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A Rigorous Approach to Debye Screening in Dilute Classical Coulomb Systems

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Abstract. The existence and exponential clustering of correlation functions for a dilute classical coulomb system are proven using methods from constructive quantum field theory, the sine gordon transformation and the Glimm, Jaffe, Spencer expansion about mean field theory. This is a vindication of a belief, of long standing amongst physicists, known as Debye screening. This states that, because of special properties of the coulomb potential, the configurations of significant probability are those in which the long range parts of r^{-1} are mostly cancelled, leaving an "effective" exponentially decaying potential acting between charge clouds.

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1. Introduction

1.1. Background

In two previous papers [1, 2], the author and Paul Federbush have studied the quantum statistical mechanics of continuous systems with pair potentials such as the Yukawa $r^{-1}e^{-\alpha r}$, $\alpha>0$. Rigorous results on the existence and clustering of correlation functions were obtained using a type of cluster expansion which is convergent for a region of parameters physically associated with the plasma phase. The reason for studying such potentials is that they provide a first step towards obtaining the same type of results for the matter system, a system of positive and negative charges, one species of which is fermions, interacting by the coulomb law r^{-1}. They have in common the difficulties that arise from the singularity of the potential at the origin. Correlation functions are the next most obvious quantities to inquire after, following the papers of Dyson and Lenard [3] on the stability of matter and Lieb and Lebowitz [4] on the existence of the thermodynamic limit.

If one applies the cluster expansion of [1, 2] to the matter system, even individual terms in the series are divergent because r^{-1} is not integrable. The coloumb force, however, has a property, which is probably special to a very small class of long range potentials, that it "screens" itself. Debye and Hückel [5] in their theory of dilute ionic solutions gave a physical argument to the effect that the force between most pairs of distant particles is almost entirely cancelled by the forces from the remaining particles in the most probable configurations. The zeroth order approximation is that r^{-1} may be replaced by an "effective force" $r^{-1}e^{-r/l_D}$ between clumps of particles called charge clouds.

This paper provides a rigorous proof that for an approximate coulomb system Debye screening holds, in the sense that the infinite volume correlation functions exist and cluster exponentially. The approximations are (1) the system is treated in classical statistical mechanics (2) r^{-1} is replaced by a potential that falls off as r^{-1} as $r\to\infty$ but which is constant in the interior of (small) cubes of side l filling $\mathbb{R}^3$. Thus each particle interacts as if it were at the centre of the cube containing it. (2) may not be a bad approximation to more physical methods of mollifying the singularity of r^{-1} at the origin because the potential approximates r^{-1} as $l\to 0$ and the results hold in a region where the average interparticle spacing is sufficiently large compared with l at fixed temperature, i.e. arbitrarily dilute plasmas at fixed temperature. Lebowitz and Stell [16], taking careful account of short range forces, unlike this paper, have conjectured exponential decay. Stillinger and White [18] have suggested an r^{-8} decay. Modulo the short range forces, this paper settles the question in favour of Lebowitz and Stell.

The techniques used in the proof are the sine gordon transformation, which has already proved an important technique in the two dimensional coulomb gas [7] and the Glimm-Jaffe-Spencer expansion about mean field theory [8] from constructive field theory. Since neither of these are yet standard techniques for this problem, there follows a brief discussion of these topics and the main technical obstacles.

1.2. The Sine Gordon Transformation

The partition function for a two component system of charges in a finite region $\Lambda \subset \mathbb{R}^3$ interacting by a pair potential $v(x,y)$ is

$$Z=\sum_{N=0}^{\infty} z'^N/N!\sum_{(e)_N}\left(\int d^N P e^{-1/2\beta \sum_{j=1}^{N} P_j^2}\right)\left(\int_{\Lambda^N} d^N x e^{-\beta\mathfrak{V}}\right)$$

$$\mathfrak{V}=\mathfrak{V}((x)_N,(e)_N)=\sum_{1\leq j<k\leq N}\mathfrak{V}(x_j,x_k)e_j e_k$$

$(P)_N=(P_1,\dots,P_N)$ are the momenta, the mass of all particles is set to 1, $(x)_N$ are the coordinates, $(e)_N$ are the charges, $e_j=\pm 1$, $j=1,\dots,N$. z' is the activity, $\beta=(\text{temperature})^{-1}$. The P integrals may be done explicitly and absorbed into $(z')^N$ to obtain

$$Z=\sum_{N=0}^{\infty} z^N/N!\sum_{(e)_N}\int_{\Lambda^N} d^N x e^{-\beta\mathfrak{V}}.$$

If $(f,g)\equiv \int dx dy f(x)v(x,y)g(y)$ is a positive definite bounded bilinear form on $L^2(\mathbb{R}^3)$, there exists a unique gaussian measure $d\phi_v$ on $S'(\mathbb{R}^3)$ with covariance $v(x,y)$ and

$$\exp\{-1/2(f,f)\}=\int d\phi_v \exp\{i\phi(f)\}.$$

This follows from Minlo's theorem. If v is sufficiently regular, $\sup v(x,x)<\infty$ and continuity is necessary, one can take $f(x)=\beta^{1/2}\sum_{j=1}^{N} e_j\delta(x-x_j)$, so that

$$\exp\{-\beta\mathfrak{V}\}=\exp\left\{1/2\beta\sum_{j=1}^{N} v(x_j,x_j)\right\}$$
$$\cdot\int d\phi_v \exp\left\{i\beta^{1/2}\sum_{j=1}^{N} e_j\phi(x_j)\right\}.$$

Let $dk(x)=\exp\{1/2\beta v(x,x)\}dx$. Then it follows that

$$Z=\int d\phi_v \exp\left\{2z\int_\Lambda dk(x)\cos\beta^{1/2}\phi(x)\right\}$$
$$=\int d\phi_v \exp\left\{2z\int_\Lambda dx :\cos\beta^{1/2}\phi(x):\right\}$$

because $:\exp\{i\beta^{1/2}\phi(x)\}: =\exp\{1/2\beta v(x,x)\}\cdot\exp\{i\beta^{1/2}\phi(x)\}$. Normal ordering is defined by normal ordering the power series termwise.

It is not possible to set $v(x,y)=|x-y|^{-1}$ in this transformation because (a) the associated form is not bounded (b) $|x-y|^{-1}$ is not bounded at $x=y$. (b) is a genuine obstruction in the sense that the partition function Z is divergent in finite volume for $v(x,y)=|x-y|^{-1}$. The physical interpretation is that such a gas will collapse into neutral particles. In order to make Z finite, there must be some form of cutoff on the short distance singularity of r^{-1}. In the ϕ variables this is reflected by the fact that the $\cos\phi$ field theory is nonrenormalizable in three dimensions.

The most natural cutoff for physics would be a repulsive potential such as a hard core between all particles, but this does not pass through the sine gordon transformation in a simple way. The next best choice is perhaps $v(x-y)=\int dx f(x-u)|u-w|^{-1}f(w-y)$ where $\int du dw f(u)|u-w|^{-1}f(w)<\infty$. This can be interpreted as a coulomb interaction between two charge clouds at x and y. With more work, the theorem can almost certainly be proved for this potential by the same method.

The easiest cutoff, which is chosen in this paper, is to replace $(4\pi r)^{-1}$ which is the kernel of $(-\text{Laplacian})^{-1}$ by the kernel of $(-\Delta_l)^{-1}$ where Δ_l is the finite difference Laplacian with respect to a cubic lattice $\Pi_l \subset \mathbb{R}^3$ of spacing l. This is a function on Π_l but it may in an obvious way be identified with a potential $v^\sim(x,y)$, constant for x, y in the interior of cubes of side l centered on Π_l. The sine gordon transformation is formally the same but the spaces are different, e.g., $v^\sim(x,y)$ is now a bilinear form on $l^2(\Pi_l)$, $d\phi_{v^\sim}$ is a measure on $\bigotimes_{\Pi_l}\mathbb{R}$, $\phi(x)$ is a function on Π_l, and $\int dx :\cos\beta^{1/2}\phi:$ becomes $\sum_{x\in\Pi_l\cap\Lambda} l^3 :\cos\beta^{1/2}\phi(x):$.

Problem (a), which has so far been ignored, is dealt with in this paper by replacing $v^\sim$ by $v_\Lambda=\text{kernel of } (-\Delta_l+2z\beta\,\xi_{\sim\Lambda})^{-1}$, where $\xi_{\sim\Lambda}(x)=1$ if $x\in\Pi_l\cap\sim\Lambda$, $=0$ otherwise. This gives rise to a bounded bilinear form on $l^2(\Pi_l)$. It converges to $(-\Delta_l)^{-1}$, as $\Lambda\nearrow\mathbb{R}^3$ suitably, both pointwise and as a form.

1.3. The Glimm-Jaffe-Spencer Expansion about Mean Field Theory

By the sine gordon transformation,

$$Z=\int d\phi_{v_\Lambda}\exp\{-2zU(\beta)\}$$

$$U(\beta)=\sum_{x\in\Pi_l\cap\Lambda} l^3(:\cos\beta^{1/2}\phi(x):-1).$$

The partition function has been normalized so that $Z=1$ when $\beta=0$. Set $l_D=(2z\beta)^{-1/2}$. Consider the limit $\beta\to 0$, l_D fixed. Formally,

$$2z(:\cos\beta^{1/2}\phi:-1)=-1/2l_D^{-2}:\phi^2:+1/4:l_D^{-2}\beta:\phi^4:-\dots.$$

The first term can be absorbed into the measure which, up to normalization, becomes the gaussian measure with covariance $v_1=(l_D^{-2}-\Delta_l)^{-1}$. The kernel of this is approximately the effective potential $r^{-1}\exp(-r/l_D)$ found by Debye. This will be the basis of an approximation provided the non-quadratic terms give small contributions as $\beta\to 0$. A limit at fixed l is rather easy to apply, however this gives no control over zl^3 because $\beta\to 0\Rightarrow :z\to\infty$. zl^3 controls the number of particles in a cube of side l which is a dimensionless way of measuring density. Therefore this approximation will only yield high density results—which is where the type of cutoff used is significant.

To obtain results which are valid for arbitrarily low density at fixed β, it is necessary to show that the non-quadratic terms give small contributions as $\beta\to 0$, $l\to 0$ with l_D fixed and β/l bounded. It is easy to check this in perturbation theory: after the quadratic term is put in the measure, the interaction is $\sum_{x\in\Pi_l\cap\Lambda} l^3$

$(2z:\cos\beta^{1/2}\phi(x):+1/2l_D^{-2}:\phi^2(x):-2z)$. Expand the exponential and then the cosine in power series. (The resulting expansion is similar to the resummed high temperature expansion used by physicists, e.g., [11, 16, 17].) The worst behaved contributions are $(l_D^{-2}\beta^{n-1})^2 \sum_{x,y\in\Pi_l\cap\Lambda} l^6 v_1^{2n}(x,y)$ for $n=2,3,\ldots$. These are $o(\beta^{1/4})$ in this limit.

The preceding argument is flawed because it singles out the period of the cosine containing $\phi=0$, whereas only the quantity $\xi_{\sim\Lambda}$ in v_Λ, a "boundary term" prevents $d\phi_{v_\Lambda}$ from being translation invariant. In other words if $x\in\Lambda$, dist $(x,\partial\Lambda)$ large, there is nothing in the argument above to tell one why the cosine should not have been expanded around $\phi(x)=2\pi\beta^{-1/2}n$ for some $n\neq 0$, $n\in\mathbb{Z}$. The idea of the Glimm-Jaffe-Spencer expansion (a semiclassical approximation) is to partition the space of all field configurations into subsets labelled by "mean fields" $h=h(x)$. These are piecewise constant functions taking values in $2\pi\beta^{-1/2}\mathbb{Z}$, vanishing outside Λ. They serve to label the cosine period to which the preceeding approximation is to be applied. The choice of length scale L for the lower bound over which h is piecewise constant is heavily constrained. If L is too large, one looses control over fluctuations of ϕ away from h. If L is too small, the sum over h diverges.

The intuitive reason for the existence of a length scale L which works is a conspiracy between the cosine interaction which "repells" $\phi(x)$ away from $\pi\beta^{-1/2}\mathbb{Z}$ and the measure $d\phi_{v_\Lambda}$ which assigns significant probability only to field configurations $\phi(x)$ with some continuity properties. Thus the interacting measure only assigns significant probability to configurations $\phi(x)$ such that $\bar{\phi}(x)\approx h(x)$ for some h. $\bar{\phi}$ is an average over volume L^3.

The sum over h will be seen to reflect an underlying Ising model at low temperature whose spin takes values in $2\pi\beta^{-1/2}\mathbb{Z}$. The control over fluctuations away from h is a substantial problem (as it was in [8]) which is discussed next.

1.4. Technical Aspects, Size of Constants

The most difficult problem arising in the convergence proof is to show that the vacuum energy/unit volume is uniformly bounded as $\beta,l\to 0, z\to\infty$ with $l_D=(2z\beta)^{-1/2}$ fixed, β/l bounded. For purpose of illustration, take $l_D=1$. The required inequality is of the form

$$\int d\phi_0\chi(\bar{\phi})\exp\left\{2z\sum_{x\in\Pi_l\cap\Omega}(l^3:\cos\beta^{1/2}\phi(x):-1)\right\}\leqq\text{const}. \tag{A}$$

Ω is a cube of side $O(l_D)$. χ: characteristic function of the interval $[-\pi\beta^{-1/2},\pi\beta^{-1/2}]$. $\bar{\phi}$ is the average of ϕ over Ω. χ arises from the phase boundary expansion, and in general could localise $\bar{\phi}$ in some other period of the cosine. $d\phi_0$ is a massless gaussian measure, e.g.,

$$d\phi_0=d\phi_{v_1}\exp\left\{1/2\sum_{x\in\Pi_l}l^3:\phi^2(x):\right\}.$$

This measure, being massless, is invariant under the change of variables $\phi\to\phi+$ const, which means that $\int d\phi_0\chi(\bar{\phi})=0(\beta^{-1/2})$ as $\beta\to 0$. However if in (A) the product of χ and the exponent converges to $\exp\{-1/2\sum l^3:\phi^2(x):\}$, then the left hand side will be bounded uniformly. Conversely, the uniform bound is required in order to prove convergence or to use the limit as the basis for an expansion.

This bound is not trivial because the exponent is the Riemann sum of the difference of two quantities diverging as β^{-1}. The normal ordering changes cosine by a multiplicative constant so that $:\cos: -1 \not\leq 0$. Thus a cancellation must be performed.

The situation resembles $P(\phi)_2$ in that the only divergence to be cancelled is a vacuum energy and so it is possible to use some standard techniques such as L_p estimates (because they are homogeneous). However the strength of the divergence and additional complications from working in three dimensions require one to exploit properties special to the cosine. The basic philosophy is to use the gas picture to resolve short distance problems ($\leq l_D$) and ϕ variables to resolve long distance problems ($\geq l_D$). The same philosophy lay behind Frohlichs work on the two dimensional coulomb gas [7].

In comparison with numbers applying to typical plasmas or ionic solutions, the values of the constants controlling the region of convergence of this expansion are ludicrously small. This defect is common to almost all rigorous investigations of physical problems, especially at the beginning of their development. The actual radius of convergence of this expansion is presumably much larger but the estimates to prove it are lacking.

The convergence of this expansion is both a good and a bad sign for the resummed high temperature expansions used by physicists. It means that these expansions will almost certainly be asymptotic but not convergent to the correct results because they do not take into account the phase boundaries which are of infinite order in the expansion parameter.

2. Definitions and the Result

Let $\Pi_l \subset \mathbb{R}^3$ be the cubic lattice with spacing l centred on the origin. The set of closed cubes, side l, centred on lattice points is denoted $\bar{\omega}$. Let $\bar{\Delta}$ be the set of closed unit cubes with disjoint interiors such that $\mathbb{R}^3 = \bigcup_{\Delta \in \bar{\Delta}} \Delta$ with one $\Delta \in \bar{\Delta}$ centred on the origin. In order that each $\Delta \in \bar{\Delta}$ be a union of $\omega \in \bar{\omega}$, it is assumed that $l = (\text{integer})^{-1}$.

Associated with Π_l are the finite difference gradient ∇ and Laplacian Δ, both of which have l dependence. Throughout this paper functions on the lattice Π_l will be identified without comment with functions on $\mathbb{R}^3$, constant in ω^{INT} for each $\omega \in \bar{\omega}$. Thus integrals over $\mathbb{R}^3$ are frequently used instead of sums over Π_l.

