gamma(2-\alpha_i^{k+1})}\left(u(x_i,t_{k+1})-u(x_i,t_k)\right. \\ &+\sum_{j=1}^{k}[u(x_i,t_{k+1-j})-u(x_i,t_{k-j})]\left[(j+1)^{1-\alpha_i^{k+1}}-(j)^{1-\alpha_i^{k+1}}\right].\end{aligned} \tag{4.4}$$

It is well known that Burgers' equation has been employed to model gas dynamics, traffic flow, turbulence and fluid mechanics, etc. Tavares et al. [114] applied the approximation techniques to solve 1D linear inhomogeneous VO fractional Burgers' equation as follows

$$\begin{cases} {}_0^C D_t^{\alpha(t)} u(x,t) + \frac{\partial u}{\partial x}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + 2x - 2,\ t \in [0,1], \\ u(x,0) = x^2,\ x \in (0,1). \end{cases} \tag{4.5}$$

Bhrawy and Zaky [11] introduced Jacobi spectral collocation method as an efficient alternative approach to solve 2D VO fractional nonlinear cable equation in the following form

$$\frac{\partial u(x,y,t)}{\partial t} = {}_0D_t^{1-\gamma_1(x,y,t)} \Delta u(x,y,t) - \mu_0 D_t^{1-\gamma_2(x,y,t)} \Delta u(x,y,t) + f(x,y,t). \tag{4.6}$$

Jiang and Liu [53] proposed a new numerical method based on reproducing kernel theory and collocation method for the time VO fractional mobile-immobile advection-dispersion model

$$\beta_1 \frac{\partial C(x,t)}{\partial t} + \beta_2 D_t^{\gamma(x,t)} C(x,t) = -v \frac{\partial C(x,t)}{\partial t} + D \frac{\partial^2 C(x,t)}{\partial t^2} + f(x,t), \tag{4.7}$$

where v is the flow velocity, and D denotes the diffusion coefficient.

Furthermore, in terms of the numerical methods, the FDM is used to study the VO nonlinear fractional wave equation in [112]. Similarly, the numerical approximation of VO has been developed by Zayernouri and Karniadakis [133] by using FDMs, and various FDMs for VO fractional diffusion equations have been proposed. Sierociuk et al. [94] demonstrated a numerical scheme for a VO derivative based on matrix approach. Fu et al. [38] applied the method of approximate particular solutions for fractional diffusion model. Wei et al. [121] employed the local radial basis function method to solve the VO time fractional diffusion equation. Li and Wu [63] proposed a reproducing kernel method; afterward, they solved VO fractional boundary value problems for fractional differential equations based on the reproducing kernel theory. Chen et al. [24] demonstrated the Bernstein polynomials to seek the numerical solution of the VO fractional equation. Hafez and Youssri [43] developed the shifted Jacobi collocation method to solve the VO fractional linear subdiffusion and nonlinear reaction-subdiffusion equations. Based on the linear B-spline approximation with the Caputo sense and the Du Fort-Frankel algorithm, Moghaddam and Machado [72] proposed a stable three-level explicit FDM for conducting the nonlinear time VO-FDEs. Tayebi et al. [115] proposed an accurate and robust meshless method to solve the VO time fractional advection-diffusion equation model on 2D arbitrary domains.

From theoretical analysis aspect, the stability and convergence of numerical methods are discussed. Atangana [5] investigated the telegraph equation with VO fractional derivatives, and the stability as well as the convergence analysis are successfully verified. Subsequently, Jiang and Li [52] proposed a space-time spectral collocation method for the 2D VO fractional percolation equation, and then the exponential convergence was well confirmed. Chen [20] proposed a numerical simulation method with second-order temporal accuracy and fourth-order spatial accuracy on account of the second-order compact approximation formula of first-order derivative, to simulate the modified fractional diffusion equation. Umarov et al. [117] proved the existence and uniqueness theorem with respect to the Cauchy problem for VO fractional pseudo-differential equations. Lin et al. [65] studied the stability and convergence of an explicit finite-difference approximation for the VO nonlinear fractional diffusion equation. Zhang et al. [137] proposed a VO time fractional mobile-immobile advection-dispersion model and established an implicit Euler approximation which was proved to be unconditionally stable. Subsequently, an implicit numerical method for the 2D VO fractional percolation equation in non-homogeneous porous media was explored while the stability and convergence of the proposed method were discussed [22]. Moreover, Jia et al. [51] relied on the simplified reproducing kernel method to solve the efficient numerical scheme for VO fractional equation and verified its convergence.

4.2. **Numerical methods for VO space FDEs.** The relationship between the Riemann-Liouville and Grünwald-Letnikov definitions is important for the numerical approximation of FDEs with meaningful initial- and boundary-values, respectively. For example, using the relationship between Riemann-Liouville and Grünwald-Letnikov derivatives, a discrete approximation to the space fractional derivative terms $D_{a+}^{\alpha(x,t)}u(x,t)$ and $D_{b-}^{\alpha(x,t)}u(x,t)$ may be defined from the standard Grünwald-Letnikov formula [139]

$$D_{a+}^{\alpha(x,t)}u(x,t) = \lim_{M_1\to\infty}(h_1)^{-\alpha(x,t)}\sum_{j=0}^{M_1} g_{\alpha(x,t)}^{(j)}u(x-jh_1,t), \tag{4.8}$$

and

$$D_{b-}^{\alpha(x,t)}u(x,t) = \lim_{M_2\to\infty}(h_2)^{-\alpha(x,t)}\sum_{j=0}^{M_2} g_{\alpha(x,t)}^{(j)}u(x+jh_2,t), \tag{4.9}$$

where $M1$ and $M2$ are positive integers, $h_1 = (x-a)/M1$, $h_2 = (b-x)/M2$, and the normalized Grünwald-Letnikov weights are defined by

$$g_{\alpha(x,t)}^{(0)} = 1, \tag{4.10}$$

$$g_{\alpha(x,t)}^{(j)} = -\frac{\alpha(x,t) - j + 1}{j} g_{\alpha(x,t)}^{(j-1)} \quad \text{for} \quad j = 1, 2, 3, ... \tag{4.11}$$

Yang et al. [128] proposed the VO-FDEs depending on the reproducing kernel splines method. Most importantly, this method is able to successfully reduce computational cost and provided accurate approximate solutions. The VO-FDE is stated as

$$D_x^{\alpha(x)} u(x) + a(x)u'(x) + b(x)u(x) + c(x)u(\tau(x)) = f(x), \quad x \in [0, 1], \tag{4.12}$$

where $D_x^{\alpha(x)}$ is the VO fractional derivative in Caputo sense, and $\alpha(x) \in [1, 2)$. In addition, Zeng et al. [134] presented a new spectral collocation method to effectively solve the VO-FDE with high accuracy. The equation is built as follows

$$\begin{cases} {}^C D_x^{\alpha(x)} u(x) + C(x)u(x) = f(x), \ x \in (x_L, x_R), \ 0 < \alpha(x) < 1, \\ u(x_L) = 0. \end{cases} \tag{4.13}$$

