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A Right and Left Truncated Gamma Distribution with Application to the Stars

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Abstract

The gamma density function is usually defined in interval between zero and infinity. This paper introduces an upper and a lower boundary to this distribution. The parameters which characterize the truncated gamma distribution are evaluated. A statistical test is performed on two samples of stars. A comparison with the lognormal and the four power law distribution is made.

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1 Introduction

A probability distribution function (PDF) which models a given physical variable is usually defined in the interval $0 \le x < \infty$. As an example the exponential , the gamma, the lognormal, the Pareto and the Weibull PDFs are defined in such interval, see [1]. We now briefly review the status of the research on the truncated gamma distribution (TG). A first attempt to deduce the parameters of a TG can be found in [2], [3] derived the minimum variance unbiased estimate of the reliability function associated with the TG distribution which is right truncated, [4, 5] estimated the parameters of a TG distribution over $0 \le x < t$, adopting the maximum likelihood estimator(MLE), [6] studied the properties of TG distributions and derived the simulation algorithms which dominate the standard algorithms for these distributions, [7] considered a doubly-truncated gamma random variable restricted by both a lower (l) and upper (u) truncation.

On adopting an astronomical point of view the left truncation is connected with the minimum mass of a star, $\approx 0.02 M_{\odot}$ and the right truncation with the maximum mass of a star, $\approx 60 M_{\odot}$, see [8]. This paper first review the gamma PDF, introduces the right and left truncated gamma PDF and finally analyzes two samples of stars and brown dwarfs (BD).

2 The various gamma distributions

This Section reviews the gamma PDF, introduces the truncated gamma PDF and analyzes the data of two astronomical samples.

2.1 The gamma distribution

Let X be a random variable taking values x in the interval $[0, \infty]$; the *gamma* PDF is

$$f(x; b, c) = \frac{\left(\frac{x}{b}\right)^{c-1} \mathrm{e}^{-\frac{x}{b}}}{b\Gamma\left(c\right)} \tag{1}$$

where

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad , \tag{2}$$

is the gamma function, $b > 0$ is the scale and $c > 0$ is the shape, see formula (17.23) in [5]. Its expected value is

$$E(x; b, c) = bc \quad , \tag{3}$$

and its variance,

$$Var(x; b, c) = b^2 c \quad . \tag{4}$$

The mode is at

$$m(x; b, c) = bc - b \quad when\, c > 1 \quad . \tag{5}$$

The distribution function (DF) is

$$DF(x; b, c) = \frac{\gamma(c, \frac{x}{b})}{\Gamma\left(c\right)} \quad , \tag{6}$$

where

$$\gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt, \tag{7}$$

is the lower incomplete gamma function, see [9, 10]. The two parameters can be estimated by matching the moments

$$b = \frac{s^2}{\bar{x}} \tag{8}$$

$$c = (\frac{\bar{x}}{s})^2 \quad , \tag{9}$$

where s^2 and $\bar{x}$ are the sample variance and the sample mean. More details can be found in [1].

2.2 The truncated gamma distribution

Let X be a random variable taking values x in the interval $[x_l, x_u]$; the truncated gamma (TG) PDF is

$$f(x; b, c, x_l, x_u) = k \, \left(\frac{x}{b}\right)^{c-1} \mathrm{e}^{-\frac{x}{b}} \tag{10}$$

where the constant k is

$$k = \frac{c}{b\Gamma\left(1+c, \frac{x_l}{b}\right) - b\Gamma\left(1+c, \frac{x_u}{b}\right) + \mathrm{e}^{-\frac{x_u}{b}} b^{-c+1} {x_u}^c - \mathrm{e}^{-\frac{x_l}{b}} b^{-c+1} {x_l}^c} \quad , \tag{11}$$

where

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt, \tag{12}$$

is the upper incomplete gamma function, see [9, 10]. Its expected value is

$$E(b, c, x_l, x_u) = -b^2 k \left(-\Gamma\left(1+c, \frac{x_l}{b}\right) + \Gamma\left(1+c, \frac{x_u}{b}\right)\right) \quad . \tag{13}$$