The partition function is

$$Z^{\sim} = Z_0^{-1} \sum_{N=1}^{\infty} z^N/N! \sum_{(e)_N} \int d^N x e^{-\beta \mathfrak{V}^{\sim}} \tag{2.1}$$

where $(e)_N = (e_1, \ldots, e_N)$ is summed over $e_i = \pm 1$ for each $i = 1, \ldots, N$. Z_0 is chosen to normalize $Z^{\sim} = 1$ at $\beta = 0$.

$$\mathfrak{V}^{\sim} = \mathfrak{V}^{\sim}(x_1, \ldots, x_N, (e)_N) = \sum_{1 \leq i < j < N} e_i e_j v^{\sim}(x_i, x_j) \tag{2.2}$$

$$v^{\sim}(x, y) = \text{kernel of } (-\Delta)^{-1}.$$

Three variables with the dimension "length" may be formed from l, z, β, the parameters in $Z^{\sim}$. These are (1) the cut off length l, (2) the Landau length $l_L \equiv \beta$ and

(3) the Debye length $l_D=(2z\beta)^{-1/2}$. By a choice of units, one may assume with no loss of generality that $l_D\equiv 1$. With this assumption, $Z^\sim$ may be rewritten

$$Z^\sim = Z^\sim(\Lambda) = Z_0(\Lambda)^{-1}\sum_{N=0}^{\infty}(l_L N!)^{-1}\sum_{(e)_N}\int_{\Lambda^{N'}} d^N x e^{-l_L \mathfrak{V}^\sim} \tag{2.3}$$

where $\Lambda\subset\mathbb{R}^3$ is a union of cubes $\Delta\in\bar{\Delta}$.

Given a sequence $A\equiv A_0, A_1(x_1,e_1), A_2(x_1,e_1;x_2,e_2),\dots$ of functions, define the expectation

$$\langle A\rangle_\Lambda^\sim = Z^{-1}\sum_{N=0}^{\infty}(l_L N!)^{-1}\sum_{(e)_N}\int_{\Lambda^N} d^N x \cdot A_N((x)_N,(e)_N)e^{-l_L\mathfrak{V}^\sim}. \tag{2.4}$$

The infinite volume expectation, when it exists, is defined as

$$A^\sim = \lim_{\Lambda\nearrow\mathbb{R}^3}\langle A\rangle_\Lambda^\sim \tag{2.5}$$

where the limit is taken through a sequence of rectangular parallelopipeds whose smallest side tends to ∞. In this paper attention is restricted to expectations of A's with the following special form

$$\begin{aligned} A &= \prod_{i=1}^{n'}\varrho(y_i)\cdot y_1,\dots,y_{n'}\in\prod\nolimits_l \text{ distinct}.\\ \varrho(y) &= (\varrho_N(y;x_1,\dots,x_N;e_1,\dots,e_N))_{N=1,\dots} = 0\cdot N=0\\ \varrho_N(y;(x,e)_N) &= \sum_{j=1}^{N} e_j\delta(y-x_j)\end{aligned} \tag{2.6}$$

n' is arbitrary. Define $\operatorname{supp} A=\{y_1,\dots,y_n\}\subset\mathbb{R}^3$.

A considerably larger class of A's could be treated with minor changes in the proof.

The Sine Gordon Transformation. $v^\sim$ is not the kernel of a bounded operator on $l^2(\Pi_l)$, as is necessary for the Sine Gordon transformation. Let $\xi_{\sim\Lambda}$ be the characteristic function of $\sim\Lambda$ and set $v_\Lambda=$kernel of $(-\Delta+\xi_{\sim\Lambda})^{-1}$. Define $\mathfrak{V}, Z(\Lambda), \langle\ \rangle_\Lambda, \langle\ \rangle$ by replacing $v^\sim$ by v_Λ in all definitions. It is conceivable that $\langle\ \rangle\neq\langle\ \rangle^\sim$ when both limits exist, but most unlikely. This will not be discussed in this paper. Formally one is still obtaining the Coulomb system in the infinite volume limit because $\lim_{\Lambda\nearrow\mathbb{R}^3} v_\Lambda=v^\sim$ pointwise and in several other senses also. v_Λ is the kernel of a bounded positive operator on $l^2(\Pi_l)$ and it makes sense to define $d\phi_\Lambda^{(0)}$, the lattice gaussian measure with covariance v_Λ. By the Sine Gordon transformation

$$\begin{aligned} Z(\Lambda) &= \int d\phi_\Lambda 1\\ \left\langle\prod_{i=1}^{n'}\varrho(y_i)\right\rangle_\Lambda &= Z(\Lambda)^{-1}\int d\phi_\Lambda\prod_{i=1}^{n'} l_L^{-1} :\sin l_L^{1/2}\phi(y_i):\\ d\phi_\Lambda &\equiv d\phi_\Lambda^{(0)}\exp\{-U(\Lambda)\}\\ U(\Lambda) &= l_L^{-1}\int_\Lambda dx(:\cos l_L^{1/2}\phi(x): -1).\end{aligned} \tag{2.7}$$

The easiest way to obtain the expectation of ϱ's is to use

$$\langle e^{i\varrho(f)}\rangle_\Lambda = Z(\Lambda)^{-1}\int d\phi_\Lambda^{(0)}\exp\, l_L^{-1}\int_\Lambda dx(:\cos l_L^{1/2}(\phi(x)+f(x)):-1)\ . \tag{2.8}$$

The result of this paper is

Theorem 2.1. *Given any $c_1>0$, there exists $c_2>0$ such that for $\beta/l\leqq c_1$, $(2z\beta^3)^{1/2}\leqq c_2$, $\lim_{\Lambda\ \mathbb{R}^3}\left\langle\prod_{i=1}^{n'}\varrho(y_i)\right\rangle_\Lambda$ exists and clusters exponentially, i.e., there are strictly positive constants $c=c(z,\beta,l)$, $c'=c'(n')$, such that for $n_1<n'$*

$$\left|\left\langle\prod_{i=1}^{n_1}\varrho(y_i)\prod_{j=n_1+1}^{n'}\varrho(y_j+a)\right\rangle-\left\langle\prod_{i=1}^{n_1}\varrho(y_i)\right\rangle\left\langle\prod_{j=n_1+1}^{n'}\varrho(y_j)\right\rangle\right|$$

$$\leqq c'\exp\left\{-c\inf_{i\leqq n_1<j\leqq n'}|y_j+a-y_i|\right\}.$$

Notation for Constants. Constants labelled $c_1, c_2, \ldots$ denote strictly positive numbers and keep fixed values throughout the paper. Constants denoted c, c', c'', etc. are also strictly positive but need not refer to the same number in different equations.

Remark. It should be possible, by following constants through the paper more carefully to show that $c\to l_D^{-1}$ as $\beta\to 0$, with $(2z\beta)^{-1/2}=l_D$ fixed, β/l bounded.

3. The Expansion

3.1. The Peierls Expansion

Let $\bar\Omega$ be the set of closed cubes, Ω, with disjoint interior, side L, such that $\Lambda=\bigcup_{\Omega\in\bar\Omega}\Omega$. L is assumed to be an integral multiple of l and each Ω is a union of cubes $\in\bar\omega$. Furthermore L is assumed to be 1/integer, so that each $\Delta\in\bar\Delta$ is a union of cubes Ω. L will be chosen $\ll 1$ independently of l_L, l in the region $l_L\leqq c_1 l$.

Define

$$\chi(\xi)=\pi^{-1/2}\int_{l_L^{1/2}\pi}^{l_L^{1/2}\pi} dt\, e^{-(\xi-t)^2}$$

$$\bar\phi(\Omega)=L^{-3}\int_\Omega dx\,\phi(x) \tag{3.1}$$

$$\chi_h(\xi)=\chi(\xi-h)\,.$$

Then

$$1=\prod_{\Omega\in\bar\Omega}\sum_h\chi_h(\bar\phi(\Omega))$$

where h is summed over integral multiples of $2\pi l_L^{-1/2}$.

Change notation so that $h=h(\Omega)$ is a function on $\bar\Omega$ taking values which are integer multiples of $2\pi l_L^{-1/2}$. Set $\chi_h=\prod_{\Omega\in\bar\Omega}\chi_{h(\Omega)}(\bar\phi(\Omega))$. Then the Peierls expansion is

$$d\phi_\Lambda=\sum_h\chi_h d\phi_\Lambda\,. \tag{3.2}$$

3.2. Translation of ϕ

Identify $h=h(\Omega)$ with the function $h(x)$ defined a.e. on $\mathbb{R}^3$ such that $h(x)=h(\Omega)$ for $x\in\Omega^{\mathrm{INT}}$, $h=0$ in $\sim\Lambda$. The class of functions obtained in this way is denoted $\bar{H}$. Let $\Sigma\subset\Lambda$ be the closed set where $h(x)$ has a step discontinuity. Σ is called the Peierls contour for h. It is a union of faces of cubes $\Omega\in\bar{\Omega}$.

$L'\gg 1$ is a length, to be chosen later independently of l, l_L in the region $l_L\leqq c_1 l$, characterising the distance away from Σ where the effects of the contour become small. Given $Z\subset\mathbb{R}^3$, set

$$\hat{Z}=\{x\in\Delta:\Delta\in\bar{\Delta},\quad \operatorname{dist}(Z,\Delta)\leqq L'\}. \tag{3.3}$$

Given h, a function $g=g(h)$ on $\mathbb{R}^3$ will be chosen so that

$$\psi=\phi-g \tag{3.4}$$

has small mean and $g\equiv h$ outside $\hat{\Sigma}$.

Choice of g. Let η_Λ be the function on Π_l defined by $\eta(x)=\eta, x\in\Lambda, =1 x\notin\Lambda$. η will be chosen later independently of $l, l_L, 1\geqq\eta>0$.

$$\begin{aligned}&\mathfrak{L}_c(x,y)=\text{kernel of } (\eta_\Lambda-\Delta)^{-1}\eta_\Lambda\\&g_c(x)=(\mathfrak{L}_c h)(x)\end{aligned} \tag{3.5}$$

where $\mathfrak{L}_c$ is the operator on $l^2(\Pi_l)$ with kernel $\mathfrak{L}_c(x,y)$. Choose a family of functions $\zeta_{L'}$ on Π_l, indexed by L' such that (1) $\zeta_{L'}(x)=0$ if $|x|\geqq L'$, (2) $\zeta_{L'}(x)=1$ if $|x|\leqq L-1$, (3) $\zeta_{L'}$ together with finite difference derivatives up to order 2 are bounded in absolute value by constants independent of L'. Define

$$g(x)=(\mathfrak{L}h)(x) \tag{3.6a}$$

$$(\mathfrak{L}h)(x)=\zeta'_{L'}(x)\sum_{y\in\Pi_l} l^3\mathfrak{L}_c(x,y)\zeta_{L'}(x-y)h(y) \tag{3.6b}$$

$$\zeta'_{L'}(x)=\left\{\sum_{y\in\Pi_l} l^3\mathfrak{L}_c(x,y)\zeta_{L'}(x-y)\right\}^{-1}. \tag{3.6c}$$

It is left to the reader to prove that $g=h$ outside $\hat{\Sigma}$.

The easiest way to understand this translation is to perceive that if the exponential of $l_L^{-1}\int(:\cos l_L^{1/2}\phi:-1)$ is replaced by a kind of periodised gaussian,

$$\sum_h \exp\left\{-\eta/2\int_\Lambda dx:(\phi-h)^2:\right\}$$

translation by g removes the linear term up to a small error whose size is controlled by L'. It is technically advantageous to live with this small error in exchange for having $g=h$ outside $\hat{\Sigma}$.

The measure $d\phi_\Lambda$ is translated by using the identity

$$d\phi_\Lambda^{(0)}=\exp\left\{-1/2\int_{\sim\Lambda}g^2-1/2\int|\nabla g|^2-\int_{\sim\Lambda}g\psi+\int\psi\Delta g\right\}d\psi_\Lambda^{(0)}. \tag{3.7}$$

The measure $d\psi_\Lambda^{(0)}$ may be expressed in terms of $d\psi_1$, the Gaussian measure with covariance $v(x,y)=$ kernel of $(1-\Delta)^{-1}$, by

$$d\psi_\Lambda^{(0)}=N(\Lambda)\exp\left[1/2\int_\Lambda:\psi^2:\right]d\psi_1. \tag{3.8}$$

$N(\Lambda)$ is the normalisation.

In order to express the result of the Peierls expansion followed by translation, define

$$V(\Lambda,h)=U(\Lambda)-(\eta/2)\int_\Lambda :(\phi-h)^2: -((1-\eta)/2)\int_\Lambda :\psi^2: \tag{3.9a}$$

$$F(\Lambda,h)=\eta/2\int_\Lambda :(\phi-h)^2: +(1-\eta)/2\int_\Lambda :\psi^2: + \int_{\sim\Lambda} g\psi$$
$$-\int\psi\Delta g+1/2\int_{\sim\Lambda} g^2+1/2\int|\nabla g|^2 \tag{3.9b}$$

$$Q(\Lambda,h)=V(\Lambda,h)+F(\Lambda,h). \tag{3.9c}$$

Then the Peierls expansion, followed by translation, yields the identity

$$d\phi_\Lambda=N(\Lambda)\sum_h \chi_h e^{-Q(\Lambda,h)}d\psi_1. \tag{3.10}$$

The definition of g may be used to show that (3.9b) can be rewritten in the simple form

$$F(\Lambda,h)=F_1(\Lambda,h)+F_2(\Lambda,h) \tag{3.11a}$$
$$F_1(\Lambda,h)=1/2\int|\nabla g|^2+1/2\int\eta_\Lambda(g-h)^2 \tag{3.11b}$$
$$F_2(\Lambda,h)=\int[(\eta_\Lambda-\Delta)(g-g_c)]\psi. \tag{3.11c}$$

This is left to the reader.

For future reference, define $V(\Lambda,X,h)$, $F_1(\Lambda,X,h)$, $F_2(\Lambda,X,h)$, $F(\Lambda,X,h)$, $Q(\Lambda,X,h)$ by replacing each range of integration by range $\cap X$, X any union of cubes $\in\bar{\Delta}$.

3.3. The Cluster Expansion

The cluster expansion is an identity that applies to a quantity of the form

$$\int d\psi_1 A W(\Lambda') \tag{3.13}$$

where Λ' is a finite union of cubes $\in\bar{\Delta}$ and $W(\Lambda')$ belongs to the σ algebra generated by fields supported in Λ' and has the property that it factors across cubes $\Delta\in\bar{\Delta}$, e.g., if $X\subset\Lambda'$ is a union of cubes $\Delta\in\bar{\Delta}$, $W(\Lambda)=W(X)W(\Lambda'\sim X)$.

Let $\bar{Y}$ be a collection of closed subsets $Y\subset\Lambda'$ with disjoint interiors whose union is Λ'. $\bar{Y}$ contains a distinguished element Y_1. (The cluster expansion will be written down only for the case $Y_1\supset\operatorname{supp}A$. The case $Y_1\not\supset\operatorname{supp}A$ is left to the reader. In regard to this, note that A factors across cubes $\Delta\in\bar{\Delta}$.) Every other element of $\bar{Y}$ is assumed to be connected. The letter $\bar{y}$ will be used to denote a finite sequence $Y_1, Y_2, \ldots, Y_n$, n arbitrary, of distinct elements of $\bar{Y}$. For a given $\bar{y}$ and parameters $s\equiv(s_1,\ldots,s_{n-1})\in[0,1]^{n-1}$, define

$$p(x,y,s)=\sum_{i\leqq i<j\leqq n+1} s_i s_{i+1}\ldots s_{j-1}\xi_i(x)\xi_j(y)$$
$$+\sum_{1\leqq j<i\leqq n+1} s_j s_{j+1}\ldots s_{i-1}\xi_i(x)\xi_j(y)$$
$$+\sum_{1\leqq i\leqq n+1}\xi_i(x)\xi_i(y) \tag{3.14}$$

where ξ_i is the characteristic function of Y_i for $i \leqq n$, ξ_{n+1} is the characteristic function of $\sim \bigcup_{i \leqq n} Y_i$ and s_n is set to zero. Define $v(x, y, s) = v(x-y)p(x, y, s)$. Let $d\psi_s$ be the Gaussian measure with covariance $v(x, y, s)$. This is the kernel of a positive operator on l^2 because it is a linear combination with positive coefficients of operators of the form $\xi v \xi$, where ξ is a characteristic function.