Subsequently, Zaky et al. [132] developed Laguerre spectral collocation methods to solve the VO fractional initial value problem

$$a(x)u'(x) + b(x)^C D_x^{\alpha(x)} u(x) + c(x)u(x) = f(x), \tag{4.14}$$

where $\alpha(x) \in (0, 1)$. Furthermore, the VO fractional derivative has been employed to model the solute transport process in porous media

$$\frac{\partial c(x,t)}{\partial t} = k(x,t) R_{a(x,t)} c(x,t) - v(x,t) \frac{\partial c(x,t)}{\partial x} + f(c, x, t), \tag{4.15}$$

where $k(x, t)$ denotes the dispersion coefficient, and $R_{\alpha(x,t)}$ is the spatial fractional derivative of Riesz style [58]

$$\begin{cases} -(-\Delta)^{\alpha(x,t)/2} f(x) = -\frac{1}{2\cos\frac{\pi\alpha(x,t)}{2}} \left[{}_a D_x^{\alpha(x,t)} f(x) + {}_x D_b^{\alpha(x,t)} f(x) \right], \\ \qquad (m - 1 < \alpha(x,t) < m), \\ {}_{a+} D_x^{\alpha(x,t)} f(x) = \sum_{j=0}^{m-1} \frac{f^{(i)}(a)(x-a)^{j-\alpha(x,t)}}{\Gamma(-\alpha(x,t)+j+1)} + \frac{1}{\Gamma(m-\alpha(x,t))} \int_a^x \frac{f^{(m)}(\eta)}{(x-\eta)^{\alpha(x,t)-m+1}} d\eta, \\ {}_x D_{b-}^{\alpha(x,t)} f(x) = \sum_{j=0}^{m-1} \frac{(-1)^{m-j} f^{(i)}(b)(b-x)^{j-\alpha(x,t)}}{\Gamma(-\alpha(x,t)+j+1)} \\ \qquad + \frac{1}{\Gamma(m-\alpha(x,t))} \int_x^b \frac{f^{(m)}(\eta)}{(\eta-x)^{\alpha(x,t)-m+1}} d\eta. \end{cases} \tag{4.16}$$

Then Kameni et al. [55] presented the VO fractional advection-dispersion equation for modeling the movement of groundwater pollution and used the Fourier transform to solve this equation

$$R\frac{\partial C(x,t)}{\partial t} = -v \frac{\partial^{\alpha(y)} C(x,t)}{\partial x^{\alpha(y)}} + D \frac{\partial^2 C(x,t)}{\partial x^2} - \lambda R C(x,t), \ 0 < \alpha(y) \leq 1, \tag{4.17}$$

where λ represents the radioactivity decay rate, R denotes the retardation coefficient. Furthermore, an operational matrix method has been demonstrated to solve the VO fractional biharmonic equation [47]. The fractional Fourier transform has been investigated through VO fractional differential operators by Tseng [116]. Atangana and Cloot [7] studied the stability and convergence of the Crank-Nicolson difference scheme for the space VO fractional Schrödinger equation with Caputo derivative.

5. Applications of VO-FDE models

5.1. **VO fractional diffusion equation models.** To our best knowledge, anomalous diffusion is ubiquitous phenomena and the growth rate or shape of the particle distribution is not comply with Gaussian distribution [35]. For instance, some typical examples include heat conduction, solute transport, groundwater pollution, gas flow in highly heterogeneous fractured or disordered porous media, relaxation in synthetic or biopolymers, propagation of seismic waves, for more details, see [4, 18, 48, 93, 97, 119, 129]. These applications motivate the development of the new mathematical and physical models. However, how to deal with the diffusion process in which the diffusion pattern changes with time evolution, spatial variation or system parameters, is still an open topic in anomalous diffusion modeling. Recently, in order to overcome the drawbacks of the integral order fractional models, the VO fractional derivative models are employed to provide a robust and rigorous approach for describing the memory effects, hereditary properties, and the delay behavior in physical applications [61, 80]. Hence, the VO derivative models have become an important research tool in complex anomalous diffusion modeling.

Obembe et al. [80] investigated the fluid flow through VO time fractional diffusion models in fractal geometry or heterogeneous media. Then, a generalized time dependent non-local flux law is employed at different scales as follow

$$u = -\frac{\beta K_a}{\mu_0} D_t^{\alpha(t)}(\nabla p), \tag{5.1}$$

where K_a is the pseudo-permeability, β is the transmissibility conversion factor, μ_0 is the oil viscosity. The VO derivative models can be used to describe the diffusion process with time dependence, the non-uniform medium particle migration and diffusion process of intermittent turbulence [98]. A VO fractional diffusion equation for describing the liquid infiltration in porous media is proposed [40]

$$\frac{\partial^{\alpha(U)} U}{\partial t^{\alpha(U)}} = \frac{\partial}{\partial x}\left(K(U)\frac{\partial U}{\partial x}\right), \tag{5.2}$$

where U is a liquid content, and then

$$\frac{\partial^{\alpha(U(x,t))}U(x,t)}{\partial t^{\alpha(U(x,t))}} = \lim_{\tau\to 0}\frac{\sum\limits_{k=0}^{\infty} A_k U(x,t-k\tau)}{\tau^{\alpha(U(x,t))}}. \tag{5.3}$$

Furthermore, the definition of the VO fractional derivative, can also consider another description of the system memory. The governing equation of time-dependent fractional anomalous diffusion can be expressed as the following form [103]

$${}^{C}D_{0+}^{\alpha(t)}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2}, \;\; 0 < \alpha(t) < 1, \tag{5.4}$$

where K is the dispersion coefficient. Based on the basic concept of definition, the order $\alpha(t)$ can be substituted by other variables. The VO time-space dependent anomalous diffusion model is defined by

$$D_t^{\alpha(x,t)}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2}, \;\; 0 < \alpha(x,t) < 1. \tag{5.5}$$

Straka [101] derived a VO fractional Fokker-Planck equation with variable anomalous exponent

$$D_t^{\beta(x)}P(x,t) = K\frac{\partial^2 P(x,t)}{\partial x^2}. \tag{5.6}$$

In consequence, the VO-FDE models can efficiently describe anomalous diffusion process in complex anisotropic medium. Space-dependent VO-FDE model is used to describe location-dependent diffusion process. Additionally, Chen et al. [28] proposed a concentration-dependent VO fractional diffusion equation model to describe the coupled chloride diffusion-binding processes in reinforced concrete

$$\frac{\partial^{p(C_f)}C_f(x,t)}{\partial t^{p(C_f)}} = K\frac{\partial^2 C_f(x,t)}{\partial x^2}, \; 0 < x < +\infty, \; t > 0, \tag{5.7}$$

where $0 < p(C_f) < 1$ is the VO of time-fractional derivative.

