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A ROBUST NONCONFORMING H^2-ELEMENT

TRYGVE K. NILSSEN, XUE-CHENG TAI, AND RAGNAR WINTHER

Abstract. Finite element methods for some elliptic fourth order singular perturbation problems are discussed. We show that if such problems are discretized by the nonconforming Morley method, in a regime close to second order elliptic equations, then the error deteriorates. In fact, a counterexample is given to show that the Morley method diverges for the reduced second order equation. As an alternative to the Morley element we propose to use a nonconforming H^2-element which is H^1-conforming. We show that the new finite element method converges in the energy norm uniformly in the perturbation parameter.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and let $\partial\Omega$ denote the boundary. The purpose of this paper is to discuss finite element methods for elliptic singular perturbation problems of the form

$$\begin{cases} \varepsilon^2\Delta^2 u - \Delta u = f & \text{in } \Omega \\ u = 0, \quad \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Δ is the Laplace operator, $\partial/\partial n$ denotes the normal derivative on $\partial\Omega$, and ε is a real parameter such that $0 < \varepsilon \leq 1$. In particular, we are interested in the regime when ε is close to zero. We observe that if ε tends to zero the differential equation in (1.1) formally degenerates to Poisson's equation. Hence, we are studying a plate model which may degenerate toward an elastic membrane problem.

When fourth order problems like (1.1) are discretized by a finite element method the standard variational formulation will require function spaces which are subspaces of the Sobolev space $H^2(\Omega)$. Hence, we need piecewise smooth functions which are globally C^1. However, it is well known that in order to construct C^1-functions, which are piecewise polynomials with respect to a given triangulation of Ω, we are forced to use polynomials of degree five or higher. Alternatively, we can use a macroelement technique like in the Hsieh–Clough–Tocher method. We refer to [3, Chapter 6] for a discussion of these issues.

In order to avoid high order polynomials or macroelements, a common approach is to use nonconforming finite elements for such problems, i.e., the C^1-continuity

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requirement is violated. The simplest nonconforming element for fourth order problems is the Morley element. The Morley space consists of piecewise quadratic functions with respect to a given triangulation of Ω. The elements of this function space are not even continuous functions, but still this space leads to a convergent nonconforming finite element method. For discussions on properties of the Morley method we refer, for example, to [1], [4], [6], and [9].

However, if the Morley method is applied to a nearly second order problem of the form (1.1) with ε close to zero, then the convergence rate of the method will deteriorate. In fact, if the Morley element is applied to a second order equation like Poisson's equation then the method will diverge. The main reason for this degeneracy of the Morley method is the fact that the finite element space is not a subspace of $H^1(\Omega)$, or more precisely, the Morley space is not a proper nonconforming finite element space for second order elliptic equations. This is of course in contrast to the conforming case, since any subspace of $H^2(\Omega)$ is also a subspace of $H^1(\Omega)$.

We will discuss the degeneracy described above for the Morley method in §3. Then, in §4, we will propose an alternative nonconforming finite element method which is robust with respect to the parameter ε. The new function space consists of continuous functions which locally belongs to a nine dimensional subspace of quartic polynomials, constructed by the use of the "cubic bubble function." The global dimension of the new space, corresponding to a fixed triangulation and the boundary conditions given in (1.1), is the sum of the number of interior vertices and twice the number of interior edges. As a comparison, the dimension of the Morley space is the the sum of interior vertices and edges. In §5 we will derive proper a priori bounds for the solution of the model (1.1). These bounds lead to an error estimate for the proposed nonconforming method which is uniform with respect to parameter ε.

We should finally mention that there is some similarity between the study presented here and the results of [8], where finite element methods for second order singular perturbation problems, degenerating to a zero order problem, are discussed.

2. Preliminaries

The inner product on $L^2 = L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$. For $m \geq 0$ we shall use $H^m = H^m(\Omega)$ to denote the usual Sobolev space of functions with partial derivatives of order less than or equal m in L^2, and the corresponding norm by $\|\cdot\|_m$. Furthermore, the notation $\|\cdot\|_{m,K}$ is used to indicate that the norm is defined with respect to a domain K, different from Ω. The seminorm derived from the partial derivatives of order equal m is denoted by $|\cdot|_m$, i.e., $|\cdot|_m^2 = \|\cdot\|_m^2 - \|\cdot\|_{m-1}^2$. The space H_0^m is the closure in H^m of $C_0^\infty(\Omega)$. Alternatively, we have

$$H_0^1 = \{v \in H^1 : v|_{\partial\Omega} = 0\} \qquad \text{and} \qquad H_0^2 = \{v \in H^2 \cap H_0^1 : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\},$$

where the restrictions to $\partial\Omega$ are taken in the sense of traces. Finally, $H^{-m} \supset L^2$ is the dual of H_0^m with respect to the L^2-inner poduct.