The cluster expansion is the identity

$$\int d\psi_1 AW(\Lambda') = \sum_X K(X) \int d\psi_1 W(\Lambda' \sim X) \tag{3.15}$$

where X is summed over all unions of $Y \in \bar{Y}$ containing Y_1. Since Λ is finite, this sum is finite.

$$K(X) = \sum_{\bar{y}} \int_{[0,1]^{n-1}} ds \int d\psi_s \kappa(\bar{y}, s) AW(X) \tag{3.16}$$

$$\kappa(\bar{y}, s) = \prod_{i=1}^{n-1} \kappa(i) \tag{3.17}$$

$$\kappa(i) = 1/2 \int_{Y_{i+1}} dx \int_{\bigcup_{i' \leqq i} Y_{i'}} dy (d/ds_i) v(x, y, s) \cdot (\delta/\delta\psi(x))(\delta/\delta\psi(y)).$$

For $n=1$, $K(X) = \int d\psi_s AW(Y_1)$ if $X = Y_1$, zero otherwise. The sum over $\bar{y}$ extends over all $\bar{y}$ such that $\bigcup_{i \leqq n} Y_i = X$, n arbitrary. This is a finite sum.

Proof. Let $d\psi'_{s, s_n}$, $s = (s_1, \ldots, s_{n-1})$, be the measure defined in the same way as $d\psi_s$ except $s_n \in [0, 1]$ instead of being set to zero. *Inductive hypothesis:*

$$\int d\psi_1 AW(\Lambda') = \sum_X [K(X) \int d\psi_1 W(\Lambda' \sim X) + R_n(X)]$$

$$R_n(X) = \int_0^1 ds_n \frac{d}{ds_n} \sum_{\bar{y}} \int_{[0,1]^{n-1}} ds \int d\psi'_{s, s_n} \prod_{i=1}^{n-1} \kappa(i) AW(\Lambda')$$

where X is summed over sets of the form $\bigcup_{i=1}^{m} Y_i$, $m \leqq n$ and $\bar{y}$ is summed over all sequences $(Y_1, \ldots, Y_n)$ such that $X = \bigcup_{i=1}^{n} Y_i$. $R_n(X) = 0$ if there is no such $\bar{y}$. For $n=1$, this hypothesis reduces to

$$\int d\psi_1 AW(\Lambda') = \int d\psi_1 AW(Y_1) \int d\psi_1 W(\Lambda' \sim Y_1) + \int_0^1 \frac{d}{ds_1} \int d\psi'_{s_1} AW(\Lambda')$$

and this is the fundamental theorem of calculus. To prove the hypothesis for n replaced by $n+1$ and thereby complete the induction, evaluate R_n using integration by parts (see [10]).

$$\frac{d}{ds_n} \int d\psi'_{s, s_n}(\cdot) = \sum_{Y_{n+1}} \int d\psi'_{s, s_n} \kappa(n)(\cdot)$$

where $Y_{n+1} \in \bar{Y}$, $Y_{n+1} \cap Y_i = \phi$ for $i < n+1$, followed by the fundamental theorem of calculus

$$d\psi'_{s,s_n} = d\psi_{s,s_n} + \int_0^1 ds_{n+1} \frac{d}{ds_{n+1}} d\psi'_{s,s_n,s_{n+1}} .$$

The second term becomes R_{n+1}. When n is sufficiently large, depending on Λ', $R_n = 0$ and the inductive hypothesis becomes the cluster expansion.

Remark. This is an expansion in clusters of regions contained in $\mathbb{R}^3$ as opposed to clusters of particles. It may be that this is an essential feature of an expansion approach to Debye screening, because screening involves the statistical behavior of large numbers of particles.

3.4. *The Combined Expansion*

Choice of $\bar{Y}$. Let $X_0 \subset \Lambda\hat{}$ be a given union of $\Delta \in \bar{\Delta}$. Let $\tilde{Y} = \tilde{Y}(h)$ be the set of subsets of $\Lambda\hat{}$ whose elements are the connected components $\Sigma\hat{}_c \subset \Sigma\hat{}$ together with $\Delta \in \bar{\Delta}$, $\Delta \subset \Lambda\hat{} \sim \Sigma\hat{}$. Take $Y_1 = Y_1(h)$ to be the smallest union of elements of $\tilde{Y}$ containing X_0. The remaining elements of $\bar{Y}$ are the elements of $\tilde{Y}$ not contained in Y_1. $\bar{Y} = \bar{Y}(h)$.

Only the case $X_0 \supset \operatorname{supp} A$ is considered here. The case $X_0 \not\supset \operatorname{supp} A$ is left to the reader. Apply the cluster expansion (3.15) to each term in (3.10). $\Lambda' = \Lambda\hat{}$.

$$\int d\phi_\Lambda A = \sum_h \sum_X K(X,h) N(\Lambda) \int d\psi_1 \chi_h(\Lambda \sim X) \exp\{-Q(\Lambda, \Lambda\hat{} \sim X, h)\} \tag{3.18}$$

$$K(X,h) = \sum_{\bar{y}} \int ds \int d\psi_s \kappa(\bar{y}, s) \chi_h(X) \exp\{-Q(\Lambda, X, h)\} A . \tag{3.19}$$

where, if Z is a union of cubes Δ, $\chi_h(Z) = \prod_{\Omega \in \bar{\Omega}, \Omega \subset Z} \chi_{h(\Omega)}(\bar{\phi}(\Omega))$. The next step is a resummation of h outside X. This yields the final form of the expansion. Note that if $X \not\supset A$, $K(X,h)$ has A dependence which has not been made explicit.

Resummation Outside X. Decompose $h = h_X + h_{\sim X}$. The decomposition is uniquely fixed by the requirements (1) on the interior of the connected component $(\sim X)^\infty$ of $\sim X$ containing the point at ∞, $h_{\sim X} = h$. (2) For each connected component X_c, $(\sim X)_c$ of X, $\sim X$ respectively, h_X is constant on the interior of $((\sim X)_c)\hat{}$ $h_{\sim X}$ is constant of $X\hat{}_c$.

Lemma 3.1. (a) $K(X,h) = K(X, h_X)$.

(b) $\int d\psi_1 \chi_h(\Lambda \sim X) \exp\{-Q(\Lambda, \Lambda\hat{} \sim X, h)\}$

$= \int d\psi_1 \chi_{h_{\sim X}}(\Lambda \sim X) \exp\{-Q(\Lambda, \Lambda\hat{} \sim X, h_{\sim X})\}$.

By virtue of this lemma, one may write $\sum_h = \sum_{h_X} \sum_{h_{\sim X}}$ and perform each sum independently. Define

$$Z'(\Lambda, X) = \sum_{h_{\sim X}} \int d\psi_1 \chi_{h_{\sim X}}(\Lambda \sim X) \exp\{-Q(\Lambda, \Lambda\hat{} \sim X, h_{\sim X})\} . \tag{3.20}$$

Lemma 3.2. *Let* $\delta X = (\partial X)\hat{} \cap (\Lambda \sim X)$. *Define*

$$M(\Lambda, X) = \sum_{h_{\delta X}} \chi_{h_{\delta X}}(\delta X) \exp\left\{-1/2 \int_X :(\phi - h_{\delta X})^2:\right\}$$

where $h_{\delta X}$ is defined on $(\delta X)\cup X$, is constant on the interior of connected components of $(\delta X)\cup X$ and vanishes in any connected component of $\delta X\cup X$ that intersects $\sim\Lambda$. Then

$$Z'(\Lambda,X)=\int d\phi_\Lambda^{(0)}\exp\{-U(\Lambda\sim X)\}M(\Lambda,X)\,.$$

The last lemma is not used in this paper but it provides some intuition for Z'. After applying Lemma 3.1 the expansion can be written in its final form

$$\int d\phi_\Lambda A=\sum_X \mathfrak{K}(X)Z'(\Lambda,X) \tag{3.21a}$$

$$\mathfrak{K}(X)=\sum_{h,\bar{y}}\int ds\int d\psi_s\kappa(y,s)\,.$$

$$\chi_h(X)\exp\{-Q(\Lambda,X,h)\}A \tag{3.21b}$$

$(h,\bar{y})$ is summed over the set defined by the relations (1) $h\in\bar{H}$ (2) $\bar{y}=(Y_1,\dots,Y_n)$, n arbitrary, $Y\in\bar{Y}(h)$, $i=1,\dots,n$. (3) $\bigcup_{i\leq n}Y_i\supset\Sigma^\wedge(h)$. (4) $X=\bigcup_{i\leq n}Y_i$. For $n=1$, (3.21b) reduces to zero unless $X\supset X_0$ in which case

$$\mathfrak{K}(X)=\sum_h\int d\psi_1\chi_h(X)\exp\{-Q(\Lambda,X,h)\}A \tag{3.21c}$$

where h is summed over the set $h\in\bar{H}$, $X=Y_1(h)$, $X\supset\Sigma^\wedge(h)$. If this set is ϕ, $\mathfrak{K}(X)=0$.

Proof of Lemma 3.1. (a) Given X_c such that $X_c\cap\sim\Lambda\neq\phi$, it is claimed that $Q(\Lambda,X_c,h)=Q(\Lambda,X_c,h_X)$. It is sufficient to prove that on X_c^{INT}, $h_X=h$, $h_X=g\equiv h$. This in turn is implied by $h=h_X$ on $(X_c^\wedge)^{\mathrm{INT}}$, which is implied by $h_{\sim X}=0$ on $(X_c^\wedge)^{\mathrm{INT}}$. Since (2) states that $h_{\sim X}$ is constant on $(X_c^\wedge)^{\mathrm{INT}}$, this is implied by $h_{\sim X}=h$ in $(\sim X)^\infty$. This proves claim. Next it is claimed that as functions of ψ $Q(\Lambda,X_c,h)=Q(\Lambda,X_c,h_X)$ also when $X_c\subset\Lambda$. For this it is sufficient to show that on X_c^{INT} $h-h_X=g-\mathfrak{L}h_X=2\pi l_L^{-1/2}n$, for some $n\in\mathbb{Z}$. By the same arguments as above this follows from $h_{\sim X}=2\pi l_L^{-1/2}n$ on $(X_c^\wedge)^{\mathrm{INT}}$, which proves the second claim. By summing over $X_c\subset X$, $Q(\Lambda,X,h)=Q(\Lambda,X,h_X)$. As a function of ψ, A is unchanged by $h\to h_X$ because it is periodic. This completes the proof of (a).

(b) Follows by the same arguments.

Proof of Lemma 3.2. Undo the translation in (3.20) by setting $\psi=\phi-\mathfrak{L}h_{\sim X}$. If X_c is a connected component of X intersecting $\sim\Lambda, h_{\sim X}=0$ on $(X_c^\wedge)^{\mathrm{INT}}$, so $\mathfrak{L}h_{\sim X}=h_{\sim X}=0$ on X_c^{INT}. Sum over all $h_{\sim X}$ such that $h_{\sim X}=h_{\delta X}$ in $[(\delta X)\cup X]^{\mathrm{INT}}$. Proof of Lemma 3.2 concluded.

3.5. *Kirkwood Salsburg Equations*

These equations will be used to obtain bounds on the ratio Z'/Z, obtained when the expansion is normalised.

$$Z'(\Lambda,X)=\sum_{W\supset X^*}\mathfrak{K}(W\sim X)Z'(\Lambda,W) \tag{3.22a}$$

$$\mathfrak{K}(W\sim X)=\sum_{h,\bar{y}}\int ds\int d\psi_s\cdot\kappa(\bar{y},s)\chi_h(W\sim X)\exp\{-Q(\Lambda,W\sim X,h)\}\,. \tag{3.22b}$$

The cubes $\Delta \in \bar{\Delta}$ are ordered in some arbitrary way. Given X, let Δ_1 be the first cube in $\Lambda^{\hat{}} \sim X$. Set $X^* = X \cup \Delta_1$.

Proof. Apply the cluster expansion to each term in the definition of Z' with $X_0 = \Delta_1$. Resum as before.

Equation (3.22) can be written as an equation for Z' on a Banach space. Let $P = \{X \subset \Lambda^{\hat{}} : X$ is a union of $\Delta \in \bar{\Delta}\}$. $|X|$ denotes the volume of $X \in P$. B is the space of functions $f: P \to \mathbb{R}$ with $\|f\|_b = \sup_{X \in P} b^{|X|} |f(X)|$. $\mathfrak{K}: B \to B$ is the linear operator defined by $(\mathfrak{K}f)(X) = \sum_{W \supset X^*} \mathfrak{K}(W \sim X) f(W)$. Set $\varrho(X) = Z'(\Lambda, X)$ if $X \neq \phi$, $= 0$ otherwise. Equation (3.22) is, in this notation,

$$Z(\Lambda)I + \varrho = \mathfrak{K}\varrho \tag{3.24}$$

where $I(\phi) = 1$, $I(X) = 0$ if $X \neq \phi$. The operator norm of $\mathfrak{K}$ depends on b and will be denoted $|||\mathfrak{K}|||_b$.

3.6. *Results on Convergence*

The expansion is determined once L, L', η are chosen. Theorem 3.3, given below, assumes a specific choice of L, L', η.

Theorem 3.3. *Given $c > 0$, there exist $c'c'' > 0$ such that for $l_L \leqq c_1 l$, $l_L \leqq c'$, $W \subset \Lambda^{\hat{}}$, union of $\Delta \in \bar{\Delta}$,*

$$\sum_{X: X \cap W \neq \phi} |\mathfrak{K}(X)| c^{|X|} \leqq c'' \exp\{-1/2 \operatorname{dist}(X_0, W)\}$$

c'' depends on A, the observable, as well as c', c_1.

4. Proof of Theorem 2.1 (Using Section 3.6)

By a choice of units, one may, without loss of generality, set $l_D = 1$. Then the hypothesis $(2z\beta^3)^{1/2} \leqq c_2$ becomes $l_L \leqq c_2$ because $(2z\beta^3)^{1/2} = l_L / l_D$.

To illustrate the principles involved, a proof, using an idea of Glimm et al. that

$$|\langle \varrho(x) \varrho(y) \rangle_\Lambda| \leqq c e^{-1/2|x-y|} \tag{4.1}$$

uniformly in Λ, is first presented. In ϕ language, provided $x \neq y$, $\varrho(x)(\varrho(y) \to l_L^{-2} : \sin l_L^{1/2} \phi(x) : \sin l_L^{1/2} \phi(y) :$. In the expansion, take X_0 to be the cube $\Delta_x \in \bar{\Delta}$, $\Delta_x \ni x$. Set Δ_y equal to the cube $\in \bar{\Delta}$ containing y. By the symmetry $\phi \to -\phi$, $\mathfrak{K}(X) = 0$ unless $X \cap \Delta_y \neq \phi$, therefore the expansion (3.21) reduces to

$$|\langle \varrho(x) \varrho(y) \rangle_\Lambda| = \left| \sum_{X: X \cap \Delta_y \neq \phi} \mathfrak{K}(X) Z'(\Lambda, X) Z^{-1}(\Lambda) \right| . \tag{4.2}$$

By Theorem 3.3, (4.2) implies (4.1) provided, for some constant c

$$|Z'(\Lambda, X) Z^{-1}(\Lambda)| \leqq c^{|X|} \tag{4.3}$$

uniformly in Λ. By Theorem 3.3 again

$$\sum_{W \supset X^*} |\mathfrak{K}(W \sim X)| c'^{|W \sim X|} \leqq 1/2$$

if c' is taken sufficiently small, because $|W\sim X|\geqq 1$. This implies that $|||\mathfrak{K}|||_{c'}\leqq 1/2$. Equation (3.24) implies $\varrho=Z(\Lambda)(\mathfrak{K}-1)^{-1}I$, so

$$Z(\Lambda)^{-1}\|\varrho\|_{c'}\leqq|||(\mathfrak{K}-1)^{-1}|||_{c'}\leqq 2\,. \tag{4.4}$$

This is exactly (4.3). $(\mathfrak{K}-1)^{-1}$ is defined and bounded by the Neuman series.