The mode is at

$$m(x; b, c, x_l, x_u) = bc - b \quad when \, c > 1 \quad , \tag{14}$$

but in order to exist the inequality $x_l < m < x_u$ should be satisfied. The distribution function is

$$DF(x; b, c, x_l, x_u) = \\ k\left(b\Gamma\left(1+c, \frac{x_l}{b}\right) - b\Gamma\left(1+c, \frac{x}{b}\right) + \mathrm{e}^{-\frac{x}{b}} b^{-c+1} x^c - \mathrm{e}^{-\frac{x_l}{b}} b^{-c+1} {x_l}^c\right) \quad . \tag{15}$$

A random number generation can be implemented by solving for x the following nonlinear equation

$$DF(x; b, c, x_l, x_u) - \mathbf{R} = 0 \quad , \tag{16}$$

where we have a pudendum number generator giving random numbers **R** between zero and one, see [11]. A simple derivation of the lower and upper boundaries gives

$$\tilde{x_l} = minimum\ of\ sample \quad \tilde{x_u} = maximum\ of\ sample \quad . \tag{17}$$

A first approximate derivation of $\tilde{b}$ and $\tilde{c}$ is through the standard estimation of parameters of the gamma distribution. We compute the χ^2 with these first values of $\tilde{b}$ and $\tilde{c}$ and we search a numerical couple which gives the minimum χ^2. The χ^2 is computed according to the formula

$$\chi^2 = \sum_{i=1}^{n} \frac{(T_i - O_i)^2}{T_i} , \tag{18}$$

where n is the number of bins, T_i is the theoretical value, and O_i is the experimental value represented by the frequencies. The merit function χ^2_{red} is evaluated by

$$\chi^2_{red} = \chi^2/NF \quad , \tag{19}$$

where $NF = n - k$ is the number of degrees of freedom, n is the number of bins, and k is the number of parameters. The goodness of the fit can be expressed by the probability Q, see equation 15.2.12 in [12], which involves the degrees of freedom and the χ^2. The Akaike information criterion (AIC), see [13], is defined by

$$AIC = 2k - 2ln(L) \quad , \tag{20}$$

where L is the likelihood function and k the number of free parameters in the model. We assume a Gaussian distribution for the errors and the likelihood function can be derived from the χ^2 statistic $L \propto \exp(-\frac{\chi^2}{2})$ where χ^2 has been computed by Equation (18), see [14], [15]. Now the AIC becomes

$$AIC = 2k + \chi^2 \quad . \tag{21}$$

2.3 Data analysis

A first test is performed on the low-mass initial mass function in the young cluster NGC 6611, see [16]. Table 1 shows the values of χ^2_{red}, the AIC, the probability Q, of the astrophysical fits and the results of the K-S test, the maximum distance, D, between the theoretical and the astronomical DF as well the significance level P_{KS} , see [17, 18, 19, 12]. Figure 1 shows the fit with the TG distribution of NGC 6611 and Figure 2 visually compares the three types of fits for NGC 6611.

A second test is performed on low-mass stars in NGC 2362, see [20]. Table 2 shows the statistical parameters which characterize the astrophysical fits. Figure 3 shows the fit with the TG distribution of NGC 2362 and Figure 4 visually compares the three types of fits for NGC 2362.