In certain circumstances, the diffusion behavior of some physical, chemical and biological fields are affected by the solute concentration which determines the diffusion or migration process [29]. Consequently, the concentration-dependent VO-FDE model has very important application potentials, and has been concerned in some chemical or biology diffusion processes. This application background makes us to conduct further research on concentration-dependent characteristics of VO fractional diffusion model

$${}^{C}D_{0+}^{\alpha[c(x,t)]}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2}, \;\; < \alpha[c(x,t)] < 1. \tag{5.8}$$

In addition, in some practical diffusion processes, physical parameters, such as the porosity, Reynolds number, fractal dimension and Hurst numbers may change over time or have different values in different space position

$$ {}^{C}D_{0+}^{\alpha[f(x,t)]}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2}, \quad 0 < \alpha[f(x,t)] < 1. \tag{5.9}$$

In the above equation, the $f(x,t)$ represents an independent variable-function.

In order to make the readers intuitively understand those models, we summary the VO-FDE models in Table 1.

Authors	Model name	Governing equation	Physical meaning
Gerasimov, Kondratieva & Sinkevich (2010)[40]	Content-dependent anomalous diffusion model	$\frac{\partial^{\alpha(U)}U}{\partial t^{\alpha(U)}} = \frac{\partial}{\partial x}(K(U)\frac{\partial U}{\partial x})$	Exploring liquid infiltration in porous media
Sun, Chen & Chen (2009)[103]	VO time-space dependent anomalous diffusion model	$D_t^{\alpha(x,t)}c(x,t) = K\frac{\partial^2 c(x,t)}{\partial x^2}$ $0 < \alpha(x,t) < 1$	Depicting concentration breakthrough curve exhibits diverse anomalous behaviors
Straka (2018)[101]	VO fractional Fokker-Planck equation	$D_t^{\beta(x)}P(x,t) = K\frac{\partial^2 P(x,t)}{\partial x^2}$	Describing spatial heterogeneity in complex anisotropic medium
Chen, Zhang & Zhang (2013)[28]	Concentration-dependent VO fractional diffusion equation model	$\frac{\partial^{p(C_f)}C_f(x,t)}{\partial t^{p(C_f)}} = K\frac{\partial^2 C_f(x,t)}{\partial x^2}$ $0 < x < \infty$, $t > 0$	Describing the coupled chloride diffusion -binding processes in reinforced concrete

TABLE 1. The applications of different VO fractional diffusion models.

Regarding the more complex transient dispersion in heterogeneous media, we used the left and right spatial VO fractional differential operators to offer a generalized model

$$\frac{\partial^{\alpha(x,t)}}{\partial t^{\alpha(x,t)}}u(x,t) = -v\frac{\partial}{\partial x}u(x,t)+D_+\frac{\partial^{\beta(x,t)}u(x,t)}{\partial_+ x^{\beta(x,t)}}+D_-\frac{\partial^{\beta(x,t)}u(x,t)}{\partial_- x^{\beta(x,t)}}, \quad (5.10)$$

where $D+$ and $D-$ are the positive and negative diffusion coefficients, respectively.

As a remark, we can conclude that the VO fractional diffusion equation model is a developed and promising approach to characterize time-dependent, space-dependent or concentration-dependent anomalous diffusion process in heterogeneous porous media [104].

5.2. **VO fractional viscoelasticity constitutive models.** There are many viscoelastic materials in engineeing, including polymers, metal material, non-Newtonian fluid, plastic, rubber, soil, oil, concrete and so on. These materials are widely used in chemical, petroleum engineering, biology, medicine, environmental engineering, and other fields [104]. For viscoelastic materials, stress is a functional strain [85]. The two kinds of situations collectively known as the rheological phenomenon have shown as: relaxation, meaning that stress decreases under the condition of constant strain; creep, that is, deformation continuously increases under the phenomenon of constant stress.

Then the generalized time-dependent model for viscoelastic deformation has been proposed [49]

$$\sigma(t) = \mu D_t^{\alpha}\varepsilon(t), \quad (5.11)$$

where σ is the stress, ε is the strain, μ is the dimensional coefficient and D^{α} is the fractional differential operator. Bagley [9] offered that the polymer linear viscoelastic stress relaxation with a given fixed temperature can be well described by FDE models. A practical fractional derivative rheological model has been documented by Smit and DeVries [96]

$$\sigma = E\frac{d^{\alpha}\varepsilon(t)}{dt^{\alpha}}, \quad (5.12)$$

where E is the material parameter and $0 < \alpha < 1$. When $\alpha = 0$, the above function becomes Hooke's law; when $\alpha = 1$, it is the Newton viscous law. However, it is clear that the variation of material feature has not been well considered by the CO fractional derivative models. From the implementation point of view, Sweilam et al. [113] thought the VO-FDE is an important tool to study some systems, such as the control of nonlinear viscoelasticity oscillator. In order to deal with the variable mechanical behaviors depending on time, space variation or system parameters, the

VO constitutive models have received much attention in the viscoelasticity fields. The constitutive model can be written as [70]

$$\sigma = E\frac{d^{\alpha(t)}\varepsilon(t)}{dt^{\alpha(t)}}. \tag{5.13}$$

The dimension changes with $\alpha(t)$, and any intermediate value between 0 and 1 can be captured with the function $\alpha(t)$ [100]. Furthermore, the constitutive model can capture accurately mechanical responses and represent the transition of mechanical property. Hence, for a viscoelastic material, it is exhibited that the stress is a function not only of the actual strain and strain rate at the current stage of the deformation process, but also of the previous strain history [49]. Thereupon, a differential operator of VO in constitutive relation for viscoelastic material has been introduced by Suzdalnitsky and Ingman [110]. Pedro et al. [81] employed a VO derivative to account for the strong non-linearity flow. In addition, the evidence of the VO nature of the particles dynamics flows was provided by the behavior of an oscillating particle in a fluid [31, 32]. Subsequently, Bouras et al. [14] developed a novel non-linear thermo-viscoelastic rheological model based on VO time fractional derivative for high temperature creep in concrete, which can be expressed as

$$D_t^{\alpha(T(t))}\varepsilon(t) = \frac{\sigma(t)}{\eta\alpha(T(t))}. \tag{5.14}$$

Li et al. [62] argued that a VO-FDE model of the shape-memory behavior is more suitable than CO-FDE models in terms of modeling the memory behavior of shape-memory polymer. The viscoelastic behavior of a single particle oscillating in a non-Newtonian fluid can capture the macroscopic viscoelasticity of suspensions of colloidal particles. Normally, the fleld of rheology (including the behavior of suspensions and polymers) is mainly devoted to study the stress-strain relationship of materials by VO fractional model [100].

In addition, Wu et al. [123] proposed a creep model based on VO fractional derivative for describing the time-dependent mechanical property of rock during the creep. The stress-strain relationship of the Abel dashpot is written as

$$\varepsilon(t) = \frac{\sigma}{\eta_0}\frac{t^\beta}{\Gamma(1+\beta)}, \quad 0 \le \beta \le 1. \tag{5.15}$$

In this way, the order of the VO fractional derivative can be regarded as a function of time

$$\sigma(t) = \eta_{\alpha(t)}\,{}_{t_{k-1}}^{C}D_t^{\alpha(t)}\varepsilon(t), \quad 0 \le \alpha(t) \le 1,\ t_{k-1} \le t \le t_k. \tag{5.16}$$

According to different creep stages of the experimental results, it is found that the improved creep model based on VO fractional derivative agrees well with the experimental data. It shows that the evolution of mechanical properties of materials can be well characterized by changing the derivative order during the whole process.