Proof of Theorem 2.1. First it is proved that the finite volume correlation functions cluster exponentially, uniformly in Λ. The technique is due to Ginibre. A new expectation $\langle\ \rangle'_\Lambda$ is constructed using the doubled partition function $Z_1(\Lambda)\times Z_2(\Lambda)$ where $Z_1(\Lambda), Z_2(\Lambda)$ are copies of $Z(\Lambda)$. For any quantities A, B of the form (2.6)

$$\langle AB\rangle_\Lambda-\langle A\rangle_\Lambda\langle B\rangle_\Lambda=(1/2)\langle(A_1-A_2)(B_1-B_2)\rangle'_\Lambda\,. \tag{4.5}$$

$A_i=A$ as a function of the fields or coordinates in $Z_i(\Lambda)$, $i=1,2$. $\langle(A_1-A_2)(B_1-B_2)\rangle'_\Lambda$ is expanded using a double phase boundary expansion with terms labelled by (h_1,h_2), (Σ_1,Σ_2). Set $\Sigma=\Sigma_1\cup\Sigma_2$. The cluster expansion is performed on both factors of the product measure simultaneously. Take $X_0=$ smallest union of $\Delta\in\bar{\Delta}$ containing supp$A$. The resummation is essentially unchanged[1]. The symmetry $1\leftrightarrow 2$, in the same way as the $\phi\to-\phi$ symmetry above, is used to show

$$\langle AB\rangle'_\Lambda=\sum_{\substack{X:X\cap\operatorname{supp}B\\ \neq\phi}}\mathfrak{K}'(X)[Z'_1(\Lambda,X)Z'_2(\Lambda,X)Z_1(\Lambda)^{-1}Z_2(\Lambda)^{-1}]\,. \tag{4.6}$$

The reader is referred to the proof of Theorem 3.3 to see that with a different constant the same bound holds for $\mathfrak{K}'(X)$. The proof of exponential clustering uniformly in Λ is completed as for (4.1).

To complete the proof of Theorem 2.1, it remains to be shown that $\lim\limits_{\Lambda\to\mathbb{R}^3}\langle A\rangle_\Lambda$ exists for all A of the form (2.6). By applying the expansion with $X_0\supset\operatorname{supp}A$, this reduces to proving that $\lim\limits_{\Lambda\to\mathbb{R}^3}Z'(\Lambda,X)Z^{-1}(\Lambda)$ exists since $\mathfrak{K}(X)$ is independent of Λ for $X\subset\Lambda$. Theorem 3.3 provides the required uniform bounds.

Proof that $\lim\limits_{\Lambda\to\mathbb{R}^3}Z'(\Lambda,W)Z^{-1}(\Lambda)$ *exists for all* $W\subset\mathbb{R}^3$, a union of $\Delta\in\bar{\Delta}$: assume $\Lambda,\Lambda'\supset W$,

$$\begin{aligned}&Z'(\Lambda,W)Z^{-1}(\Lambda)-Z'(\Lambda',W)Z^{-1}(\Lambda')\\&\quad=\{Z'(\Lambda,W)Z(\Lambda')-Z'(\Lambda',W)Z(\Lambda)\}[Z(\Lambda)Z(\Lambda')]^{-1}\,.\end{aligned} \tag{4.7}$$

Construct a doubled Kirkwood Salsburg expansion for the quantity in curly brackets taking $X_0=W$. All terms cancel in pairs except those for which either $X\cap\sim\Lambda\neq\phi$ or $X\cap\sim\Lambda'\neq\phi$, i.e., $X\cap(\sim\Lambda\cup\sim\Lambda')\neq\phi$. The proof is concluded using Theorem 3.3 to show that $(4.7)\to 0$, as $\Lambda,\Lambda'\nearrow\mathbb{R}^3$. This also concludes the proof of Theorem 2.1.

---

[1] This proof of clustering is easier than that in [8] because unlike the situation in [8], the observable A has the same symmetry (periodic in ϕ) as the interaction and this has been exploited in setting up the expansion

5. Combinatorics

The proofs of Theorem 3.3 and 3.4 involve some combinatorics, such as in the sums over $\bar{y}, h$. The Lemma 5.1 given below is addressed to this aspect of the convergence proof. First a series of definitions which will be used throughout the convergence proof are presented.

Given an integer $n \geqq 2$, a tree graph on n vertices is defined to be a set of pairs (i,j), $i<j$, of integers $1 \leqq i,j \leqq n$ such that each integer j, $1<j \leqq n$ occurs once and only once in a pair (i,j). This is not the standard definition. Given $\bar{y}=(Y_1,\ldots,Y_n)$, it is convenient to rewrite $\kappa(\bar{y},s)$ as

$$\kappa(\bar{y},s)=\Bigl(\prod_{(i,j)\in T}|Y_i|\Bigr)^{-1}\sum_T\sum_\alpha q(T,s,\bar{y})\kappa(T,\alpha) \tag{5.1}$$

where T is summed over tree graphs on n vertices. α is summed over maps $\alpha: T\to\bar{\Delta}\times\bar{\Delta}, (i,j)\mapsto(\alpha'(i),\alpha''(j))$ such that $\alpha'(i)\subset Y_i, \alpha''(j)\subset Y_j$. Thus α is summed over a $\bar{y}, T$ dependent class which will be denoted $\bar{\alpha}=\bar{\alpha}(\bar{y},T)$.

$$q(T,s,\bar{y})=\prod_{(i,j)\in T}\left(\frac{d}{ds_{j-1}}s_{j-1}\ldots s_i\right)|Y_i| \tag{5.2}$$

$|Y|$ denotes the volume of Y.

$$\kappa(T,\alpha)=\prod_{(i,j)\in T}\kappa(i,j,\alpha)$$

$$\kappa(i,j,\alpha)=\int_{\alpha'(i)}dx'\int_{\alpha''(j)}dx''v(x',x'')\,.$$

$$(\delta/\delta\psi(x'))(\delta/\delta\psi(x''))\,. \tag{5.3}$$

Given $\bar{y}, T, \alpha$, let

$$d(T,\alpha)=\sum_{(i,j)\in T}\operatorname{dist}(\alpha'_i,\alpha''_j)$$

Lemma 5.1. *Given c_4, c_1, there exists c', c'', c_5 such that for $l_L \leqq c_1 l$, $l_L \leqq c'$*

$$\sum_{X:X\cap W\neq\phi}e^{c_4|X|}|\mathfrak{K}(X)|\leqq c''\exp\{-1/2\operatorname{dist}(X_0,W)\}$$

$$\cdot\sup_{n,T,s,\bar{y},h,\alpha}\exp\{c_5 n+(1/4)F_1(h)+(3/4)\cdot d(T,\alpha)\}$$

$$\cdot\int d\psi_s\kappa(T,\alpha)A\chi_h e^{-Q(h)}\,.$$

The supremum is over the set of $n, T, s, \bar{y}, h, \alpha$ such that $n\in\mathbb{Z}$, $n\geqq 1$, $T\in$ set of tree graphs on n vertices, which is defined to be the null set if $n=1$, $s\in[0,1]^{n-1}$. $(\bar{y},h)$ belongs to the set defined by $h\in\bar{H}$, $\bar{y}=(Y_1,\ldots,Y_n)$, $Y_i\in\bar{Y}(h)$ for $i=1,\ldots,n$, $\bigcup_{i\leqq n}Y_i\supset\Sigma^\frown(h)$, $\alpha\in\bar{\alpha}(\bar{y},T)$.

Notice that, as usual, Lemma 5.1 is written for the case $X_0\supset\operatorname{supp}A$. If $X_0\not\supset\operatorname{supp}A$, there is a parallel lemma with A replaced by the factors of A supported inside X_0. An abbreviated notation has been used in Lemma 5.1, in that $Q(h)=Q(\Lambda,X,h)$, $\chi_h=\chi_h(\Lambda\cap X)$, $X=\bigcup_{i\leqq n}Y_i$.

Proof of Lemma 5.1. The bound of Lemma 5.1 is implied by [using the decomposition (5.1)]

$$\sum_n \sum_T \int ds \sum_{\bar{y},h,\alpha:\, X\cap W\neq\phi} e^{c_4|X|} q(T,s,\bar{y}) \left(\prod_{(i,j)\in T} |Y_i|\right)^{-1}$$
$$\cdot \exp\{-c_5 n - 1/4 F_1(h) - 3/4 d(T,\alpha)\}$$
$$\leqq c'' \exp\{-1/2\,\mathrm{dist}(X_0, W)\}\ . \tag{5.4}$$

This in turn is implied by (5.5), (5.6), (5.7), (5.8), given below.

$$\sum_T \int ds q(T,s,\bar{y}) \leqq e^{|X|} \tag{5.5}$$

$$\sum_n \sup_T \sum_{\bar{y},\alpha} \exp\{-1/2(c_5 - 2c_4 - 3)|X| - 1/4 d(T,\alpha)\} \left(\prod_{(i,j)\in T} |Y_i|^{-1}\right) < \infty\ . \tag{5.6}$$

Given n, the sum over $\bar{y}$ is over all sequences $(Y_i, \ldots Y_n)$, $Y_i \in P$, $i=1,\ldots,n$. Y_i connected, $i>1$. $Y_1 \supset X_0$, each connected component of Y_1 contains a cube $\Delta \subset X_0$, $\Delta \in \bar{\Delta}$.

$$\sum_h \exp\{-1/4 F_1(h) + c_5(1/2|X| - n)\} < \infty \tag{5.7}$$

uniformly in $\bar{y}$. Given $\bar{y}=(Y_1,\ldots Y_n)$ for some n, $Y_i \in P$, h is summed over $h\in\bar{H}$ such that $Y_i \in \bar{Y}(h)$, $i=1,\ldots,n$, $X\supset \Sigma\hat{}(h)$. If there are no such h, set the left hand side $=0$.

$$\mathrm{dist}(X_0, W) \leqq |X| + d(T,\alpha) \tag{5.8}$$

on the subset of $\{n, T, s, \bar{y}, h, \alpha\}$ such that $X\cap W \neq \phi$. Inequality (5.8) is obvious using the definition of $d(T,\alpha)$, $\alpha\in\bar{\alpha}(\bar{y},T)$ and $X_0, W\subset X$.

Proof of (5.5). This estimate is implicitly contained in [9]. First observe that

$$\sum_T q(T,s,\bar{y}) = \prod_{j=2}^{n} \sum_{i<j} (d/ds_{j-1})(s_{j-1}\ldots s_i)|Y_i|\,.$$

Substitute for $q(T,s,\bar{y})$ in the right hand side of

$$\int_{[0,1]^{n-1}} ds_1 \ldots ds_{n-1} \sum_T q(T,s,\bar{y}) \leqq \int_{[0,1]^{n-1}} ds_1 \ldots ds_{n-1}$$
$$\cdot \sum_T q(T,s,\bar{y}) \exp\left\{\sum_{i<n-1} s_{n-1}\ldots s_i |Y_i|\right\}$$

and perform the integrals in the order $s_{n-1}, \ldots, s_1$, estimating each one before doing the next by

$$\int_0^1 ds x e^{sx} \leqq e^x$$

valid for $x \geqq 0$. The result is (5.5). Proof concluded.

Proof of (5.6). Fix n, T, then $\bar{y}$ has the form $(Y_1,\ldots,Y_n)$. Arrange the sum over $\bar{y},\alpha$ in the order Y_n, $\alpha''(n)$, $\alpha'(i)$, where $(i,n)\in T$, Y_{n-1}, $\alpha''(n-1)$, $\alpha'(i')$, where $(i', n-1)\in T$, etc. The sum over Y_n now takes the form

$$\sum_{Y_n:\, Y_n\supset\alpha''(n)} \exp\{-1/2(c_5 - 2c_4 - 3)|Y_n|\} \tag{5.9}$$

which, provided $n \neq 1$, is bounded by

$$\sum_{N=1}^{\infty} 6^{6N} \exp\{-1/2(c_5 - 2c_4 - 3)N\} \tag{5.10}$$

because, by an argument in [10, p. 219], the number of connected unions of N cubes in $\bar{\Lambda}$ containing a given cube is less than 6^{6N}. Choose c_5 large so that (5.10) converges and set c' equal to the sum. The sum over $\alpha''(n), \alpha'(i)$ now takes the form

$$\sum_{\alpha'(i)} |Y_i|^{-1} \sum_{\alpha''(n)} \exp\{-1/4 \operatorname{dist}(\alpha'(i), \alpha''(n))\} \tag{5.11}$$

which is bounded uniformly in $\bar{y}$. Set c'' equal to the bound. After this estimate the sum over Y_{n-1} is of the same form as (5.10) and is estimated by c' again, unless $n=1$.

The sum over Y_1 is different because Y_1 may have up to $|X_0|$ connected components, each containing a cube in $\bar{\Lambda} \subset X_0$. Thus the Y_1 sum is bounded by $c'^{|X_0|}$. Thus the sum of all terms in (5.6) with n fixed is bounded by

$$c'^{|X_0|}(c'c'')^{n-1} \,. \tag{5.12}$$

Choose c_5 large (this does not conflict with previous choice of c_5) so that $\sum_{n=1}^{\infty} (c'c'')^{n-1} < \infty$. The proof of (5.6) is complete.

The proof of Lemma 5.1 will be complete once (5.7) is proved. (5.7) is implied by Lemmas 5.2, 5.3 given below.

Lemma 5.2. *There exists* $c > 0$ *such that*

$$F_1(h) \geqq c \sum_f |\delta h(f)|^2$$

where f *is a face of a cube in* $\bar{\Omega}$, *considered as a closed subset of* $\mathbb{R}^3$. $|\delta h_f|$ *is the discontinuity in* h *across* f. *Further, if* X *is a union of cubes in* $\bar{\Lambda}$,

$$F_1(X, h) \geqq c \sum_{f^{\mathrm{INT}} \subset X^{\mathrm{INT}}} |\delta h(f)|^2$$

$$f^{\mathrm{INT}} = f \sim \partial f \,. \quad \text{(Boundary w.r.t. } \mathbb{R}^2\text{)}$$

Lemma 5.3. *Given* c, c', *if* $l_L > 0$ *is sufficiently small,*

$$\sum_h \exp\left\{-c \sum_f |\delta h(f)|^2 + c'(1/2|X| - n)\right\} < \infty$$

uniformly in $\bar{y}, n$.

Proof of Lemma 5.3. It is sufficient to prove

$$\sum_h \exp\left\{-1/2c \sum_f |\delta h(f)|^2 - 1/2c'|X|\right\} < \infty \tag{5.13}$$

$$-1/2c \sum_f |\delta h(f)|^2 + c'(|X| - n) \leqq 0 \,. \tag{5.14}$$

Proof of (5.14). The conditions on $\bar{y}, h$ imply

$$|X| - n \leqq |\Sigma(h)^{\urcorner}| \,. \tag{5.15}$$

Also

$$|\Sigma(h)^{\sim}| \leqq (2(L'+1))^3 |\{f : f \subset \Sigma(h)\}| \,. \tag{5.16}$$

The absolute value signs on the right denote cardinality. (5.16) follows from (3.3). Furthermore

$$|\{f : f \subset \Sigma(h)\}| \leqq (2\pi l_L^{-1/2})^{-2} \sum_f |\delta h(f)|^2 \,. \tag{5.17}$$

For $l_L > 0$ sufficiently small, depending on L', estimates (5.15), (5.16), (5.17) imply (5.14).