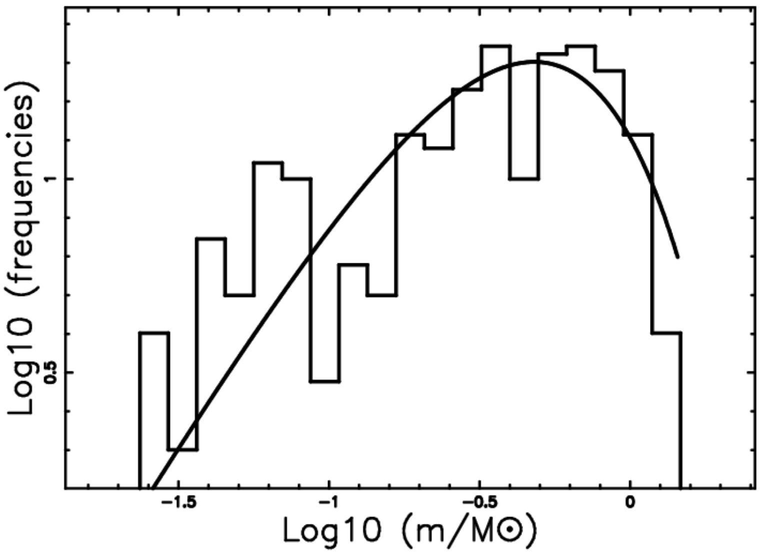

Figure 1: Logarithmic histogram of mass distribution as given by NGC 6611 cluster data (207 stars + BDs) with a superposition of the TG distribution when the number of bins, n, is 12, c = 1.287 , b = 0.372 , x_l = 0.019 and x_u =1.36. Vertical and horizontal axes have logarithmic scales.

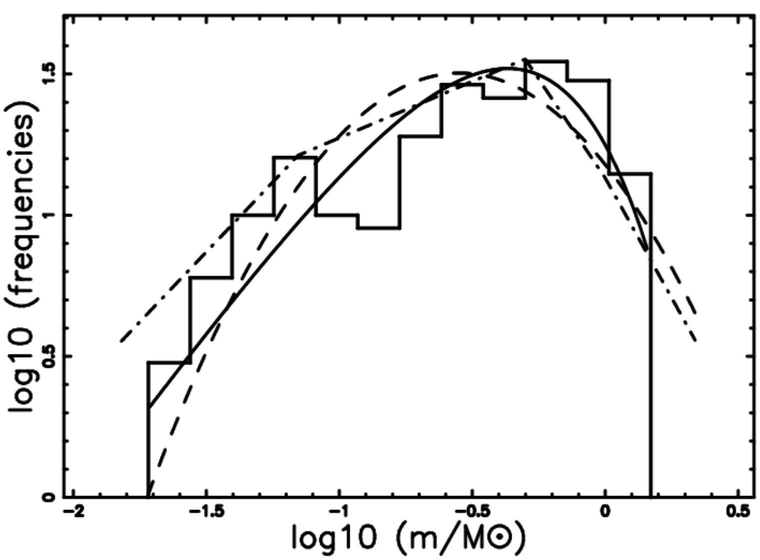

Figure 2: Histogram (step-diagram) of mass distribution as given by NGC 6611 cluster data (207 stars + BDs) with a superposition of the left TG distribution (full line), the lognormal (dashed), and the four power laws (dot-dash-dot-dash). Vertical and horizontal axes have logarithmic scales.

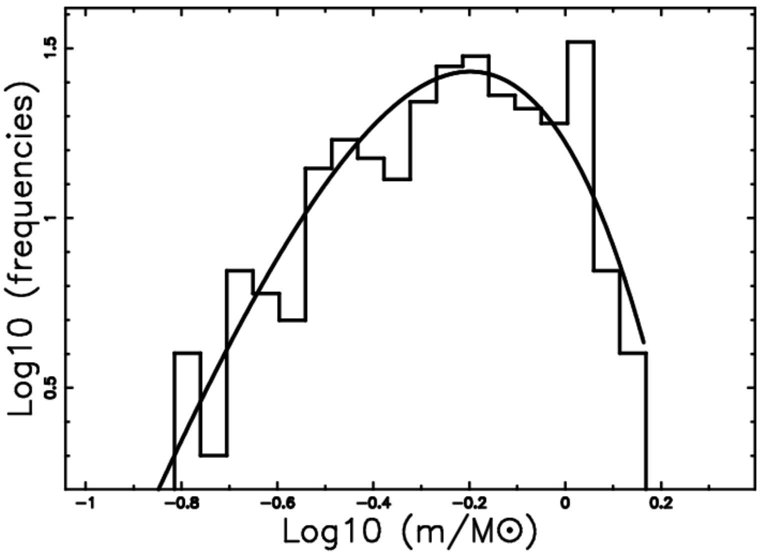

Figure 3: Logarithmic histogram of mass distribution as given by NGC 2362 cluster data (272 stars) with a superposition of the TG distribution when the number of bins, n, is 12, $b = 0.161$, $c = 3.933$, x_l= 0.12 and x_u =1.47 . Vertical and horizontal axes have logarithmic scales.