In conclusion, based on the idea of using the fractional order to characterize the mechanical property, a VO fractional viscoelastic model is derived from the corresponding fractional viscoelastic model. The VO model is expected to represent the transition of mechanical feature including strain softening behaviors through the variable fractional order [70].

5.3. **VO fractional control model.** Fractional derivative model has been recognized as a powerful modeling approach in many natural and engineering fields [28, 33]. Therefore, the VO fractional calculus has also received extensive attention in the field of control.

Diaz and Coimbra [37] proposed a control technique and applied the VO approach as the control action to stabilize a chaotic dynamical system. At the same time, the VO fractional operators can be also applied to describe VO fractional noise and estimate the VO fractional derivative of an unknown signal in noisy environment based on the wavelet analysis [26]. Heydari and Avazzadeh [46] demonstrated a new computational method based on the Legendre wavelets for solving the VO fractional optimal control problems.

5.4. **Other applications.** There are many other complex physical phenomena that can be described by the VO fractional derivatives, which will be explored in this section.

For example, Chen et al. [22] studied a 2D VO fractional percolation equation in non-homogeneous porous media via a modified Darcy's law with VO Riemann-Liouville fractional derivatives defined as follows

$$q_x = k_x \frac{\partial^{\alpha(x,y)} p}{\partial x^{\alpha(x,y)}},\ q_y = k_y \frac{\partial^{\beta(x,y)} p}{\partial y^{\beta(x,y)}},\ 0 < \alpha(x,y), \beta(x,y) < 1. \tag{5.17}$$

The VO Riemann-Liouville fractional derivative is defined as

$$\frac{\partial^{\alpha(x,y)} p}{\partial x^{\alpha(x,y)}} = \frac{1}{\Gamma(1-\alpha(x,y))} \left(\frac{\partial}{\partial \xi} \int_0^{\xi} \frac{p(s,y,t)}{(\xi - s)^{\alpha(x,y)}} ds \right)_{\xi = x}. \tag{5.18}$$

Then the 2D VO fractional percolation equation is given by

$$\begin{aligned} \tfrac{\partial p}{\partial t} = \tfrac{\partial}{\partial x}\left(A(x,y)\tfrac{\partial^{\alpha(x,y)}}{\partial x^{\alpha(x,y)}} p\right) + \tfrac{\partial}{\partial y}\left(B(x,y)\tfrac{\partial^{\beta(x,y)}}{\partial x^{\beta(x,y)}} p\right) + f(x,y,t), \\ (x,y) \in \Omega,\ 0 \le t \le T. \end{aligned} \tag{5.19}$$

Meanwhile, the VO fractional telegraph equation has been proposed in many complex processes that is an extension of CO fractional derivative [5]

$$D_t^{\alpha(x,t)}w + k\frac{\partial w}{\partial t} - a\frac{\partial^2 w}{\partial x^2} + bw = 0, \quad 1 < \alpha(x,t) \leq 2, \quad k > 0, \; b > 0, \quad (5.20)$$

where k is a natural number.

A generalized groundwater flow equation using the concept of VO fractional derivative was investigated by Atangana et al. [6], due to the groundwater flow changes in time and space. Thus, the governing equation is obtained as

$$SD_t^{\alpha(x,t)}\phi(r,t) = TD_{rr}\phi(r,t) + \frac{1}{r}D_r\phi(r,t), \quad 0 < \alpha(x,t) < 1, \qquad (5.21)$$

where ϕ is the piezometric head, and S is the storativity.

The VO fractional derivative is good at depicting memory properties that change with time or spatial location [68]. For instance, Bhrawy and Zaky [13] explored a high-order numerical scheme for soving the multi-dimensional VO fractional Schrödinger equations. Moghaddam et al. [73] developed a technique for the approximate solution in regard to the VO fractional Bagley-Torvik and Basset differential equations in the area of fluid dynamics; meanwhile, the accuracy of the proposed algorithm was properly verified. Due to many parameters involved in the existing physiological models for bone remodeling, Neto et al. [78] presented a new approach with VO derivative to simplify its structure and provide more compact models that lead to similar results. Gómez-Aguilar [42] analyzed a VO fractional alcoholism model to describe the complex dynamics and illustrated the uniqueness and existence of the solutions employing the fixed point postulate. Almeida et al. [1] proposed the Malthusian growth model with a time VO-FDE to determine the fractional order function. In fact, a great quantity of natural phenomena can be modeled by the VO-FDE models and the study of such problems has attracted much attention.

6. **Conclusions**

VO fractional calculus has been extended from the notion of CO fractional calculus with the order of differentiation or integration varying with time (t), space (x), and other variables. In general, the facilitation of VO fractional calculus enables it to better characterize many memory systems, hereditary properties, and dynamic processes. Moreover, based on previous investigations, the physical meaning of VO fractional calculus has been explored. But it should be noted that the analytical solutions of VO-FDE models are extremely difficult to derive, and the efficient numerical/approximate solutions are of great importance in practice. We hope this work can play a vital role in studying of VO fractional calculus, which is designed to unravel some uncanny features of systems and handle some hard issues in real-world applications.

Acknowledgements

We thank the organizers for their assistance and many participants for their comments at the ICFDA'18 International Conference on Fractional Differentiation and its Applications. This work was supported by the National Natural Science Foundation of China under Grants Nos. 11572112, 41628202, and 11811530069, the Postgraduate Research & Practice Innovation Program of Jiangsu Province (Grant No. KYCX18_0524), and the Fundamental Research Funds for the Central Universities under Grants nos. 2017B21614 and 2018B680X14.

References

[1] R. Almeida, N.R.O. Bastos, M.T.T. Monteiro, A fractional Malthusian growth model with variable order using an optimization approach. *Stat. Optim. Inf. Comput.* **6**, No 1 (2018), 4–11.

[2] R. Almeida, D. Tavares, D.F.M. Torres, *The Variable-Order Fractional Calculus of Variations.* SpringerBriefs in Applied Sciences and Technology, Springer, Cham (2018).

[3] R. Almeida, D.F.M. Torres, Computing Hadamard type operators of variable fractional order. *Appl. Math. Comput.* **257** (2015), 74–88.

[4] V.V. Anh, J.M. Angulo, M.D. Ruiz-Medina, Diffusion on multifractals. *Nonlinear Anal.-Theor.* **63**, No 5 (2005), 2043–2056.

[5] A. Atangana, On the stability and convergence of the time-fractional variable order telegraph equation. *J. Comput. Phys.* **293**, No C (2015), 104–114.

[6] A. Atangana, J.F. Botha, A generalized groundwater flow equation using the concept of variable-order derivative. *Bound. Value Probl.* **2013**, No 1 (2013), # 53.