Proof of (5.13). Given h, set $X_n \subset \mathbb{R}^3 = \{X : h(X) \leqq 2\pi l_L^{-1/2} n\}$ for $n \in \mathbb{Z}$. Let $\tilde{\Gamma}$ be the *family* of connected components of boundaries ∂X_n, $n \in \mathbb{Z}$. Let Γ be the *set* of connected components of boundaries ∂X_n, $n \in \mathbb{Z}$, i.e., $\tilde{\Gamma}$ may have repeated elements, Γ does not. If $\gamma \in \Gamma$, γ is a union of faces f of cubes $\in \bar{\Omega}$. Set $|\gamma| = N^0$ of faces in γ.

$$\sum_f |(\delta h(f)| = \sum_{\gamma \in \tilde{\Gamma}(h)} |\gamma| \cdot 2\pi l_L^{-1/2} \,. \tag{5.18}$$

Therefore provided $2\pi l_L^{-1/2} \geqq 1$,

$$\begin{aligned} &\sum_h \exp\left\{-c/2 \sum_f |\delta h(f)|^2\right\} \\ &\quad \leqq \sum_{\Gamma} \sum_{h:\Gamma(h)=\Gamma} \exp\left\{-c/2 \cdot 2\pi l_L^{-1/2} \sum_{\gamma \in \tilde{\Gamma}(h)} |\gamma|\right\} \\ &\quad \leqq \sum_{n=0}^{\infty} \sum_{\Gamma:|\Gamma|=n} c''^n \exp\left\{-c/4 \cdot 2\pi l_L^{-1/2} \sum_{\gamma \in \Gamma(h)} |\gamma|\right\} \end{aligned} \tag{5.19}$$

where $|\Gamma| = \operatorname{card} \Gamma$ and

$$c'' = \sum_{m=0}^{\infty} \exp(-c/4 \cdot 2\pi l_L^{-1/2} 4m) \tag{5.20}$$

(5.19) was obtained by estimating the sum over $\{h : \Gamma(h) = \Gamma\}$, $|\Gamma| = n$. The right hand side of (5.19) is less than

$$\sum_{n=0}^{\infty} c''^n / n! \left(\sum_{\gamma} \exp\{-c/4 \cdot 2\pi l_L^{-1/2} |\gamma|\}\right)^n \tag{5.21}$$

where γ is summed over all possible connected components of boundaries $\partial X'$ where $X' \subset X$ is a union of cubes $\in \bar{\Omega}$. The null set is excluded. This in turn equals

$$\exp\left(c'' \sum_{\gamma} \exp\{-c/4 \cdot 2\pi l_L^{-1/2} \cdot |\gamma|\}\right). \tag{5.22}$$

By a lemma of [12, p. 117], the number of γ with $|\gamma|$ fixed is less than $L^{-3}|X|\,|\gamma|^3 3^{|\gamma|-3}$, so (5.22) is bounded by $\exp(c'/2|X|)$ provided $l_L > 0$ is sufficiently small, depending on L', L, c', c. This completes the proof of (5.13) and hence Lemma 5.3.

[2] This is easy to prove, especially if one is content to bound the number of cycles by $c'c^{|\gamma|}$. Simply build up a cycle face by face with $\leqq 6$ possible choices per face

Proof of Lemma 5.2. This proof is taken from [8].

$$\sum_f |\delta h(f)|^2 = L^{-6} \sum_{\Omega,\Omega'} \int_\Omega \int_{\Omega'} dx dy \cdot |h(x)-h(y)|^2 \tag{5.23}$$

where $\Omega, \Omega' \in \bar{\Omega}$, the sum is over all nearest neighbors.

$$|h(x)-h(y)|^2 \leqq 3|h(x)-g(x)|^2 + 3|h(y)-g(y)|^2 + 3|g(x)-g(y)|^2 \tag{5.24}$$

substitute

$$|g(y)-g(x)|^2 = \left|\int_x^y \nabla g \cdot ds\right|^2 \leqq 4L \int_x^y |\nabla g|^2 ds \tag{5.25}$$

estimate (5.23) using (5.24), (5.25). Take the path from x, to y to be a union of three segments parallel to the x, y, z axes. The right hand side of (5.23) is thus bounded by

$$L^{-6} 3 \cdot L^3 \cdot 2 \cdot \sum_{\Omega,\Omega'} \int_\Omega |h-g|^2 + L^{-6} 3 \cdot L^4 \cdot 3 \cdot 4L \cdot \sum_{\Omega,\Omega'} \int_{\Omega \cup \Omega'} |\nabla g|^2 \tag{5.26}$$

which is less than

$$18L^{-3} \int |g-h|^2 + 216L^{-1} \int |\nabla g|^2 \leqq c^{-1} F_1(h)$$

where $c = \min(1/36\eta L^3, 1/432L)$. All these estimates continue to hold if everything is restricted to X, a union of cubes $\in \bar{\Delta}$. This completes the proof of Lemma 5.2.

6. Estimates on Functional Integrals, Gaussian Integrals

An abbreviated notation, in which dependences on $\hat{y}$, h, n, T, s, α are not all explicit will be used. The objective in this section will be to estimate

$$|\int d\psi \kappa(T,\alpha) A \chi e^{-Q}| \tag{6.1}$$

so as to show that the supremum in Lemma 5.1 is finite. This will prove Theorem 3.3.

There are $2(n-1)$ derivatives in $\kappa(T,\alpha)$, labelled by variables $(x'_1, \ldots, x'_{n-1}, x''_1, \ldots, x''_{n-1}) \equiv (x', x'')_{n-1} \equiv (x)_{n-1}$. Let π be a partition of these variables into subsets $x_{\pi(i)}$, $i = 1, 2, \ldots, r$, r arbitrary. Using an obvious notation for derivatives, (6.1) may be expanded by Leibniz rule to yield

$$\left|\sum_\pi \int d\psi \int_\alpha d(x)_n \prod_{(i,j)\in T} v(x'_i, x''_j) \cdot A'(x_{\pi(1)}) \chi'(x_{\pi(2)}) \prod_{i>2} Q'(x_{\pi(i)}) e^{-Q}\right|. \tag{6.2}$$

The subscript α on the integral indicates the region of integration $x'_i \in \alpha'(i)$, $x''_j \in \alpha''(j)$. It will later be necessary to bound the number of partitions π for which (6.2) is non zero. There are less than 2^{4n} choices for $\pi(1)$, $\pi(2)$. Since Q is a local function of ψ, $Q'(x_{\pi(i)})=0$ unless $x_{\pi(i)}$ are all given the same region of integration by α. Let $n(\alpha, \Delta)$ be the number of variables assigned the region of integration $\Delta \in \bar{\Delta}$ by α. Then the number of partitions is bounded by $2^{4n} \cdot \prod_{\Delta} \cdot (N^0$ of partitions of variables localised in Δ by $\alpha)$ which is less than

$$\prod_{\Delta} c^{n(\alpha, \Delta)} n(\alpha, \Delta)! \tag{6.3}$$

Each factor $v(x'_i, x''_j)$ in (6.2) may be replaced by $v\xi_{\alpha,i,j}(x'_i, x''_j)$, where (1) $\xi_{\alpha,i,j}(x'_i, x''_j) = \xi_{\alpha,i,j}(|x'_i - x''_j|)$ for $x'_i, x''_j \in \Pi_l$; (2) $\xi_{\alpha,i,j}=1$ for $|x'_i - x''_j| \geqq \operatorname{dist}(\alpha'(i), \alpha''(j))$, $=0$ for $|x'_i - x_j| \leqq \operatorname{dist}(\alpha'(i), \alpha''(j)) - 1$, where $x'_i, x''_j \in \Pi_l$. (3) $\xi_{\alpha,i,j}$ is bounded together with its finite difference derivatives up to second order uniformly in α, i, j.

Given a function f on Π_l, define the Sobelev norm

$$\|f\| = \left(\sum_{x, y \in \Pi_l} l^6 f(x) \bar{f}(y) v(x, y) \right)^{1/2}. \tag{6.4}$$

The corresponding Hilbert space will be denoted $H_{-1}(\Pi_l)$.

Lemma 6.1. *$v\xi_{\alpha,i,j}$ is a bounded operator on H_{-1} and*

$$\|v\xi_{\alpha,i,j}\| \leqq c \exp\{-\operatorname{dist}(\alpha'(i), \alpha''(j))\}$$

for some constant c independent of α, i, j.

Proof of Lemma 6.1. It is equivalent to bound $(1-\Delta) v\xi_{\alpha,i,j}$ as a convolution operator on $l^2(\Pi_l)$ and this is bounded by its l^1 norm by Young's inequality. Since

$$(1-\Delta)(v\xi_{\alpha,i,j}) = -2(\nabla v)(\nabla \xi_{\alpha,i,j}) - v(\Delta \xi_{\alpha,i,j}) \tag{6.5}$$

the bound of Lemma 6.1 follows using (1) $\xi_{\alpha,i,j}$ and its gradient vanishes for $|x'_i - x''_j| \leqq \operatorname{dist}(\alpha'(i), \alpha''(j)) - 1$; (2) $|v|$, $|\nabla v| = 0(\exp\{-|x'_i - x''_j|\})$ for $|x'_i - x''_j| \to \infty$. This is easily proved from the analyticity properties of the fourier transform of v. For example, see [13]. If dist $(\alpha'(i), \alpha''(j) = 0$, $\xi_{\alpha,i,j} = 1$ and the lemma is trivial. End of proof.

By taking norms on H_{-1} spaces of functions of several variables, (6.2) is smaller than

$$\sum_{\pi} \int d\psi \, \|\eta_{\alpha,\pi(1)} A'\| \, \|\eta_{\alpha,\pi(2)} \chi'\|$$
$$\cdot \left(\prod_{i>2} \|\eta_{\alpha,\pi(i)} Q'\| \right) e^{-Q} c^n e^{-d(T,\alpha)} \tag{6.6}$$

where the norms are taken on the spaces $H_{-1}(\Pi_l^{|\pi(i)|})$, $i=1, \ldots, r$. $\eta_{\pi(i)}$ is the characteristic function of the region of integration assigned to $x_{\pi(i)}$ by α, i.e., a characteristic function of a product of cubes $\in \bar{\Delta}$.

Lemma 6.2. $\|\eta_{\alpha,\pi(i)}A'\| \leqq (c' l_L^{-1})^{n'} (cn')^{|\pi(1)|}$ *for constants* c, c' *depending only on* c_1. $|\pi| = \operatorname{card}\pi$.

Proof.

$$\begin{aligned} A &= \prod_{j=1}^{n'} l_L^{-1} : \sin l_L^{1/2} \phi(y_j) : \\ &= \prod_{j=1}^{n'} l_L^{-1} \exp(l_L v_\Lambda(y_j, y_j)/2) \sin l_L^{1/2} \phi(y_j) . \end{aligned} \tag{6.7}$$

$A'(x_{\pi(1)})$ is a sum of at most $n'^{|\pi(1)|}$ terms arising from Leibniz rule. By a simple calculation using the bounds $|\sin l_L^{1/2}\phi| \leqq 1$, $|v_\Lambda(y, y)| \leqq cl^{-1}$ uniformly in y, $|v(y, y)| \leqq cl^{-1}$ uniformly in y, the proof of Lemma 6.2 may easily be completed.

By Lemma 6.2 and the Holder inequality (6.6) is less than

$$\begin{aligned} &(c' l_L^{-1})^{n'} \sum_\pi c^n e^{-d(T,\alpha)} \\ &\cdot \left(\int d\psi \|\eta_{\alpha,\pi(2)}\chi'\|^p e^{-pQ}\right)^{p^{-1}} \\ &\cdot \left(\int d\psi \prod_{i>2} \|\eta_{\alpha,\pi(i)} Q'\|^{p'}\right)^{p'^{-1}} \end{aligned} \tag{6.8}$$

$p + p' = 1$. Note that $|\pi(1)| \leqq 2(n-1)$.

Theorem 6.3. *Given* $p' \geqq 1$, *if* L' *is sufficiently large,*

$$\left(\int d\psi \prod_{i>2} \|\eta_{\alpha,\pi(i)} Q'\|^{p'}\right)^{p'^{-1}} e^{-F_1/4} \leqq l_L^{1/4m} c^n \prod_\Delta (n(\alpha, \Delta)!)^3$$

where $m = N^0$ *of distinct cubes* $\in \bar{\Lambda}$ *in which a variable in* $\bigcup_{i>2} \pi(i)$ *is localised by* $\eta_{\alpha,\pi(i)}, i > 2$.

This theorem will be used on the second integral in (6.8). The following manipulations prepare for a theorem on the first integral in (6.8), the "vacuum energy" integral.

Given a map $\beta : \pi(2) \to \bar{\Omega} \times \ldots \times \bar{\Omega}$, $|\pi(2)|$ factors, let $\eta_\beta(x_{\pi(2)})$ be the characteristic function of $\beta(\pi(2))$ considered as a set in $(\mathbb{R}^3)^{|\pi(2)|}$. Write

$$\eta_{\alpha,\pi(2)}(x_{\pi(2)}) = \sum_\beta \eta_\beta(x_{\pi(2)}) \tag{6.9}$$

where β is summed over all maps such that $\eta_\beta \leqq \eta_{\alpha,\pi(2)}$. The number of terms in the sum is bounded by $L^{-3|\pi(2)|}$. By the triangle inequality

$$\begin{aligned} \|\eta_{\alpha,\pi(2)}\chi'\|^p &\leqq \left(\sum_\beta \|\eta_\beta \chi'\|\right)^p \\ &\leqq L^{-3pp'^{-1}|\pi(2)|} \sum_\beta \|\eta_\beta \chi'\|^p \end{aligned} \tag{6.10}$$

$p^{-1} + p'^{-1} = 1$.

Define $\bar{t}(\beta,h)$ to be the set of functions $t:\bar{\Omega}\to\mathbb{R}$ such that (a) $t(\Omega)\in[h(\Omega)-\pi\beta^{-1/2},\ h(\Omega)+\pi\beta^{-1/2}]$; (b) if Ω is one of the factors in $\beta(\pi(2))$, $t(\Omega)=h(\Omega)\pm\pi\beta^{-1/2}$. Define

$$\chi_{\beta,h}^{\sim}(\phi)=\sup_{t\in t(\beta,h)}\exp\left\{-\sum_{\Omega\in\bar{\Omega}}(\bar{\phi}(\Omega)-t(\Omega))^2/2\right\}. \tag{6.11}$$

Lemma 6.4.

$$\|\eta_\beta\chi_h'\|\leqq c^{|\pi(2)|}\left(\prod_\Delta n(\alpha,\Delta)!\right)L^{5/2|\pi(2)|}\chi_{\beta,h}^{\sim}.$$

Proof. Essentially the same inequality is proved in [8]. Referring to (3.1), it is seen that $\chi(\xi)$ is the convolution of a gaussian and a step function, thus

$$\chi^{(1)}(\xi)=\pi^{-1/2}(e^{-(\xi+\beta^{-1/2}\pi)^2}-e^{-(\xi-\beta^{-1/2}\pi)^2}) \tag{6.12}$$

which is already sufficient to prove the lemma when all the factors of $\beta(\pi(2))$ are distinct. If this is not the case, one needs the following estimate on higher derivatives

$$|\chi^{(m)}(\xi)|\leqq cm!\sup_{t=\pm\beta^{-1/2}\pi}e^{-(\xi-t)^2/2} \tag{6.13}$$

which is obtained by applying the Cauchy formula for the $(m-1)^{th}$ derivative to (6.12).

(6.13) implies

$$|\eta_\beta\chi_h'|\leqq c^{|\pi(2)|}\left(\prod_\Delta n(\alpha,\Delta)!\right)\chi_{\beta,h}^{\sim}\eta_\beta. \tag{6.14}$$

The proof of the lemma is completed by the simple estimate

$$\|\eta_\beta\|=\left[\int_\Omega dx\int_\Omega dy\,v(x,y)\right]^{|\pi(2)|/2}$$
$$\leqq cL^{5/2|\pi(2)|}. \tag{6.15}$$

End of proof.

The next theorem is the subject of Section 7.

Theorem 6.5. *There exist $L>0$, $L_0'>0$, $p>1$ such that for $L'\geqq L_0'$, there exist c,c' such that*

(a) $(\int d\psi\chi_{\tilde\beta}^{\sim p}e^{-pQ})^{p^{-1}}\leqq c^n l_L^{m'}e^{-F_1/2}$;

(b) $(\int d\psi_s\|\eta_{\alpha,\pi(2)}\chi'\|^p e^{-pQ})^{p^{-1}}\leqq c'^n\left(\prod_\Delta n(\alpha,\Delta)!\right)\cdot l_L^{m'}e^{-F_1/2}$

uniformly in the region $l_L\leqq c_1 l$, where $m'=N^0$ of distinct cubes $\in\bar{\Delta}$ in which a variable in $\pi(2)$ is localised by $\eta_{\alpha,\pi(2)}\cdot\chi_{\tilde\beta}\equiv\chi_{\beta,h}$.

