A robust logic for rule-based reasoning under uncertainty

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Abstract

Reasoning with uncertain information is a problem of key importance when dealing with real life knowledge. The more information required by the procedure used to handle the knowledge, the higher the probability of failure of the reasoning system. The theory of rough sets [Pawlak 1982] is not information intensive and is thus a good basis for reasoning in domains where knowledge is sparse. We present an introduction to a logic based on rough set theory that is suitable for reasoning under uncertainty. We introduce inference rules analogous to those of classical logic, and demonstrate their effectiveness in rule based reasoning.

1. Introduction

Any system designed to reason about the real world must, perforce, be capable of dealing with uncertain information, that is information whose certainty may not be completely established, and incomplete knowledge about its domain. This is a direct consequence of the complexity of the real world and the finite size of the knowledge base that such a system has at its disposal. A number of mathematical formalisms have been developed to cope with uncertainty in knowledge base systems [Saffiotti 1987], and most have been demonstrated on a number of reasoning tasks. These formalisms suffer from a number of disadvantages. The severest of these is that they are all very *information intensive*; they all require large amounts of precise information in order to deal with uncertainty. This means that the truth value of the relations between variables are required in the form of grades of membership and probability distributions. These values are often unknown, or expensive to obtain, and methods that are not information intensive are often desirable. Rough set theory seems to solve the problem of information intensity, enabling us to avoid the paradox of performing precise calculations with imprecise data.

2. Rough set theory

Rough sets, originally introduced by Pawlak [1982], have been further developed and applied to a number of problems by various authors, [Orlowska and Pawlak 1984], [Fariñas del Cerro and Orlowska 1985], and [Wong *et al* 1986]. Here we discuss the basic ideas behind the theory.

2.1 Basic concepts of rough sets

Consider a set of elementary objects $\mathbf{A} = \{A_1,...,A_n\}$. The members of this set are used to define a set of objects $\mathbf{E} = \{E_1,...,E_m\}$ each of the A_i corresponding to a possible attribute of the E_j. Now, since the specification of the E_j must be based on the A_i alone, it could well be the case that some of the specifications of the E_j are indistinguishable since the values that distinguish them are not identified by the A_i. For instance a blue egg and a green egg will be indistinguishable if the only concepts that can be used to describe them are 'red', 'ball' and 'egg'. Thus the use of a finite set **A** implies the existence of an equivalence relation and a consequent partition on **E**:

$$\mathbf{P} = \{P_1,...,P_r\} \quad \text{where} \quad \bigcup P_i = \mathbf{E} \quad \text{and} \quad P_i\text{fi}P_j = \emptyset \quad \text{for } i \neq j = 1,...,r \qquad [1]$$

and each $\mathbf{P}_s = (E_{s_1},...,E_{z_s})$ is an equivalence class. Let $T‹\mathbf{S}$ be an object, whose attributes are $T_A⁄\mathbf{A}$, that we wish to describe in terms of the set of partitioned attributes E_i. Let:

$$T^c(\mathbf{P}, \mathbf{E}) = \{e{:}e \in P_i,\ P_i‹T \neq \emptyset\} \qquad [2]$$

$$T^e(\mathbf{P}, \mathbf{E}) = \{e{:}\ e \in P_i,\ P_i\text{fi}T \neq \emptyset\} \qquad [3]$$

where $T^c(\mathbf{P}, \mathbf{E})$ is the core[1] of T based on **E** and **P**, the set of all partitioned attributes that T possesses, and $T^e(\mathbf{P}, \mathbf{E})$ is the envelope of T based on **E** and **P**, the set of all partitioned attributes of which at least one is possessed by T. The pair $[T^c(\mathbf{P}, \mathbf{E}), T^e(\mathbf{P}, \mathbf{E})]$ is a *rough set.*

Let the set of all rough sets that may be defined using **E** partitioned as **P** be denoted by **R**. Consider $R = [R^c, R^e]$, $R' = [R^{c'}, B^e] \in \mathbf{R}$. It is simple to show that the following set theoretic relations hold where the symbol '~' stands for complement[2]:

$(R\text{fl}R')^c$	¤	$R^c\text{fl}R'^c$	$(R\text{fl}R')^e$	=	$R^e\text{fl}R'^e$
$(R\text{fi}R')^c$	=	$R^c\text{fi}R'^c$	$(R\text{fi}R')^e$	⁄	$R^e\text{fi}R'^e$
$(\sim R)^c$	=	$\sim(R^e)$	$(\sim R)^e$	=	$\sim(R^c)$

[4]

2.2 Combining rough sets

The degree to which a concept A may be defined within (**E**, **A**) depends on the cardinality of A^c and A^e. We have the following [Pawlak 1982]:

If	$A^c = A^e$	then A is precisely defined by (E, **A**)
If	$A^c \neq A^e$ and $A^e \neq \emptyset$	then A is roughly defined by (**E**, **A**)
If	$A^c = \emptyset$	then A is internally undefined by (**E**, **A**) [5]
If	$A^e = K$	then A is externally undefined by (**E**, **A**)
If	$A^c = \emptyset$ and $A^e = K$	then A is totally undefined by (**E**, **A**)

We can determine the degree to which logical combinations of roughly defined objects may themselves be defined. The basic logical operations of disjunction, conjunction and negation may be defined in terms of set operations on the core and envelope of the objects concerned, and relations such as material implication defined from them:

$(A£B)^c$	¤	$A^c\text{fl}B^c$	$(A\$B)^c$	=	$A^c\text{fi}B^c$
$(A£B)^e$	=	$A^e\text{fl}B^e$	$(A\$B)^e$	⁄	$A^e\text{fi}B^e$

[6]

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[1] In Pawlak's original work on rough sets the envelope and core were named 'upper approximation' and 'lower approximation' respectively.

[2] Note that $\sim R^c$ is equivalent to $\sim(R^c)$, the complement of the core of R and should be distinguished from $(\sim R)^c$, the core of the complement of R

$(\neg A)^c$	=	$\sim(A^e)$	$(A\rightarrow B)^c$	¤	$\sim A^e$flB^c
$(\neg A)^e$	=	$\sim(A^c)$	$(A\rightarrow B)^e$	=	$\sim A^c$flB^e

3. A logic of rough truth values

As stated above, it is often advantageous to deal with formalisms that are not information intensive, especially in domains in which only sparse information is typically available. In such domains we want robust reasoning mechanisms that are capable of absorbing large amounts of ill known information whilst still returning accurate answers where possible. In this section we present a simple quantified logic that meets these requirements.

Let us consider a Boolean algebra of propositions (P, ∨, ∧, ¬). We define a rough measure R on P such that Åp ∈ P, R(p) = [p^c, p^e]. R(p) is thus an estimate of the degree to which p is defined by the set of descriptors **A**, and this may be related to the truth of the proposition p by careful choice of **A** [Parsons *et al* 1991]. We can distinguish the following limiting values of R(p), for Ø‹X‹**A**, and Ø‹Y‹**A**, which correspond to the definitions of [5]:

If	R(p)	=	[**A**, **A**] then p is true	
If	R(p)	=	[X, **A**] then p is roughly true	
If	R(p)	=	[Ø, **A**] then p is of unknown value	[7]
If	R(p)	=	[Ø, Y] then p is roughly false	
If	R(p)	=	[Ø, Ø] then p is false	

These rough values form a lattice, ordered by set inclusion⁄, giving the following:

[Ø, Ø]	⁄	[Ø, X]	⁄	[Ø, **A**]	⁄	[Y,**A**]	⁄	[**A, A**]	[8]
false		roughly false		unknown		roughly true		true	

This suggests the introduction of a rough truth measure RV over P which identifies which of these five ordered states the rough value of each p ∈ P takes on. The advantages of such a measures are its extreme simplicity and robustness, a direct result of the simple conditions used to define the values. The axioms of the rough truth measure may be stated as:

RV (p£q)	=	max (RV(p), RV(q))	[9]
RV (p$q)	=	min (RV(p), RV(q))	

These may be easily verified by considering the set operations on the rough measure for each proposition. Similar considerations will validate the following results for the negation of a rough truth value:

RV(p)	true	roughly true	unknown	roughly false	false
RV (Ÿ p)	false	roughly false	unknown	roughly true	true

[10]

3.1 Rough inference rules

In order to use our rough valued logic for practical reasoning purposes, we need to provide a set of rules for propagating inference. We can adapt the reasoning patterns of classical logic for rough valued logic. Firstly, modus ponens:

$$\frac{RV(p\rightarrow q) = \alpha \qquad RV(p) = \beta}{\alpha \geq RV(q) \geq \min(\alpha, \beta)} \qquad [11]$$

The upper limit is obtained by realising that RV (p→q) is the maximum of RV (Ÿp) and RV (q). The lower limit stems from the fact that min (α, β) = RV ((p →q)$p) = RV (p$ q) ≤ RV (q). A similar line of reasoning gives us the pattern for modus tollens:

$$\frac{\text{RV}(p \rightarrow q) = \alpha \qquad \text{RV}(\neg q) = \beta}{\text{RV}(\neg p) \geq \min(\alpha, \beta)} \qquad \alpha \geq \qquad [12]$$