[7] A. Atangana, A.H. Cloot, Stability and convergence of the space fractional variable-order Schrödinger equation. *Adv. Differ. Equ.* **2013**, No 1 (2013), 1–10.

[8] A.A. Awotunde, R.A. Ghanam, N.E. Tatar, Artificial boundary condition for a modified fractional diffusion problem. *Bound. Value Probl.* **2015**, No 1 (2015), # 20.

[9] R.L. Bagley, The thermorheologically complex material. *J. Acoust. Soc. Am.* **90**, No 7 (1991), 797–806.

[10] P. Balasubramaniam, P. Muthukumar, K. Ratnavelu, Theoretical and practical applications of fuzzy fractional integral sliding mode control

for fractional-order dynamical system. *Nonlinear Dynam.* **80**, No 1-2 (2015), 249–267.

[11] A.H. Bhrawy, M.A. Zaky, Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation. *Nonlinear Dynam.* **80**, No 1-2 (2015), 101–116.

[12] A.H. Bhrawy, M.A. Zaky, Numerical algorithm for the variable-order Caputo fractional functional differential equation. *Nonlinear Dynam.* **85**, No 3 (2016), 1815–1823.

[13] A.H. Bhrawy, M.A. Zaky, An improved collocation method for multi-dimensional space-time variable-order fractional Schrödinger equations. *Appl. Numer. Math.* **111** (2017), 197–218.

[14] Y. Bouras, D. Zorica, T.M. Atanacković, Z. Vrcelj, A non-linear thermo-viscoelastic rheological model based on fractional derivatives for high temperature creep in concrete. *Appl. Math. Model.* **55** (2018), 551–568.

[15] D.W. Brzeziński, P. Ostalczyk, About accuracy increase of fractional order derivative and integral computations by applying the Grünwald-Letnikov formula. *Commun. Nonlinear Sci.* **40** (2016), 151-162.

[16] W. Cai, W. Chen, J. Fang, S. Holm, A survey on fractional derivative modeling of power-law frequency-dependent viscous dissipative and scattering attenuation in acoustic wave propagation. *Appl. Mech. Rev.* **70**, No 3 (2018), # 030802.

[17] J. Cao, Y. Qiu, A high order numerical scheme for variable order fractional ordinary differential equation. *Appl. Math. Lett.* **61** (2016), 88–94.

[18] A. Chang, H. Sun, Time-space fractional derivative models for CO2 transport in heterogeneous media. *Fract. Calc. Appl. Anal.* **21**, No 1 (2018), 151–173; DOI: 10.1515/fca-2018-0010; https://www.degruyter.com/view/j/fca.2018.21.issue-1/issue-files/fca.2018.21.issue-1.xml.

[19] A.V. Chechkin, R. Gorenflo, I.M. Sokolov, Fractional diffusion in inhomogeneous media. *J. Phys. A: Gen. Phys.* **38**, No 42 (2005), 679–684.

[20] Y. Chen, C. Chen, Numerical simulation with the second order compact approximation of first order derivative for the modified fractional diffusion equation. *Appl. Math. Comput.* **320** (2018), 319–330.

[21] C. Chen, F. Liu, V. Anh, I. Turner, Numerical schemes with high spatial accuracy for a variable-order anomalous subdiffusion equation. *SIAM J. Sci. Comput.* **32**, No 4 (2010), 1740–1760.

[22] S. Chen, F. Liu, K. Burrage, Numerical simulation of a new two-dimensional variable-order fractional percolation equation in non-homogeneous porous media. *Comput. Math. Appl.* **68**, No 12 (2014), 2133–2141.

[23] Y. Chen, L. Liu, B. Li, Y. Sun, Numerical solution for the variable order linear cable equation with Bernstein polynomials. *Appl. Math. Comput.* **238**, No 7 (2014), 329–341.

[24] Y. Chen, L. Liu, D. Liu, D. Boutat, Numerical study of a class of variable order nonlinear fractional differential equation in terms of Bernstein polynomials. *Ain Shams Eng. J.* **9**, No 4 (2018), 1235–1241.

[25] H. Chen, H. Wang, Numerical simulation for conservative fractional diffusion equations by an expanded mixed formulation. *J. Comput. Appl. Math.* **296** (2015), 480–498.

[26] Y. Chen, Y. Wei, D. Liu, D. Boutat, X. Chen, Variable-order fractional numerical differentiation for noisy signals by wavelet denoising. *J. Comput. Phys.* **311** (2016), 338–347.

[27] Y. Chen, Y. Wei, D. Liu, H. Yu, Numerical solution for a class of nonlinear variable order fractional differential equations with Legendre wavelets. *Appl. Math. Lett.* **46** (2015), 83–88.

[28] W. Chen, J. Zhang, J. Zhang, A variable-order time-fractional derivative model for chloride ions sub-diffusion in concrete structures. *Fract. Calc. Appl. Anal.* **16**, No 1 (2013), 76–92; DOI: 10.2478/s13540-013-0006-y; https://www.degruyter.com/view/j/fca.2013.16.issue-1/issue-files/fca.2013.16.issue-1.xml.

[29] T.S.Y. Choong, T.N. Wong, T.G. Chuah, A. Idris, Film-pore-concentration-dependent surface diffusion model for the adsorption of dye onto palm kernel shell activated carbon. *J. Colloid Interf. Sci.* **301**, No 2 (2006), 436–440.

[30] C.F.M. Coimbra, Mechanics with variable-order differential operators. *Ann. Der Phys.* **12**, No (11-12) (2003), 692–703.

[31] C.F.M. Coimbra, D. L'Esperance, R.A. Lambert, J.D. Trolinger, R.H. Rangel, An experimental study on stationary history effects in high-frequency Stokes flows. *J. Fluid Mech.* **504**, No 504 (2004), 353–363.

[32] C.F.M. Coimbra, R.H. Rangel, Spherical particle motion in harmonic Stokes flows. *AIAA J.* **39**, No 9 (2015), 1673–1682.

[33] G.R.J. Cooper, D.R. Cowan, Filtering using variable order vertical derivatives. *Comput. Geosci.* **30**, No 5 (2004), 455–459.

[34] A. Dabiri, B.P. Moghaddam, J.A.T. Machado, Optimal variable-order fractional PID controllers for dynamical systems. *J. Comput. Appl. Math.* **339** (2018), 40–48.

[35] W. Deng, Numerical algorithm for the time fractional Fokker-Planck equation. *J. Comput. Phys.* **227**, No 2 (2007), 1510–1522.

[36] G. Diaz, C.F.M. Coimbra, Nonlinear dynamics and control of a variable order oscillator with application to the van der Pol equation. *Nonlinear Dynam.* **56**, No 1 (2009), 145–157.

[37] G. Diaz, C.F.M. Coimbra, Dynamics and control of nonlinear variable order oscillators. Ch. 6, in: *Nonlinear Dynamics* (Ed. T. Evans), InTech (2010), 129–144.

[38] Z. Fu, W. Chen, L. Ling, Method of approximate particular solutions for constant- and variable-order fractional diffusion models. *Eng. Anal. Bound. Elem.* **57** (2015), 37–46.