The second inequality is implied by the first, (6.10) and Lemma 6.4. The result of applying Theorems 6.3, 6.5–6.8 implies: there exists c,c' such that

$$|\int d\psi\kappa(T,\alpha)A\chi e^{-Q}|\leqq(c'l_L^{-1})^{n'}e^{-d(T,\alpha)}e^{-F_1/4}$$
$$\cdot l_L^{(n-n')/4}c^n\prod_\Delta(n(\alpha,\Delta)!)^5. \tag{6.16}$$

The sum over π has been dominated by (6.3). $n-n'\geqq m+m'$ because every region Y_i in $\bar{y}$ which does not contain a factor of A must have at least one variable in $\bigcup_{i>1}\pi(i)$ localised within it. Inequality (6.16) holds, provided L, L' are chosen in accordance with Theorems 6.3 and 6.5.

Proof of Theorem 3.3. Compare (6.16) with Lemma 5.1. The supremum over n, T, s, $\bar{y}$, h, α will be finite if

$$e^{-d(T,\alpha)/4}\, l_L^{-n/4} c^n e^{c_5 n}\cdot \prod_{\Delta}(n(\alpha,\Delta)!)^5 \leqq \text{const} \tag{6.17}$$

uniformly in T, α, n. This will hold if $l_L>0$ is sufficiently small by virtue of:

Lemma 6.6.

$$\left(\prod_{\Delta}(n(\alpha,\Delta)!)^5\right) e^{-d(T,\alpha)/4} \leqq c^n .$$

Proof. The construction of the cluster expansion is such that for each $\Delta\in\bar{\Delta}$, $d(T,\alpha)$ must contain $n(\alpha,\Delta)-1$ terms, $\operatorname{dist}(\Delta,\alpha''(j))$, with $\alpha''(j)$ all distinct. Therefore it is sufficient to show

$$(n!)^5 \prod_{\Delta'\in\bar{\Delta}} \exp\{-\operatorname{dist}(\Delta,\Delta')/4\} \leqq c' \tag{6.18}$$

where the product is over any $n-1$ distinct cubes $\Delta'\in\bar{\Delta}$. Evidently the left hand side is maximised if the cubes Δ' are packed as close to Δ as possible. The proof of (6.18) for this case is left to the reader. End of proof of Lemma 6.6 and Theorem 3.3.

Proof of Theorem 6.3. Q' contains terms arising from differentiating the linear terms in ψ. The control of these is the immediate objective. Define

$$F_2'(x)=(\eta_\Lambda-\Delta)(g-g_c)(x). \tag{6.19}$$

Lemma 6.7. *Given $c>0$, if L' is sufficiently large, depending on L, c*

$$\int_X dx\, F_2'^2(x) \leqq cL'^{-3} F_1(X^\frown)$$

for all X, union of cubes $\in\bar{\Delta}$.

Proof. It is sufficient to prove

$$\int_\Omega F_2'^2\cdot dx \leqq c' F_1(\Omega^\frown) \tag{6.20}$$

where $c'=0(e^{-\eta L'})$ as $L'\to\infty$ and recover the result in the lemma by summing over $\Omega\subset X$. The operator $\mathfrak{L}-\mathfrak{L}_c$ annihilates constants, therefore $h(x)$ may be replaced by $h'(x)-h(\Omega)$ in F_2' without changing anything. Let $\xi(x)=1$ inside $\Omega^\frown$, zero outside. An easy argument shows that F_2' restricted to Ω does not depend on $h'(x)-h(\Omega)$ outside $\Omega^\frown$, therefore this may be replaced by $h^\sim=[h'-h(\Omega)]\xi$. Thus

$$F_2'(x)=((\eta_\Lambda-\Delta)(\mathfrak{L}-\mathfrak{L}_c)h^\sim)(x). \tag{6.21}$$

By definition of $\mathfrak{L}_c$, for $x, y \in \Pi_l$

$$
\begin{aligned}
(\eta_\Lambda - \Delta)(\mathfrak{L} - \mathfrak{L}_c)(x, y) = &\eta_\Lambda(x)(\zeta'(x) - 1)\delta(x - y) \\
&- 2(\nabla\mathfrak{L}_c)(x, y) \cdot \nabla(\zeta'(x)\zeta(x - y)) - \mathfrak{L}_c(x, y) \\
&\cdot \Delta(\zeta'(x)\zeta(x - y)). \qquad (6.22)
\end{aligned}
$$

Subscripts L' on ζ', ζ have been dropped. The ∇, Δ apply to the x variable. Substitute (6.22) into (6.21) and calculate the square of the $l^2(\Pi_l)$ norm as in the right hand side of (6.20). It is sufficient to calculate this for the terms corresponding to the three kernels in (6.22) separately. The following, to be verified below, will be used

$$
|\zeta' - 1|\, |\nabla\zeta'|, |\Delta\zeta'| \leqq c' \qquad (6.23)
$$

uniformly in x, Λ where $c' = 0$ (exp $(-\eta L')$), together with

$$
\mathfrak{L}_c(x, y), |\nabla\mathfrak{L}_c(x, y)| \leqq c'' \exp(-\eta|x - y|) \quad \text{for} \quad |x - y| \geqq 1. \qquad (6.24)
$$

For example, the operator norm on l^2 of the second term in (6.22) is bounded by

$$
\begin{aligned}
\|(\nabla\mathfrak{L}_c) \cdot \nabla(\zeta'\zeta)\| &\leqq \|\nabla\zeta' \cdot \nabla\mathfrak{L}_c\| + \|(\nabla\mathfrak{L}_c)\nabla(\zeta'(\zeta - 1))\|_{H-S} \\
&\leqq c' \|\nabla\mathfrak{L}_c\| + \|(\nabla\mathfrak{L}_c) \cdot \nabla(\zeta'(\zeta - 1))\|_{H-S}
\end{aligned}
$$

which is bounded by $0(\exp(-\eta L'))$. $H - S$ stands for the Hilbert Schmidt norm, which is small because $(\zeta - 1)(x - y)$ vanishes for $|x - y| \leqq L' - 1$. The other terms in (6.22) may also be bounded in operator norm to yield

$$
\int_\Omega F_2'^2\, dx \leqq c'^2 \int_{\Omega^\frown} h^{\sim 2}\, dx
$$

where

$$
\begin{aligned}
c' &= 0(\exp(-\eta L')) . \\
&= c'^2 L^3 \sum_{\Omega' \subset \Omega^\frown} |h(\Omega') - h(\Omega)|^2 \\
&\leqq c'^2 L^3 \sum_f |\delta h(f)|^2 \cdot 3L'L^{-1} \cdot L'^3 L^{-3} . \qquad (6.25)
\end{aligned}
$$

The sum is over f such that $f^{\text{INT}} \subset (\Omega^\frown)^{\text{INT}}$. Estimate (6.20) is now obtained by taking L' large depending on L and appealing to Lemma 5.2.

The claims (6.23), (6.24) will now be justified. The following fact about the kernel $v(x, y)$ of $(\eta - \Delta)^{-1}$ will be used: $v(x, y) = v(x - y)$ for $x, y \in \Pi_l$, $v(x)$ and its derivatives are $0(e^{-\eta|x|})$ as $|x| \to \infty$ away from the origin. See for example [13].

The kernels of $(\eta_\Lambda - \Delta)^{-1}$, $\mathfrak{L}_c(x, y)$ are positive and bounded by $v(x, y)$. This may be seen by using the Feyman Kac formula for $\exp(t(\Delta - \eta_\Lambda))$. By the resolvent equation

$$
(\eta_\Lambda - \Delta)^{-1} = (\eta - \Delta)^{-1} - (\eta - \Delta)^{-1}(\eta_\Lambda - \eta)(\eta_\Lambda - \Delta)^{-1}
$$

and the exponential decay of $(\eta - \Delta)^{-1}$, $(\eta_\Lambda - \Delta)^{-1}$, the derivatives up to second order with respect to x of $\mathfrak{L}_c(x, y)$ are $0(e^{-\eta|x-y|})$ as $|x - y| \to \infty$.

By definition

$$\zeta'^{-1}(x) = \sum_{y \in \Pi_l} l^3 \mathfrak{L}_c(x,y)\zeta(x-y)$$

$$= 1 + \sum_{y \in \Pi_l} l^3 \mathfrak{L}_c(x,y)(\zeta(x-y)-1)$$

$$= 1 - 0(e^{-\eta L'}) \tag{6.26}$$

which shows that $\zeta'(x) = 1 + 0(e^{-\eta L'})$ uniformly in x. By differentiating, it follows that $|\nabla\zeta'(x)|$, $|\Delta\zeta'(x)|$ are bounded uniformly in Λ by $0(\exp(-\eta L'))$ because derivatives of $\zeta(x-y)$ vanish for $|x-y| \leqq L'-1$. The proof of Lemma 6.7 is finished.

Lemma 6.8 *(Checkerboard Estimate). Let F_i, $i=1, \ldots, N$, belong to the σ algebra generated by fields $\psi(x)$ supported in Δ_i, $\Delta_i \in \bar{\Delta}$ distinct. There exists $p>0$ independent of N, s such that*

$$\left| \int d\psi_s \prod_{i=1}^{N} F_i \right| \leqq \prod_{i=1}^{N} \|F_i\|_p$$

uniformly in s, where $\|(\cdot)\|_p$ is the $L^p(d\psi_s)$ norm.

Proof. The covariance of $d\psi_s$, $v(x,y,s)$, is a convex combination of covariances of the form $v_{\partial X}(x,y) = v(x,y)$ if $x, y \in X$ or $x, y \in \sim X$, $=0$ otherwise. X is any union of cubes $\in \bar{\Delta}$. First it is claimed that the checkerboard estimate holds for the gaussian measure $d\psi_{\partial X}$ with covariance $sv_{\partial X}$ where $0 \leqq s \leqq 1$. This is because if F is any function of fields $\psi(x)$ supported in X

$$\int d\psi_{\partial X} F = \int d\psi_{sv} F \tag{6.27}$$

where $d\psi_{sv}$ has covariance sv. The same is true if F is supported in $\sim X$. It is not necessary to be concerned about ∂X because $\Pi_l \cap \partial X = \phi$. Therefore

$$\int d\psi_{\partial X} \prod_{i=1}^{N} F_i$$

factors across ∂X into two parts involving $d\psi_{sv}$ for which the checkerboard estimate is standard. See [13]. The checkerboard estimate for $d\psi_s$ can now be built up by writing, for a polynomial F

$$\int d\psi_s F(\psi) = \int d\psi_1 \times \ldots \times d\psi_r F(\psi_1 + \ldots + \psi_r) \tag{6.28}$$

where $d\psi_i$ are gaussian measures with the covariances of the form $s_{\partial X} v_{\partial X}$ occuring in the convex combination making up $v(x,y,s)$. The checkerboard estimate holds for each $d\psi_i$ with uniform p. The lemma is proved for the special case F_i = polynomial by iterating the checkerboard estimate for $d\psi_1, \ldots, d\psi_r$. p may be chosen uniformly in r. The general case follows. End of proof.

To begin with, consider the terms

$$\prod_{\substack{i>2 \\ |\pi(i)|>2}} \|\eta_{\alpha,\pi(i)} Q'_{\pi(i)}\| \tag{6.29}$$

in the left hand side of Theorem 6.3. Each $Q'_{\pi(i)}$ has the form

$$-\delta^r/\delta^r\psi(x)l_L^{-1}:\cos l_L^{1/2}\psi(x): \tag{6.30}$$

where $r=|\pi(i)|\geqq 3$. By performing the derivatives and unnormal ordering, (6.30) is bounded in absolute value by

$$l_L^{r/2-1}\exp\{l_L v_\Delta(x,x)\} \tag{6.31}$$

and (6.29) is less than

$$\prod_{\substack{1>2\\|\pi(i)|>2}} c^{|\pi(i)|}l_L^{1/4} \tag{6.32}$$

where c depends only on c_1.

Given $\Delta\in\bar{\Delta}$, let $S(\Delta)$ be the set of $\pi(i)$ such that $i>2$, $|\pi(i)|\leqq 2$, and the variables in $\pi(i)$ are localised by α in Δ. Lemma 6.8 and (6.29)–(6.32) show that Theorem 6.3 is implied by

$$\left(\int d\psi_s \prod_{\pi(i)\in S(\Delta)} \|\eta_{\alpha,\pi(i)}Q'_{\pi(i)}\|^{2p}\right)^{1/2p} c^{-L'^{-3}F_1(\Delta^\frown)/4} \leqq c^{n(\alpha,\Delta)}(n(\alpha,\Delta)!)^2 l_L^{1/4} \tag{6.33}$$

for cubes Δ with $S(\Delta)\neq\phi$. Without loss, p may be increased to an even integer.

Case 1. $\Delta\not\subset\Sigma^\frown$, i.e., $g=h$ and $\nabla g=0$. In this case either ($|\pi(i)|=1$)

$$\begin{aligned}Q'_{\pi(i)}(x_{\pi(i)}) &= l_L^{-1/2}:\sin l_L^{1/2}\psi(x_{\pi(i)}):-\psi(x_{\pi(i)})\\ &= \int_0^1 dt(1-t)\frac{d^2}{dt^2}l_L^{-1/2}:\sin l_L^{1/2}t\psi(x_{\pi(i)}):\end{aligned} \tag{6.34}$$

or ($|\pi(i)|=2$)

$$\begin{aligned}Q'_{\pi(i)}(x_{\pi(i)}) &= (:\cos l_L^{1/2}\psi(x'):-1)\delta(x',x'')\\ &= \int_0^1 dt\frac{d}{dt}:\cos l_L^{1/2}t\psi(x'):\delta(x',x'')\end{aligned} \tag{6.35}$$

where $x_{\pi(i)}=(x',x'')$ in (6.35). Substituting (6.34), (6.35) into (6.33) reduces the $d\psi_s$ integral to evaluating an integral of the form

$$\int d\psi_s \prod_\gamma :\sin l_L^{1/2}t_\gamma\psi(x_\gamma): \prod_{\gamma'} :\cos l_L^{1/2}t_{\gamma'}\psi(x_{\gamma'}):. \tag{6.36}$$

This integral can be evaluated explicitly as a sum over $e_j=\pm 1$ for $1\leqq j\leqq N$ of terms

$$\pm(1/2)^N \exp\left\{-l_L\sum_{\mu\neq\nu} e_\mu e_\nu t_\mu t_\nu v(x_\mu,x_\nu)/2\right\} \tag{6.37}$$

where $1\leqq\mu$, $\nu\leqq N$, $N=2p|S(\Delta)|$.

After the t derivatives, indicated by (6.34) and (6.35) have been performed, (6.33) is obtained by taking the supremum $\times N^0$ of terms. The N^0 of terms is dominated by the $c^{n(\alpha,\Delta)}(n(\alpha,\Delta)!)^2$ in (6.33). The supremum is estimated using

$$\sum_{\mu\neq\nu} e_\mu e_\nu t_\mu t_\nu v(x_\mu,x_\nu)\geqq -\sum_\mu t_\mu^2 v(x_\mu,x_\mu)=-0(l) \tag{6.38}$$

together with the method of taking norms on H_{-1} discussed at the beginning of this section. This leads to a product of quantities of the form

$$\left(\int_\Delta dx \int_\Delta dy\, v^m(x,y)\, l_L^{m-2}\right)^{1/2} = o([c_1^{m-5/2}\, l_L^{1/2}]^{1/2}) \tag{6.39}$$

$m \geqq 3$. This is the origin of the factor in l_L in (6.33). Case 1 concluded.

Case 2. $\Delta \subset \Sigma\hat{}$. In this case either ($|\pi(i)|=1$)

$$Q'_{\pi(i)} = \int_0^1 dt(1-t)\frac{d^2}{dt^2}\, l_L^{-1/2} :\sin l_L^{1/2} t \cdot (\psi+g-h)(x_{\pi(i)}): + \eta(g-h)(x_{\pi(i)}) + F'_2(x_{\pi(i)}) \tag{6.38}$$

or ($|\pi(i)|=2$)

$$Q'_{\pi(i)} = \int_0^1 dt \frac{d}{dt} :\cos l_L^{1/2}\, t(\psi+g-h)(x'):\delta(x', x''). \tag{6.39}$$

Proceed as in Case 1. The second and third terms in (6.38) are bounded in H_{-1} norm by

$$\eta\left(\int_\Delta |g-h|^2\right)^{1/2} + F_1^{1/2}(\Delta\hat{}) \leqq 2F_1^{1/2}(\Delta\hat{}) \tag{6.40}$$

using Lemma 6.7. Terms in $(g-h)$ which arise in the course of doing the t derivatives are likewise bounded by $F_1^{1/2}(\Delta\hat{})$. These terms are bounded using part of the factor $\exp(-L'^{-3}F_1(\Delta\hat{})/4)$ in (6.33), e.g., $\exp(-L'^{-3}F_1(\Delta\hat{})/8)$. This requires another factor $n(\alpha, \Delta)!$ on the right hand side of (6.33). Finally, the bound in Lemma 5.2 shows that

$$F_1(\Delta\hat{}h) \geqq c \sum_{\substack{f \subset \Sigma(h) \\ f^{\mathrm{INT}} \subset \Delta^{\wedge \mathrm{INT}}}} |\delta h(f)|^2 \geqq c 4\pi^2 l_L^{-1}$$

because, by assumption $\Delta \subset \Sigma\hat{}$ so that the sum is non empty. Therefore $\exp(-L'^{-3}F_1(\Delta\hat{})/8) \leqq c l_L^{1/4}$. This provides the factor $l_L^{1/4}$ on the right of (6.33) in Case 2. The treatment of (6.39) is similar. This concludes Case 2 and Theorem 6.3.