4. An example

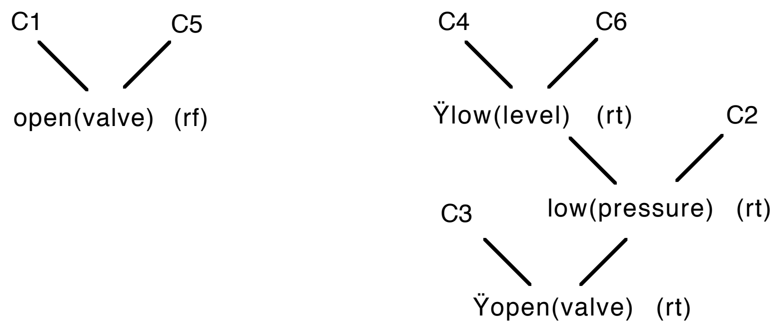
To illustrate the kind of reasoning possible with our rough valued logic, consider the following simple example. An intelligent system has a set of rules which it uses to determine when it is appropriate to open and close a pressure regulating valve. These rules are expressed as logical sentences, each of which is quantified with a rough truth value:

C1	high(pressure)	→	open(valve)	(t)
C2	Ÿlow(pressure)	→	low(level)	(t)
C3	low(pressure)	→	Ÿopen(valve)	(t)
C4	high(temperature)	→	Ÿlow(level)	(rt)

The system is also equipped with two sensors, one of which detects temperature with a high degree of accuracy, the other pressure with a lower degree of accuracy:

C5	high(pressure)	(rf)
C6	high(temperature)	(rt)

From this data the system can deduce two conflicting statements about the valve using the rules of modus ponens [11] and modus tollens [12]:



The impasse may be resolved since Ÿopen(value) is deduced with rough truth value rt a higher truth value than that with which the conclusion open(valve) is reached. The system can thus conclude that the valve should not be opened.

5. Robust reasoning in rule based systems

For most practical purposes, intelligent knowledge-based systems are rule-based , with knowledge encoded in the form of 'if...then...' rules. In this section we consider the use of rough valued logic in such systems. In many domains detailed numerical estimates of the certainty of rules and facts may be impossible to obtain, and the reasoning mechanism adopted must be capable of dealing with vague estimates. Especially important is the robustness of the mechanism— its ability to deal with rules and facts whose certainty is unknown. In this section we analyse the robustness of rough valued logic in the context of rule based reasoning.

5.1 Forward chaining

The knowledge base of a typical rule-based system consists of a series of rules of the form ‘if p then q’ with a certainty value attached to each. In forward chaining inference starts with one or more facts, also with an associated certainty, which match the antecedents of particular rules. These rules are fired to obtain their consequents, with the certainty of the consequent being determined by a combination of the certainties of rule and antecedent, and the consequents used to fire more rules. This process continues until there are no facts that match the antecedents of unfired rules, or the goal fact has been deduced.

If we assume that rules of the form ‘if p then q’ are translated by use of material implication into logical statements of the form p → q, then the mechanism of forward chaining is the rule of modus ponens [11]. This allows us to establish when consequent values of unknown certainty will be generated, taking the lower bound on the value of q:

t		

p		
	t	

No combinations of values of antecedent and rule other than those shown, have either antecedent, consequent, or rule valued as unknown.The truth tables show that the value of the consequent can be determined if the value of the rule and the antecedent is given, or, if one has an unknown value the other has the value false (f) or roughly false (rf).

5.2 Backward chaining

A similar analysis may be performed for backward chaining. Here we are interested in determining the antecedent of a rule from the rule and its consequent. Reasoning proceeds from the goal, and continues until a known fact is identified as the antecedent of a rule that must be fired in order to generate the goal. Using the pattern for rough modus tollens [12] we obtain a similar truth tables to those above for the lowest bound on the value of p:

		p
t		

	t	

LOOK FOR THE MISSING Ÿ SIGN

Once again, no combinations of values of antecedent and rule other than those shown, have either antecedent, consequent, or rule valued as unknown.The tables show that a fact of known value may be deduced from a fact and a rule of known value, or from a fact of unknown value and a rule that is true (t) or roughly true (rt), or from a rule of unknown value and a fact that is false or roughly false. In both these latter cases, the deuced fact has value false or roughly false.

Thus, in both forward and backward chaining, uncertainty can be absorbed by the logic at the cost of reducing the certainty value of the facts deduced. In situations in which the certainty of conclusions is secondary to the need to continue to operate in the face of degraded information such behaviour will be an advantage.

6. Conclusion

We have presented a symbolically quantified logic for reasoning under uncertainty that is based upon the concept of rough sets. This mathematical model provides a simple yet sound basis for a robust reasoning system. We have supplied rules of inference analogous to those of classical logic, and shown how they might be used by a reasoning system to determine the best action under conditions of uncertain knowledge. An analysis of the robustness of the logic in rule based reasoning has also been presented.

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