[39] Z. Ge, C. Ou, Chaos synchronization of fractional order modified duffing systems with parameters excited by a chaotic signal. *Chaos Soliton. Fract.* **35**, No 4 (2008), 705–717.

[40] D.N. Gerasimov, V.A. Kondratieva, O.A. Sinkevich, An anomalous non-self-similar infiltration and fractional diffusion equation. *Phys. D Nonlinear Phenom.* **239**, No 16 (2010), 1593–1597.

[41] W.G. Glöckle, T.F. Nonnenmacher, A fractional calculus approach to self-similar protein dynamics. *Biophys. J.* **68**, No 1 (1995), 46–53.

[42] J.F. Gómez-Aguilar, Analytical and numerical solutions of a nonlinear alcoholism model via variable-order fractional differential equations. *Phys. A* **494** (2018), 52–75.

[43] R.M. Hafez, Y.H. Youssri, Jacobi collocation scheme for variable-order fractional reaction-subdiffusion equation. *Comput. Appl. Math.* **37**, No 4 (2018), 5315–5333.

[44] G. He, M. Luo, Dynamic behavior of fractional order Duffing chaotic system and its synchronization via singly active control. *Appl. Math. Mech.* **33**, No 5 (2012), 567–582.

[45] R. Herrmann, Uniqueness of the fractional derivative definition. *arXiv Preprint*, arXiv:1303.2939 (2013).

[46] M.H. Heydari, Z. Avazzadeh, A new wavelet method for variable-order fractional optimal control problems. *Asian J. Control* **20**, No 5 (2017), 1–14.

[47] M.H. Heydari, Z. Avazzadeh, An operational matrix method for solving variable-order fractional biharmonic equation. *Comput. Appl. Math.* **37**, No 4 (2018), 4397–4411.

[48] Y. Hong, J. Lin, W. Chen, Simulation of thermal field in mass concrete structures with cooling pipes by the localized radial basis function collocation method. *Int. J. Heat Mass Tran.* **129** (2019), 449–459.

[49] D. Ingman, J. Suzdalnitsky, Control of damping oscillations by fractional differential operator with time-dependent order. *Comput. Method. Appl. Mech. Eng.* **193**, No 52 (2004), 5585–5595.

[50] D. Ingman, J. Suzdalnitsky, M. Zeifman, Constitutive dynamic-order model for nonlinear contact phenomena. *J. Appl. Mech.* **67**, No 2 (2000), 383–390.

[51] Y. Jia, M. Xu, Y. Lin, A numerical solution for variable order fractional functional differential equation. *Appl. Math. Lett.* **64** (2017), 125–130.

[52] W. Jiang, H. Li, A space-time spectral collocation method for the two-dimensional variable-order fractional percolation equations. *Comput. Math. Appl.* **75**, No 10 (2018), 3508–3520.

[53] W. Jiang, N. Liu, A numerical method for solving the time variable fractional order mobile-immobile advection-dispersion model. *Appl. Numer. Math.* **119** (2017), 18–32.

[54] S. Jiang, J. Zhang, Z. Qian, Z. Zhang, Fast evaluation of the Caputo fractional derivative and its applications to fractional diffusion equations. *Commun. Comput. Phys.* **21**, No 3 (2017), 650–678.

[55] S.N. Kamenia, J.D. Djidaa, A. Atangana, Modelling the movement of groundwater pollution with variable order derivative. *J. Nonlinear Sci. Appl.* **10** (2017), 5422–5432.

[56] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* Elsevier (2006).

[57] Y.L. Kobelev, L.Y. Kobelev, Y.L. Klimontovich, Statistical physics of dynamic systems with variable memory. *Dokl. Phys.* **48**, No 6 (2003), 285–289.

[58] Y.L. Kobelev, L.Y. Kobelev, Y.L. Klimontovich, Anomalous diffusion with time-and coordinate-dependent memory. *Dokl. Phys.* **48**, No 6 (2003), 264–268.

[59] P. Kumar, S.K. Chaudhary, Analysis of fractional order control system with performance and stability. *Int. J. Eng. Sci. Tech.* **9**, No 5 (2017), 408–416.

[60] T.A.M. Langlands, B.I. Henry, Fractional chemotaxis diffusion equations. *Phys. Rev. E* **81** (2010), # 051102.

[61] J.R. Leith, Fractal scaling of fractional diffusion processes. *Signal Process.* **83**, No 11 (2003), 2397–2409.

[62] Z. Li, H. Wang, R. Xiao, S. Yang, A variable-order fractional differential equation model of shape memory polymers. *Chaos Soliton. Fract.* **102** (2017), 473–485.

[63] X. Li, B. Wu, A numerical technique for variable fractional functional boundary value problems. *Appl. Math. Lett.* **43** (2015), 108–113.

[64] X. Li, B. Wu, A new reproducing kernel method for variable order fractional boundary value problems for functional differential equations. *J. Comput. Appl. Math.* **311** (2016), 387–393.

[65] R. Lin, F. Liu, V. Anh, I. Turner, Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. *Appl. Math. Comput.* **212**, No 2 (2009), 435–445.

[66] C.F. Lorenzo, T.T. Hartley, Variable order and distributed order fractional operators. *Nonlinear Dynam.* **29**, No 1 (2002), 57–98.

[67] C.F. Lorenzo, T.T. Hartley, Initialization, conceptualization, and application in the generalized fractional calculus. *Crit. Rev. Biomed. Eng.* **35**, No 6 (2007), 477–553.

[68] R.L. Magin, O. Abdullah, D. Baleanu, X.J. Zhou, Anomalous diffusion expressed through fractional order differential operators in the Bloch-Torrey equation. *J. Magn. Reson.* **190**, No 2 (2008), 255–270.

[69] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations. *Appl. Numer. Math.* **51**, No 1 (2006), 80–90.

[70] R. Meng, D. Yin, C. Zhou, H. Wu, Fractional description of time-dependent mechanical property evolution in materials with strain softening behavior. *Appl. Math. Model.* **40**, No 1 (2016), 398–406.

[71] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* **339**, No 1 (2000), 1–77.

[72] B.P. Moghaddam, J.A.T. Machado, A stable three-level explicit spline finite difference scheme for a class of nonlinear time variable order fractional partial differential equations. *Comput. Math. Appl.* **73**, (2017), 1262–1269.

[73] B.P. Moghaddam, J.A.T. Machado, H. Behforooz, An integro quadratic spline approach for a class of variable-order fractional initial value problems. *Chaos Soliton. Fract.* **102**, Suppl. C (2017), 354–360.

[74] P. Muthukumar, P. Balasubramaniam, Feedback synchronization of the fractional order reverse butterfly-shaped chaotic system and its application to digital cryptography. *Nonlinear Dynam.* **74**, No 4 (2013), 1169–1181.

[75] P. Muthukumar, P. Balasubramaniam, K. Ratnavelu, Synchronization and an application of a novel fractional order King Cobra chaotic system. *Chaos* **24**, No 3 (2014), # 033105.