7. The Vacuum Energy Estimate (Theorem 6.5)

The first step is to use a conditioning inequality (see [13]) to reduce the theorem to an estimate for a single cube $\Omega \in \bar{\Omega}$. Let $d\psi_N$ denote the gaussian measure with covariance $(1-\Delta_N)^{-1}$, where Δ_N is the Laplacian with Neuman boundary conditions on all faces f of cubes in $\bar{\Omega}$ along with cubes of side L and disjoint interiors filling $\mathbb{R}^3 \sim \Lambda$. $v(x,y,s)$ is a convex combination of covariances $v_{\partial X}(x,y)$, (see Lemma 6.8), each of which are easily seen to be bounded in the sense of bilinear forms by the kernel $w_1(x,y)$ of $(1-\Delta_N)^{-1}$, because $(1-\Delta)^{-1} \leqq (1-\Delta_N)^{-1}$. Construct two independent fields ψ_s, $\psi_{\delta N}$ with covariances $v(x,y,s)$, $w_1(x,y) - v(x,y,s)$. Then

$$\int d\psi_N \tilde{\chi}^p_{\beta,h} e^{-pQ} = \int d\psi_s d\psi_{\delta N} \tilde{\chi}^p_{\beta,h} e^{-pQ}. \tag{7.1}$$

On the right hand side, $\chi^{\sim}$, $\exp(-pQ)$ become functions of ψ_s, $\psi_{\delta N}$ by writing $\psi=\psi_s+\psi_{\delta N}$. Recall from (6.11) that $\chi^{\sim}_{\beta,h}$ is the supremum of exponentials. By the inequality $\int\sup\geqq\sup\int$, the $d\psi_{\delta N}$ integral is moved inside the supremum and then by convexity into the exponent, so that (7.1) is bounded below by

$$\int d\psi_s \sup_{t\in\tau(\beta,h)} \exp\left\{\int d\psi_{\delta N}\left(-\sum_{\Omega\in\bar{\Omega}} p[\bar{\psi}_s(\Omega)\right.\right.$$

$$\left.\left.+\bar{\psi}_{\delta N}(\Omega)+\bar{g}(\Omega)-t(\Omega)]^2/2-pQ(\psi_s+\psi_{\delta N})\right)\right\}. \tag{7.2}$$

By evaluating the $d\psi_{\delta N}$ integral, this shows that

$$\left(\int d\psi_s \chi^{\sim p}_{\beta,h} e^{-pQ}\right)^{p-1} \leqq \left(\int d\psi_N \chi^{\sim p}_{\beta,h} e^{-pQ}\right)^{p-1} e^{0(L^{-1})|X|} \tag{7.3}$$

where the normal ordering on the right hand side in Q is with respect to the $d\psi_N$ measure. The constant $0(L^{-1})$ comes from evaluating

$$\int d\psi_{\delta N}\bar{\psi}^2_{\delta N}(\Omega)=L^{-6}\int_\Omega dx\int_\Omega dy[w_1(x,y)-v(x,y,s)] \tag{7.4}$$

for each $\Omega\subset X=\bigcup_{i=1}^{n} Y_i$ where $\bar{y}=(Y_1,\dots,Y_n)$.

Since $d\psi_N$ factors into separate measures for each cube $\Omega\in\bar{\Omega}$, this completes the first step.

Step 2. Reduction to the case $\psi=\phi-h$, i.e., $\Omega\not\subset\Sigma^{\sim}$. Define

$$W(X)=-\int_{X\cap\Lambda}\{l_L^{-1}(:\cos l_L^{1/2}\phi:-1)+\eta/2:(\phi-h)^2:\} \tag{7.5}$$

$$d\psi_{N,M}=d\psi_N \exp\left\{(1-M)/2\int_{X\cap\Lambda}:\psi^2:\right\} \tag{7.6}$$

$d\psi_{N,M}$ is not normalised.

By the Holder inequality,

$$\begin{aligned}\left(\int d\psi_N(\chi^{\sim}_{\beta,h}e^{-Q})^p\right)^{p^{-1}} &\leqq \left(\int d\psi_{N,0}(\chi^{\sim}_{\beta,h}e^{-W})^{pp_1}\right)^{p^{-1}p_1^{-1}}\\ &\cdot\left(\int d\psi_{N,M}e^{-pp_2F_2}\right)^{p^{-1}p_2^{-1}}e^{-F_1}\end{aligned} \tag{7.7}$$

$M=p_2(1-p(1-\eta))$, $p_1^{-1}+p_2^{-1}=1$, $p_1>1$, $p_2>1$. Choose $p>1$ small so that $M>0$. Given $\Omega\subset X$, $\Omega\in\bar{\Omega}$, $l^2(\Pi_l\cap\Omega)$ is invariant for Δ_N. Let P be the projection onto the subspace complementary to the zero eigenvector in $l^2(\Pi_l\cap\Omega)$. Let $w_0(x,y)$ be the kernel of $(-\Delta_N)^{-1}$ restricted to $Pl^2(\Pi_l\cap\Omega)$ and let $d(\delta\psi(\Omega))$ be the gaussian measure with covariance w_0 indexed by $Pl^2(\Pi_l\cap\Omega)$. Then

$$d\psi_{N,0}=N(\bar{X})\prod_{\substack{\Omega\in\bar{\Omega}\\ \Omega\subset X}}\int d\bar{\psi}(\Omega)\,d(\delta\psi(\Omega)) \tag{7.8}$$

where $d\psi(\Omega)$ is lebesque measure (corresponding to the zero eigenvector)

$$\begin{aligned}N(X)=&\prod_{\substack{\Omega\in\bar{\Omega}\\ \Omega\subset X}}\left(\int d\bar{\psi}(\Omega)e^{-1/2:\bar{\psi}^2(\Omega):}\right)^{-1}\\ &\cdot\left(\int d(\delta\psi(\Omega))e^{-1/2\int_\Omega:\delta\psi^2:}\right)^{-1}.\end{aligned} \tag{7.9}$$

The normal ordering is with respect to $(-\Delta_N+1)^{-1}$.
(7.8) is abbreviated as

$$d\psi_{N,0}=N(X)d\bar{\psi}d(\delta\psi). \tag{7.10}$$

Translate this measure by setting $\psi=\phi-g$

$$=N(X)d\bar{\phi}d(\delta\phi)\exp\left\{-\int_X(\nabla_N g)^2/2-\int_X(\Delta_N g)\phi\right\}. \tag{7.11}$$

By (7.11) and the Cauchy Schwarz inequality

$$\begin{aligned}&\left(\int d\psi_{N,0}(\chi_{\beta,h}^{\sim}e^{-W})^{pp_1}\right)^{p^{-1}p_1^{-1}}\\ &\leqq\left\{N(X)\int d\bar{\phi}\chi_{\beta,h}^{\sim pp_1}\left(\int d(\delta\phi)e^{-2pp_1W}\right)^{1/2}\right.\\ &\left.\cdot\left(\int d(\delta\phi)\exp\left\{-\int_X(\nabla_N g)^2-2\int_X(\Delta_N g)\delta\phi\right\}\right)^{1/2}\right\}^{p^{-1}p_1^{-1}}.\end{aligned} \tag{7.12}$$

The second $d(\delta\phi)$ can be performed explicitly. Thus

$$\int d(\delta\phi)\exp\left\{-2\int_X(\Delta_N g)\delta\phi\right\}=\exp\left\{2\int_X(\nabla_N g)^2\right\}\leqq\exp\{4F_1(X)\}. \tag{7.13}$$

Choose $p_1\gg 1$ so that $2p^{-1}p_1^{-1}\leqq 1/8$.

The $d\psi_{N,M}$ integral in (7.7) can also be integrated and estimated:

$$\begin{aligned}\int d\psi_{n,M}e^{-pp_2F_2}&=\left(\int d\psi_{N,M}1\right)\exp\left\{p^2p_2^2/2\int_X F_2'((M-\Delta_N)^{-1}F_2')\right\}\\ &\leqq\left(\int d\psi_{N,M}1\right)\exp\left\{1/2p^2p_2^2M^{-1}\int_X F_2'^2(x)dx\right\}.\end{aligned} \tag{7.14}$$

Choose $L'\geqq L_0'$ where L_0' is so large that by Lemma 6.7

$$1/2pp_2M^{-1}\int_X F_2'^2\leqq 1/8F_1(X^\frown)=1/8F_1(X). \tag{7.15}$$

Note that Δg, F_2' vanish outside X because $g=h=$constant in (connected components of $\sim X)^\frown$. Combine (7.3), (7.7), (7.8), (7.12)–(7.15)

$$\begin{aligned}&\left(\int d\psi_s(\chi_{\beta,h}^{\sim}e^{-Q})^p\right)^{p^{-1}}\leqq\exp(0(L^{-1})|X|)\\ &\cdot N(X)^{p^{-1}p_1^{-1}}\left(\int d\psi_{N,M}1\right)^{p^{-1}p_2^{-1}}e^{-3F_1/4}\\ &\cdot\left\{\int d\bar{\phi}\chi_{\beta,h}^{\sim pp_1}\left(\int d(\delta\phi)e^{-2pp_1W}\right)^{1/2}\right\}^{p^{-1}p_1^{-1}}\end{aligned} \tag{7.16}$$

p, p_1 depend only on η. Thix completes step 2.

The third step is to perform the $\delta\phi$ integral, which can be done by reversing the sine gordon transformation and estimating in such a way as to extract the $\bar{\phi}$ behavior. This is all summarised by Lemma 7.1 which is the technical foundation for the proof of Theorem 6.5. The final step, the $d\bar{\phi}$ integral, is the subject of Lemma 7.2.

Lemma 7.1. *Given $\eta>0$, $\mu>0$, $\alpha>0$, there exists $L>0$ such that for all l_L, l, $l_L\leqq c_1 l$, $\Omega\in\bar{\Omega}$,*

$$\int d(\delta\phi(\Omega))\exp\left\{\alpha l_L^{-1}\int_\Omega dx(:\cos l_L^{1/2}[\bar{\phi}(\Omega)+\delta\phi(x)]:-1)\right.$$
$$\left.+1/2\eta\alpha\int_\Omega :\delta\phi^2(x):\right\}\leqq c\exp\{\alpha l_L^{-1}L^3(\cos l_L^{1/2}\bar{\phi}(\Omega)-1)$$
$$+1/2\alpha\mu L^3\bar{\phi}(\Omega)^2\}.$$

Lemma 7.2. *There exists $\eta'>0$ so that for all α' there exists c so that*

$$\left(\int d\bar{\phi}\chi_{\beta,h}^{\sim\alpha'}\prod_{\substack{\Omega\in\bar{\Omega}\\ \Omega\subset X}}\exp\{\alpha' l_L^{-1}L^3(\cos l_L^{1/2}\bar{\phi}(\Omega)-1)\right.$$
$$\left.+1/2\alpha'\eta' L^3\bar{\phi}(\Omega)^2\}\right)^{\alpha'^{-1}}\leqq\exp(c|X|)l_L^{m'/4}$$

c is independent of l_L. η' is independent of L. m' was defined beneath Theorem 6.5.

Choose $\eta'>0$ so that Lemma 7.2 holds. Choose $\eta:0<\eta<\eta'$. Set $\mu=\eta'-\eta$, $\alpha=2pp_1$, $\alpha'=pp_1$. Choose L so that Lemma 7.1 holds.

Proof of Theorem 6.5. (assuming Lemmas 7.1, 7.2) completed. Combine (7.16), Lemmas 7.1, 7.2.

$$(\int d\psi_s(\chi_{\beta,h}^{\sim}e^{-Q})^p)^{p^{-1}}\leqq\exp(c|X|)$$
$$\cdot N(X)^{p^{-1}p_1^{-1}}(\int d\psi_{N,M}1)^{p^{-1}p_2^{-1}}e^{-3F_1/4}l_L^{m'}. \tag{7.17}$$

$N(X)$ can be bounded by Jensen's inequality

$$\int d(\delta\psi(\Omega))\exp\left(-1/2\int_\Omega :\delta\psi^2:\right)$$
$$\geqq\exp\left\{-1/2\int_\Omega dx\int d(\delta\psi):\delta\psi^2(x):\right\}$$
$$=\exp\left\{-1/2\int_\Omega dx[w_0(x,x)-w_1(x,x)]+1/2\int d\psi_N\bar{\psi}^2(\Omega)\right\}=c \tag{7.18}$$

where c is independent of l,Ω. The $d\bar{\psi}$ integral is easily evaluated by unnormal ordering

$$:\bar{\psi}(\Omega)^2:=\bar{\psi}(\Omega)^2-L^{-6}\int_\Omega dx\int_\Omega dy w_1(x,y)$$
$$=\bar{\psi}(\Omega)^2-0(L^{-1}). \tag{7.19}$$

Thus

$$N(X)\leqq e^{c|X|}. \tag{7.20}$$

Also

$$\int d\psi_{N,M}1=\int d\psi_N\exp\left\{(1-M)/2\int_X :\psi^2:\right\}$$
$$=\exp\{-1/2\mathrm{Tr}\ln(1+A)+1/2\mathrm{Tr}A\}$$
$$\leqq\exp(1/4\mathrm{Tr}A^2)\leqq\exp(c|X|) \tag{7.21}$$

where $A=(-\Delta_N+1)^{-1}(1-M)$. The inequality can be obtained by using $\ln(1+x) \geqq x-\frac{x^2}{2}$, valid for $0\leqq x\leqq 1$.

By virtue of (7.20) and (7.21) the right hand side of (7.17) is less than

$$\exp\{c|X|-3F_1(X)/4\}l_L^{m'}$$
$$\leqq \exp\{cn-1/2F_1(X)\}l_L^{m'}. \tag{7.22}$$

The final inequality is obtained from Lemma 5.2 and (5.14) and holds provided l_L is sufficiently small. This completes the proof of Theorem 6.5 assuming Lemmas 7.1 and 7.2.

Proof of Lemma 7.2. The left hand side factors into an integral for each cube $\Omega\subset X$. Each factor has the form

$$\int d\xi \sup_t \exp\{\alpha' l_L^{-1}L^3(\cos l_L^{1/2}\xi-1)$$
$$+1/2\alpha'\eta' L^3\xi^2-\alpha'/2(\xi-t)^2\}. \tag{7.23}$$

Let $I=[-\pi l_L^{-1/2}, \pi l_L^{-1/2}]$. The supremum in (7.23) is taken over I or ∂I depending on β, h. It is enough to prove that there exists $\eta'>0$ so that (a) if the sup is over I, (7.23) is bounded by a constant uniformly in l_L (b) if the sup is over ∂I, (7.23) is bounded by $0(l_L^{1/4\alpha'})$.

Case a). This is implied by: – there exists $\eta''>0$ so that for all $\zeta\in\mathbb{R}$, $t\in[-\pi,\pi]$,

$$L^3(\cos\zeta-1)+\eta''/2L^3\zeta^2-1/2(\zeta-t)^2\leqq 0. \tag{7.24}$$

This is claimed to be obvious. Moreover η'' may be chosen independently of L^3 since $L^3\leqq 1$.

Case b). Left to the reader.