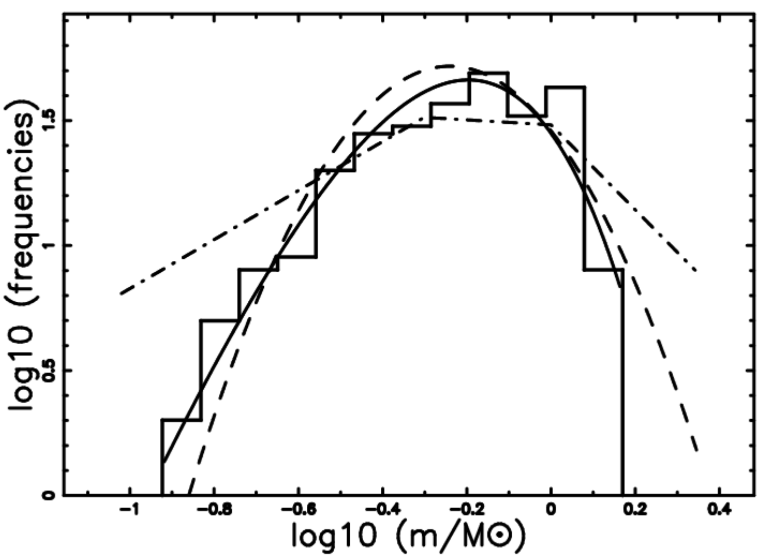

Figure 4: Histogram (step-diagram) of mass distribution as given by NGC 2362 cluster data (272 stars) with a superposition of the left TG distribution (full line), the lognornal (dashed), and the four power laws (dot-dash-dot-dash). Vertical and horizontal axes have logarithmic scales.

Table 1: Numerical values of NGC 6611 cluster data (207 stars + BDs). The number of linear bins, n, is 20.

PDF	parameters	AIC	χ^2_{red}	Q	D	P_{KS}
lognormal	σ=1.029,$\mu_{LN} = -1.258$	71.24	3.73	$1.3\,10^{-7}$	0.09366	0.04959
gamma	b=0.248 ,c = 1.717	62.83	3.26	$3.15\,10^{6}$	0.109	0.0124
truncated gamma	b=0.372 ,c =1.287 x_l=0.019, x_u=1.46	52.34	2.77	0.00017	0.09	0.061
four power laws	Eqn.(59) in Zaninetti 2013	81.39	5.18	$2.41\,19^{-9}$	0.12514	$2.72\,10^{-3}$

Table 2: Numerical values of of the NGC 2362 cluster data (272 stars). The number of linear bins, n, is 20.

PDF	parameters	AIC	χ^2_{red}	Q	D	P_{KS}
lognormal	σ=0.5,$\mu_{LN} = -0.55$	37.64	1.86	0.013	0.07305	0.10486
gamma	b=0.13 ,c =4.955	34.28	1.68	0.034	0.059	0.284
truncated gamma	b=0.161 ,c =3.933 x_l=0.12, x_u=1.47	33.88	1.61	0.055	0.071	0.122
four power laws	Eqn.(58) in Zaninetti 2013	77.608	4.89	$1.17\,10^{-8}$	0.16941	$2.6\,10^{-7}$

3 Conclusions

The right or left TG PDF has been extensively investigated in the field of mathematics , as an example [7] reports most of the mathematical details. The application of the TG PDF in astronomy represents conversely a new promising field. Here we have deduced the constant of normalization ,eqn.(11), the average value ,eqn.(13), the DF , eqn.(15), and presented an algorithm for the generation of the random numbers , (eqn.16). The application of the TG PDF to the IMF is positive and both the reduced χ^2 and the K-S test give better results in respect to the standard PDFs used by the astronomers which are the lognormal and the four power laws , see Tables 1 and 2. A comparison with the left truncated beta PDF , see Tables 1 and 2 in [21] allows to say that the left truncated beta PDF produces a better fit to the IMF in respect to the truncated gamma PDF here analyzed.

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