[76] P. Muthukumar, P. Balasubramaniam, K. Ratnavelu, Synchronization of a novel fractional order stretch-twist-fold (STF) flow chaotic system

and its application to a new authenticated encryption scheme (AES). *Nonlinear Dynam.* **77**, No 4 (2014), 1547–1559.

[77] P. Muthukumar, P. Balasubramaniam, K. Ratnavelu, Fast projective synchronization of fractional order chaotic and reverse chaotic systems with its application to an affine cipher using date of birth (DOB). *Nonlinear Dynam.* **80**, No 4 (2015), 1883–1897.

[78] J. Pinheiro Neto, R. Moura Coelho, D. Valério, S. Vinga, D. Sierociuk, W. Malesza, M. Macias, A. Dzieliński, Variable order differential models of bone remodelling. *IFAC Int. Fed. Autom. Control* **50**, No 1 (2017), 8066–8071.

[79] S. Nimmo, A.K. Evans, The effects of continuously varying the fractional differential order of chaotic nonlinear systems. *Chaos Soliton. Fract.* **10**, No 7 (1999), 1111–1118.

[80] A.D. Obembe, M.E. Hossain, S.A. Abu-Khamsin, Variable-order derivative time fractional diffusion model for heterogeneous porous media. *J. Petrol. Sci. Eng.* **152** (2017), 391–405.

[81] H.T.C. Pedro, M.H. Kobayashi, J.M.C. Pereira, C.F.M. Coimbra, Variable order modeling of diffusive-convective effects on the oscillatory flow past a sphere. *J. Vib. Control* **14**, No (9-10) (2008), 1659–1672.

[82] I. Podlubny, *Fractional Differential Equations.* Academic Press, San Diego etc. (1999).

[83] I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calc. Appl. Anal.* **5**, No 4 (2002), 230–237.

[84] Y. Povstenko, J. Klekot, The Dirichlet problem for the time-fractional advection-diffusion equation in a line segment. *Bound. Value Probl.* **2016**, No 1 (2016), # 89.

[85] L.E.S. Ramirez, C.F.M. Coimbra, A variable order constitutive relation for viscoelasticity. *Ann. Der Phys.* **16**, No 7-8 (2007), 543–552.

[86] A. Razminia, A.F. Dizaji, V.J. Majd, Solution existence for non-autonomous variable-order fractional differential equations. *Math. Comput. Model.* **55**, No 3 (2012), 1106–1117.

[87] M.D. Ruiz-Medina, V.V. Anh, J.M. Angulo, Fractional generalized random fields of variable order. *Stoch. Anal. Appl.* **22**, No 3 (2004), 775–799.

[88] S.G. Samko, Fractional integration and differentiation of variable order. *Anal. Math.* **21**, No 3 (1995), 213–236.

[89] S.G. Samko, B. Ross, Integration and differentiation to a variable fractional order. *Integr. Transf. Spec. Funct.* **1**, No 4 (1993), 277–300.

[90] S. Shen, F. Liu, J. Chen, I. Turner, V. Anh, Numerical techniques for the variable order time fractional diffusion equation. *Appl. Math. Comput.* **218**, No 22 (2011), 10861–10870.

[91] H. Sheng, H.G. Sun, Y.Q. Chen, T. Qiu, Synthesis of multifractional Gaussian noises based on variable-order fractional operators. *Signal Process.* **91**, No 7 (2011), 1645–1650.

[92] H. Sheng, H.G. Sun, C. Coopmans, Y. Chen, G.W. Bohannan, A Physical experimental study of variable-order fractional integrator and differentiator. *Eur. Phys. J. Spec. Top.* **193**, No 1 (2011), 93–104.

[93] D. Sierociuk, A. Dzieliński, G. Sarwas, I. Petras, I. Podlubny, T. Skovranek, Modelling heat transfer in heterogeneous media using fractional calculus. *Philos. Trans.* **371**, No 1990 (2013), # 20120146.

[94] D. Sierociuk, W. Malesza, M. Macias, Derivation, interpretation, and analog modelling of fractional variable order derivative definition. *Appl. Math. Model.* **39**, No 13 (2014), 3876–3888.

[95] D. Sierociuk, W. Malesza, M. Macias, On the recursive fractional variable-order derivative: Equivalent switching strategy, duality, and analog modeling. *Circ. Syst. Signal Process.* **34**, No 4 (2015), 1077–1113.

[96] W. Smit, H.D. Vries, Rheological models containing fractional derivatives. *Rheol. Acta* **9**, No 4 (1970), 525–534.

[97] I.M. Sokolov, J. Klafter, From diffusion to anomalous diffusion: a century after Einstein's Brownian motion. *Chaos* **15**, No 2 (2005), # 26103.

[98] I.M. Sokolov, J. Klafter, Field-induced dispersion in subdiffusion. *Phys. Rev. Lett.* **97**, No 14 (2006), # 140602.

[99] J.E. Solís-Pérez, J.F. Gómez-Aguilar, A. Atangana, Novel numerical method for solving variable-order fractional differential equations with power, exponential and Mittag-Leffler laws. *Chaos Soliton. Fract.* **114** (2018), 175-185.

[100] C.M. Soon, C.F.M. Coimbra, M.H. Kobayashi, The variable viscoelasticity oscillator. *Ann. Der. Phys.* **14**, No 14 (2005), 378–389.

[101] P. Straka, Variable order fractional Fokker-Planck equations derived from continuous time random walks. *Phys. A* **503** (2018), 451–463.

[102] H.G. Sun, Y. Chen, W. Chen, Random-order fractional differential equation models. *Signal Process.* **91**, No 3 (2011), 525–530.

[103] H.G. Sun, W. Chen, Y. Chen, Variable-order fractional differential operators in anomalous diffusion modeling. *Phys. A* **388**, No 21 (2009), 4586–4592.

[104] H.G. Sun, W. Chen, C. Li, Y. Chen, Finite difference schemes for variable-order time fractional diffusion equation. *Int. J. Bifurcat. Chaos* **22**, No 4 (2012), # 1250085.

[105] H.G. Sun, W. Chen, H. Sheng, Y. Chen, On mean square displacement behaviors of anomalous diffusions with variable and random orders. *Phys. Lett. A* **374**, No 7 (2010), 906–910.

[106] H.G. Sun, W. Chen, H. Wei, Y. Chen, A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems. *Eur. Phys. J. Spec. Top.* **193**, No 1 (2011), 185–192.

[107] H.G. Sun, S. Hu, Y. Chen, C. Wen, Z. Yu, A dynamic-order fractional dynamic system. *Chinese Phys. Lett.* **30**, No 4 (2013), 365–367.

[108] H.G. Sun, X. Song, Y. Chen, A class of fractional dynamic systems with fuzzy order. In: *Intelligent Control and Automation IEEE* **20**, No 1 (2010), 197–201.

[109] H.G. Sun, Y. Zhang, W. Chen, D.M. Reeves, Use of a variable-index fractional-derivative model to capture transient dispersion in heterogeneous media. *J. Contam. Hydrol.* **157** (2014), 47–58.