Proof of Lemma 7.1. Let $w(x,y)$ be the kernel of $(-\Delta_N-\eta\alpha)^{-1}$ restricted to the complement of the zero eigenspace in $l^2(\Pi_l\cap\Omega)$. If L is sufficiently small, this is the kernel of a bounded operator. Let $d_w(\delta\phi(\Omega))$ be the associated gaussian measure. It is sufficient to prove the same bound with a different constant c for

$$\int d_w(\delta\phi(\Omega))\exp\left\{\alpha l_L^{-1}\int_\Omega dx(:\cos l_L^{1/2}[\bar\phi(\Omega)+\delta\phi(\Omega)]:-1)\right\} \tag{7.25}$$

where the normal ordering is with respect to $d_w(\delta\phi(\Omega))$. For since the normal ordering is multiplicative and cosine is $\leqq 1$, the change in normal ordering changes the exponent by an additive factor of at most

$$\alpha l_L^{-1}|\exp\{l_L w_1(x,x)/2\}-\exp\{l_L w(x,x)/2\}|$$

which is bounded as $l_L\to 0$ uniformly in $x\in\Omega$ provided $\sup_x|l_L w(x,x)|\leqq c$ and $\sup_x|w_1(x,x)-w(x,x)|\leqq c'$ uniformly in l_L, l for $l_L\leqq c_1 l$. Both these assertions are proven in Lemma 7.6.

By reversing the sine gordon transformation, (7.25) may be rewritten as

$$\sum_{N=0}^{\infty} z^N/N! \sum_{(e)_N} \left(\int_{\Omega^N} d^N x \exp\{-l_L W\} \right) \exp\left(i l_L^{1/2} \bar{\phi} \sum_{j=1}^{N} e_j \right) \tag{7.26a}$$

$$W((x,e)_N) = \sum_{1 \leqq j < j' \leqq N} e_j e_{j'} w(x_j, x_{j'}) \tag{7.26b}$$

$$z = 1/2\alpha l_L^{-1}. \tag{7.26c}$$

$\bar{\phi}(\Omega)$ has been abbreviated to $\bar{\phi}$. The expression (7.26a) will be denoted Z_w.

The next lemma requires some preparation. It will be recognised as a form of the high temperature expansion, Ursell's method, used by physicists.

$$W((x,e)_N, s) = \sum_{1 \leqq j < k \leqq N} e_j e_k s_j \dots s_{k-1} w(x_j, x_k) \tag{7.27}$$

where $s = (s_1, \dots, s_{N-1}) \in [0,1]^{N-1}$.

$$W_k'((x,e)_N, s) = \sum_{1 \leqq j < k} e_j e_k \frac{d}{ds_{k-1}} (s_j \dots s_{k-1}) w(x_j, x_k). \tag{7.28}$$

The principle underlying the definition (7.27) is the same as in (3.14) and the definition of $d\psi_s$; namely, $W(s) \equiv W((x,e)_N, s)$ is a convex combination of interactions of the form $W(x_1, e_1, x_2, e_2, \dots, x_l, e_l) + W(x_{l+1}, e_{l+1}, \dots, x_N, e_N)$, $l = 1, \dots, N$, which have no interaction between pairs of particles when one belongs to the cluster labelled by coordinates $(x_j, e_j)_{j \leqq l}$ and the other belongs to the cluster $(x_j, e_j)_{j > l}$. Since $w(x,y)$ is the kernel of a positive operator

$$W((x,e)_{\leqq l}) \geqq -1/2 \sum_{j=1}^{l} w(x_j, x_j). \tag{7.29}$$

This, together with a similar estimate for $W((x,e)_{>l})$ implies for $N > 2$

$$W((x,e)_N, s) \geqq -1/2 \sum_{j=1}^{N} w(x_j, x_j). \tag{7.30}$$

Lemma 7.3. *Provided the sum over N is absolutely convergent,*

$$Z_w = \exp\left\{ \sum_{N=1}^{\infty} z^N / N l_L^{N-1} \sum_{(e)_N} e^{i l_L^{1/2} \bar{\phi} \sum_{j=1}^{N} e_j} I_N((e)_N) \right\} e^{-2z \operatorname{vol}(\Omega)}$$

where $\operatorname{vol}(\Omega) = L^3$ *(= volume of* Ω*) and*

$$I((e)_N) = \int_{[0,1]^{N-1}} ds \int_{\Omega^N} d^N x \left(\prod_{j=2}^{N} - W_j'(s) \right) e^{-l_L W(s)}$$

$$I_1(e_1) \equiv \int_{\Omega} dx = L^3.$$

The quantity in the exponent is the virial expansion. The next lemma specifies a region of convergence.

Lemma 7.4. *For all $l_L>0$, $l>0$ such that $l_L \leqq c_1 l$,*

$$\left| \sum_{(e)_N} e^{il_L^{1/2}\bar{\phi}\sum_{j=1}^N e_j} I_N((e)_N) \right| \leqq c^N L^{2(N-1)} L^3 \sin^2 l_L^{1/2}\bar{\phi} + c'^N L^{2N} l_L$$

for $N \geqq 2$.

Lemma 7.1 is a corollary of these two lemmas. By Lemmas 7.3 and 7.4,

$$Z_w \leqq \exp\{2z(\cos l_L^{1/2}\bar{\phi}-1)L^3\} \cdot \exp\left\{L^3 \sum_{N=2}^{\infty} z^N/N l_L^{N-1}(c^N L^{2(N-1)}\sin^2 l_L^{1/2}\bar{\phi} + c'^N L^{2N-3} l_L)\right\}. \tag{7.31}$$

Since $(\sin l_L^{1/2}\bar{\phi})^2 \leqq l_L \bar{\phi}^2$, the sum over N can be made to converge uniformly in l_L and the coefficient of $L^3\bar{\phi}^2$ made $\leqq 1/2\alpha\mu$ by taking L small. This concludes the proof of Lemmas 7.1, assuming Lemmas 7.3 and 7.4.

Proof of Lemma 7.4. Write I_N as the sum of two parts

$$I_{N,0}((e_N)) = \int_{[0,1]^{N-1}} ds \int_{\Omega^N} d^N x \left(\prod_{j=2}^N -W_j'(s)\right) \tag{7.32}$$

$$I_{N,1}((e_N)) = -l_L \int_0^1 dt \int_{[0,1]^{N-1}} ds \int_{\Omega^N} d^N x \cdot \left(\prod_{j=2}^N -W_j'(s)\right) W(s) e^{-tl_L W(s)}. \tag{7.33}$$

As in Section 5,

$$\prod_{j=2}^N W_j(s) = \sum_T q(T,s) \prod_{(i,j)\in T} w(x_i,x_j) e_i e_j \tag{7.34}$$

where T is summed over tree graphs on N vertices and

$$q(T,s) = \prod_{(i,j)\in T} \frac{d}{ds_{j-1}}(s_{j-1} \dots s_i). \tag{7.35}$$

By substituting (7.34) into (7.32)

$$\left| \sum_{(e)_N} \exp\left\{il_L^{1/2}\bar{\phi}\sum_{j=1}^N e_j\right\} I_{N,0}((e)_N) \right| \leqq \sum_T \int_{[0,1]^{N-1}} ds\, q(T,s) 2^N \int_{\Omega^N} d^N x \left(\prod_{(i,j)\in T} |w(x_i,x_j)|\right) \sin^2 l_L^{1/2}\bar{\phi}. \tag{7.36}$$

Lemma 7.5.

$$\sup_{x\in\Omega} \int_\Omega dy |w(x,y)| \leqq cL^2$$

$$\sup_{x\in\Omega} \int_\Omega dy |w(x,y)|^2 \leqq c'L$$

c, c' are uniform in l.

By applying the first inequality to (7.36), estimating the integrals in the order $\int dx_N, \int dx_{N-1}, \ldots, \int dx_1$, and then using

$$\int_{[0,1]^{N-1}} ds\, q(T,s) \leqq e^N \tag{7.37}$$

which is a special case of (5.5), (7.36) is less than

$$c^N L^3 L^{2(n-1)} \sin^2 l_L^{1/2} \bar{\phi} \tag{7.38}$$

which accounts for the first term in the bound in Lemma 7.4.

The estimation of (7.33): the t integral is estimated by taking the supremum over t. By (7.30)

$$\exp(-t l_L W(s)) \leqq \exp\left(\sum_{j=1}^{N} l_L w(x_j, x_j)/2 \right) \leqq \exp(cN) \tag{7.39}$$

where c is independent of l_L, l, L in the region $l_L \leqq c_1 l$. The second inequality comes from Lemma 7.6, Part (a).

Lemma 7.6.

(a) $l_L w(x,x) \leqq c$

(b) $|w_1(x,x) - w(x,x)| \leqq c'$

uniformly in $x \in \Omega$, l_L, l, in the region $l_L \leqq c_1 l$. c is also independent of L. (c' is not).

By the Cauchy Schwarz inequality

$$\int_{\Omega^N} d^N x \left| \prod_{j=2}^{N} W_j(s) \right| |W(s)| \leqq \left(\int_{\Omega^N} d^N x \left| \prod_{j=2}^{N} W_j(s) \right|^2 \right)^{1/2} \left(\int_{\Omega^N} d^N x |W(s)|^2 \right)^{1/2} \tag{7.40}$$

(7.39), (7.40), Lemma 7.5 may be assembled to prove

$$|I_{N,1}((e)_N)| \leqq c^N L^{2N} l_L . \tag{7.41}$$

Proof of Lemma 7.4 completed.

Proof of Lemma 7.5. The first inequality is implied by the second by the Cauchy Schwarz inequality. The eigenfunctions of Δ_N restricted to $l^2(\Pi_l \cap \Omega)$ are

$$f_k(x) = (L/2)^{-3/2} \begin{matrix} \{\cos k_1 x_1\} \\ \{\sin k_1 x_1\} \end{matrix} \quad \begin{matrix} \{\cos k_2 x_2\} \\ \{\sin k_2 x_2\} \end{matrix} \quad \begin{matrix} \{\cos k_3 x_3\} \\ \{\sin k_3 x_3\} \end{matrix} \tag{7.42}$$

where $x = (x_1, x_2, x_3)$, $k \equiv (k_1, k_2, k_3) \in T_l \cap \Pi_{2\pi/L} \cdot T_l = \left[-\frac{\pi}{l}, \frac{\pi}{l} \right]^3$. The reader is referred to [14] for details. If k_i is even $\cos k_i x_i$ is selected, otherwise $\sin k_i x_i$. Set

$$\hat{w}(k) = \left(4 l^{-2} \sum_{i=1}^{3} \sin^2 k_i l/2 - \eta\alpha \right)^{-1} . \tag{7.43}$$

Then

$$\begin{aligned} \int_{\Omega} dy |w(x,y)|^2 &= \int_{\Omega} dy \left| \sum_{k \neq 0} \hat{w}(k) f_k(x) f_k(y) \right|^2 \\ &= \sum_{k \neq 0} \hat{w}^2(k) f_k^2(x) \\ &\leqq \sum_{k \neq 0} \hat{w}^2(k) (L/2)^{-3} \leqq \sum_{n \neq 0,\, n \in 2\pi \mathbb{Z}^3} \hat{w}^2(L^{-1} n)(L/2)^{-3} \leqq cL . \end{aligned} \tag{7.44}$$

Proof of Lemma 7.5 concluded.

Proof of Lemma 7.6. By the eigenfunction expansion

$$l_L w(x,x) = l_L \sum_{k \neq 0} \hat{w}(k) f_k^2(x) \leq l_L (L/2)^{-3} \sum_{n \neq 0} \hat{w}(L^{-1} n) \tag{7.45}$$

where n is summed over $[-l^{-1}L\pi, l^{-1}L\pi] \cap 2\pi\mathbb{Z}$. It is an easy calculation, left to the reader to see that the right hand side is bounded uniformly in L, l, l_L for $l_L \leq c_1 l$. This completes Part (a). Part (b) may also be proved using the eigenfunction expansion and is likewise left to the reader.

Proof of Lemma 7.3. Let

$$Z_n = \sum_{(e)_N} \exp\left\{ i l_L^{1/2} \bar{\phi} \sum_{j=1}^{N} e_j \right\} \int_{\Omega^N} d^N x e^{-l_L W} \tag{7.46}$$

$$J_N = l_L^{N-1} \sum_{(e)_N} \exp\left\{ i l_L^{1/2} \bar{\phi} \sum_{j=1}^{N} e_j \right\} I((e)_N). \tag{7.47}$$

First, it is claimed that it is sufficient to prove that

$$Z_N = \sum_P \prod_{\gamma \in P} J_{|\gamma|} (|\gamma| - 1)! \tag{7.48}$$

where P is summed over all partitions of $\{1, 2, \ldots, N\}$ into subsets γ. $|\gamma| = N^0$ of elements in γ.

Verification of Claim. The number of partitions into subsets $\gamma_1, \ldots, \gamma_r$ with cardinalities $n_1, \ldots, n_r$ is $N!/(n_1! n_2! \ldots n_r!)$ so that (7.48) can be rewritten

$$Z_N = N! \sum_r (r!)^{-1} \sum_{n_1, \ldots, n_r : \Sigma n_i = N} \prod_{i=1}^{r} J_{n_i}/n_i. \tag{7.49}$$

The $(r!)^{-1}$ compensates for the elements of a partition being unordered whereas $\gamma_1, \ldots, \gamma_n$ are ordered.

$$\sum_{N=0}^{\infty} z^N/N! Z_N = \sum_{r=0}^{\infty} (r!)^{-1} \left(\sum_{n=1}^{\infty} z^n/n J_n \right)^r$$

$$= \exp\left(\sum_{n=1}^{\infty} z^n/n J_n \right). \tag{7.50}$$

Comparison of (7.50) with Lemma 7.3 shows that the claim has been vindicated.

Equation (7.48) is implied by iterating

$$Z_{|X|} = \sum_{\substack{S \subset X \\ S \ni x}} (|S| - 1)! J_{|S|} Z_{|X \sim S|} \tag{7.51}$$

where $X \subset \{1, 2, \ldots, N\}$. x is the first element in X. (7.51) is equivalent to

$$\int_{\Omega^{|X|}} d^{|X|} x \exp(-l_L W) = \sum_{\substack{S \subset X \\ S \ni x}} (|S| - 1)! I_{|S|} \int_{\Omega^{|X \sim S|}} d^{|X \sim S|} x \exp(-l_L W). \tag{7.52}$$

This is the same as the cluster expansion of Section 3.3 except that the interaction between particles is interpolated rather than regions of $\mathbb{R}^3$. Instead of repeating Section 3.3 with the appropriate substitutions, the following informal discussion is given: consider the case $X=\{1,2,\ldots,n\}$. Let $W(s_1)=s_1 W(x_1,e_1,\ldots,x_N,e_N)$ $+(1-s_1)W(x_2,e_2,\ldots,x_N,e_N)$. Then

$$\int d^N x e^{-l_L W} = \int d^N x\, e^{-l_L W(0)} + \int_0^1 ds_1 \frac{d}{ds_1} \int d^N x e^{-l_L W(s_1)}. \tag{7.53}$$

The first term corresponds to $S=\{1\}$ in (7.52). Perform the derivative. The result is

$$l_L \sum_{j>1} \int_0^1 ds_1 \int d^N x(-e_1 e_j w(x_1,x_j)) e^{-l_L W(s_1)}. \tag{7.54}$$

For each $j>1$, set $W(s_1,s_2)$ equal to

$$s_2 W(s_1,x_1,e_1,\ldots,x_N,e_N)+(1-s_2)\{W(s_1,x_1,e_1,x_j,e_j)$$
$$+W(x_2,e_2,x_3,e_3,\ldots,\hat{x}_j,\hat{e}_j,\ldots,x_N,e_N)\}.$$

The $\hat{}$ means omit the indicated variable. Express the j^{th} term in (7.54) as the sum of a term labelled by $S=\{1,j\}$ in (7.53) and an error by using the fundamental theorem of calculus on $\exp(-l_L W(s_1,s_2))$ as in (7.53). Continue until the error vanishes. End of proof of Lemma 7.3.

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Note Added in Proof. The author should have included Reference [19] in the preprint version of this paper. At the time he was unaware of this work by Lenard and Edwards in which the sine gordon transformation is applied to a one dimensional coulomb system to obtain exact results including screening.

Equation (6.5), page 333 and Equation (6.22) on page 337 are incorrect because they rely on Leibniz rule, which for a finite difference gradient reads $\nabla(fg) = (\nabla f)g + f(\nabla g) + l(\nabla f)(\nabla g)$. The final term has been overlooked. However it may easily be checked that including it does no harm.