[110] J. Suzdalnitsky, D. Ingman, Application of differential operator with servo-order function in model of viscoelastic deformation process. *J. Eng. Mech.* **131**, No 7 (2005), 763–767.

[111] N.H. Sweilam, S.M. Al-Mekhlafi, Numerical study for multi-strain tuberculosis (TB) model of variable-order fractional derivatives. *J. Adv. Res.* **7**, No 2 (2016), 271–283.

[112] N.H. Sweilam, M.M. Khader, H.M. Almarwm, Numerical studies for the variable-order nonlinear fractional wave equation. *Fract. Calc. Appl. Anal.* **15**, No 4 (2012), 669–683; DOI: 10.2478/s13540-012-0045-9; https://www.degruyter.com/view/j/fca.2012.15.issue-4/s13540-012-0045-9/s13540-012-0045-9.xml.

[113] N.H. Sweilam, T.A. Rahman Assiri, Numerical simulations for the space-time variable order nonlinear fractional wave equation. *J. Appl. Math.* **2013**, No 2013 (2013), 183–189.

[114] D. Tavares, R. Almeida, D.F.M. Torres, Caputo derivatives of fractional variable order: Numerical approximations. *Commun. Nonlinear Sci.* **35** (2016), 69–87.

[115] A. Tayebi, Y. Shekari, M.H. Heydari, A meshless method for solving two-dimensional variable-order time fractional advection-diffusion equation. *J. Comput. Phys.* **340** (2017), 655–669.

[116] C.C. Tseng, Design of variable and adaptive fractional order FIR differentiators. *Signal Process.* **86**, No 10 (2018), 2554–2566.

[117] S. Umarov, S. Steinberg, Variable order differential equations and diffusion processes with changing modes. *Z. für Anal. Und Ihre Anwend.* **28**, No 4 (2010), 431–450.

[118] D. Valério, J. Sá da Costa, Variable-order fractional derivatives and their numerical approximations. *Signal Process.* **91**, No 3 (2011), 470–483.

[119] F. Wang, Q. Hua, C. Liu, Boundary function method for inverse geometry problem in two-dimensional anisotropic heat conduction equation. *Appl. Math. Lett.* **84** (2018), 130–136.

[120] S. Wang, R. Wu, Dynamic analysis of a 5D fractional-order hyperchaotic system. *Int. J. Control Autom. Syst.* **15**, No 3 (2017), 1003–1010.

[121] S. Wei, W. Chen, Y. Zhang, H. Wei, R.M. Garrard, A local radial basis function collocation method to solve the variable-order time fractional diffusion equation in a two-dimensional irregular domain. *Numer. Method. Part. Differ. Equ.* **34**, No 4 (2018), 1209–1223.

[122] G. Wu, D. Baleanu, H. Xie, S. Zeng, Lattice fractional diffusion equation of random order. *Math. Method. Appl. Sci.* **40**, No 17 (2015), 6054–6060.

[123] F. Wu, J. Liu, J. Wang, An improved Maxwell creep model for rock based on variable-order fractional derivatives. *Environ. Earth Sci.* **73**, No 11 (2015), 6965–6971.

[124] T. Xu, S. Lü, W. Chen, H. Chen, Finite difference scheme for multi-term variable-order fractional diffusion equation. *Adv. Differ. Equ.* **2018**, No 1 (2018), # 103.

[125] W. Xu, H.G. Sun, W. Chen, H. Chen. Transport properties of concrete-like granular materials interacted by their microstructures and particle components. *Int. J. Mod. Phys. B* **32**, No 18 (2018), # 1840011.

[126] X. Yang, C. Li, T. Huang, Q. Song, Mittag-Leffler stability analysis of nonlinear fractional-order systems with impulses. *Appl. Math. Comput.* **293** (2017), 416–422.

[127] X. Yang, J.A.T. Machado, A new fractional operator of variable order: application in the description of anomalous diffusion. *Phys. A* **481** (2017), 276–283.

[128] J. Yang, H. Yao, B. Wu, An efficient numerical method for variable order fractional functional differential equation. *Appl. Math. Lett.* **76** (2018), 221–226.

[129] A. Yildirim, S.T. Mohyud-Din, Analytical approach to space- and time-fractional Burgers equations. *Chinese Phys. Lett.* **27**, No 9 (2010), 38–41.

[130] D. Yin, Y. Li, H. Wu, X. Duan, Fractional description of mechanical property evolution of soft soils during creep. *Water Sci. Eng.* **6**, No 4 (2013), 446–455.

[131] S.B. Yuste, L. Acedo, An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations. *SIAM J. Numer. Anal.* **42**, No 5 (2005), 1862–1874.

[132] M.A. Zaky, E.H. Doha, T.M. Taha, D. Baleanu, New recursive approximations for variable-order fractional operators with applications. *Math. Model. Anal.* **2**, No 2018 (2018), 1–28.

[133] M. Zayernouri, G.E. Karniadakis, Fractional spectral collocation methods for linear and nonlinear variable order FPDEs. *J. Comput. Phys.* **293**, No C (2015), 312–338.

[134] F. Zeng, Z. Zhang, G.E. Karniadakis, A Generalized Spectral Collocation Method with Tunable Accuracy for Variable-Order Fractional Differential Equations. *SIAM J. Sci. Comput.* **37**, No 6 (2015), 2710–2732.

[135] L. Zhang, S. Li, Regularity of weak solutions of the Cauchy problem to a fractional porous medium equation. *Bound. Value Probl.* **2015**, No 1 (2015), 1–6.

[136] H. Zhang, F. Liu, The fundamental solutions of the space, space-time Riesz fractional partial differential equations with periodic conditions, *Numer. Math. A J. Chinese Univ. (Engl. Ser.)* **16**, No 2 (2007), # 181.

[137] H. Zhang, F. Liu, M.S. Phanikumar, M.M. Meerschaert, A novel numerical method for the time variable fractional order mobile-immobile advection-dispersion model. *Comput. Math. Appl.* **66**, No 5 (2013), 693–701.

[138] X. Zhao, Z. Sun, G.E. Karniadakis, Second-order approximations for variable order fractional derivatives: Algorithms and applications. *J. Comput. Phys.* **293** (2015), 184–200.

[139] P. Zhuang, F. Liu, V. Anh, I. Turner, Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term. *SIAM J. Numer. Anal.* **47**, No 3 (2009), 1760–1781.

[1] *State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, College of Mechanics and Materials*
Hohai University, Nanjing 210098, CHINA
e-mail: shg@hhu.edu.cn
Received: October 1, 2018
e-mail: cal15@hhu.edu.cn
[2] *Department of Geological Sciences*
University of Alabama, Tuscaloosa, AL 35487, USA
e-mail: yzhang264@ua.edu

Please cite to this paper as published in:
Fract. Calc. Appl. Anal., Vol. **22**, No 1 (2019), pp. 27–59,
DOI: 10.1515/fca-2019-0003; at https://www.degruyter.com/view/j/fca.