We let Du be the gradient of u and $D^2u = (\partial^2 u/\partial x_i \partial x_j)_{i,j}$ the 2×2-tensor of second order partials. In order to define weak solutions of (1.1) we introduce the bilinear forms

$$a(u,v) = \int_\Omega D^2u : D^2v\, dx, \tag{2.1}$$

where the colon denotes the scalar product of tensors, and

$$b(u,v) = \int_\Omega Du \cdot Dv \, dx. \tag{2.2}$$

A function $u \in H_0^2$ is defined to be a weak solution of (1.1) if

$$\varepsilon^2 a(u,v) + b(u,v) = (f,v) \quad \forall v \in H_0^2. \tag{2.3}$$

In fact, in this weak formulation we may simplify the bilinear form a and use

$$a(u,v) = \int_\Omega \Delta u \Delta v \, dx$$

instead of the one given by (2.1), since

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega (\text{trace} D^2 u)(\text{trace} D^2 v) \, dx = \int_\Omega D^2 u : D^2 v \, dx \tag{2.4}$$

for all $u, v \in H_0^2$. However, since we will consider nonconforming finite element methods, this identity may not hold on the proper finite element spaces, and therefore we use the form given by (2.1).

It is a consequence of the regularity theory for elliptic problems in nonsmooth domains (cf. [5, Corollary 7.3.2.5]) that if $f \in H^{-1}$ and Ω is convex, then $u \in H^3$, i.e., there is a constant c independent of f such that the corresponding weak solution u of (1.1) satisfies

$$\|u\|_3 \le c \, \|f\|_{-1}.$$

However, the constant c will in general depend on ε and will blow up as ε tends to zero. This issue will be discussed further in §5.

3. The Morley method

Assume that $\{\mathcal{T}_h\}$ is a quasi-uniform and shape-regular family of triangulations of Ω, where the discretization parameter h is a characteristic diameter. We let $\mathcal{X}_h$ be the set of vertices and $\mathcal{E}_h$ the set of edges corresponding to $\mathcal{T}_h$. The corresponding Morley finite element space, M_h, consists of all piecewise quadratics which are continuous at each vertex of $\mathcal{T}_h$ and such that the normal component of the gradient is continuous at the midpoint of each edge. Furthermore, in order to approximate the boundary conditions in (1.1) the functions in M_h are zero at boundary vertices and have zero normal derivatives at the midpoints of all boundary edges. A function $w \in M_h$ is uniquely determined by the value of w at each interior vertex and by the value of the normal component of Dw at the midpoint of each interior edge; see Figure 1.

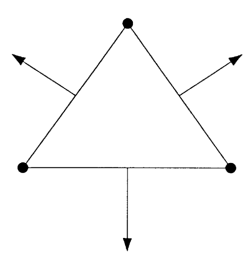

Figure 1. The six degrees of freedom of the Morley element

The finite element approximation $u_h \in M_h$ of u is now determined by the linear system

$$\varepsilon^2 a_h(u,v) + b_h(u,v) = (f,v) \quad \forall v \in M_h. \tag{3.1}$$

Here the bilinear forms a_h and b_h are obtained from a and b by summing over all triangles in $\mathcal{T}_h$, i.e.

$$a_h(u,v) = \sum_{T\in\mathcal{T}_h} \int_T D^2u : D^2v\,dx$$

and

$$b_h(u,v) = \sum_{T\in\mathcal{T}_h} \int_T Du \cdot Dv\,dx.$$

Associated with the bilinear form $\varepsilon^2 a_h + b_h$, we define a seminorm $\|\!|\cdot|\!\|_{\varepsilon,h}$ by

$$|\!|\!| w |\!|\!|_{\varepsilon,h}^2 = \varepsilon^2 a_h(w,w) + b_h(w,w).$$

Observe that this seminorm is a norm on $H_0^2 + M_h$, and, as ε tends to zero, this energy norm approaches a piecewise H_0^1-norm.

If $u \in H_0^2$ is the corresponding weak solution of (1.1), then the error $u - u_h$ can be estimated in the energy norm using the second Strang lemma (cf. [3, Theorem 4.2.2]) which states that

$$|\!|\!| u-u_h |\!|\!|_{\varepsilon,h} \le \inf_{v\in M_h} |\!|\!| u-v |\!|\!|_{\varepsilon,h} + \sup_{w\in M_h} \frac{|E_{\varepsilon,h}(u,w)|}{|\!|\!| w |\!|\!|_{\varepsilon,h}}, \tag{3.2}$$

where the consistency error $E_{\varepsilon,h}(u,w)$ is given by

$$E_{\varepsilon,h}(u,w) = \varepsilon^2 a_h(u,w) + b_h(u,w) - (f,w).$$

Assume that $f \in L^2$ and $u \in H^3 \cap H_0^2$. From the basic estimate (3.2) and by following the approach of [9] to estimate $E_{\varepsilon,h}(u,w)$ it is straightforward to obtain an error estimate of the form

$$|\!|\!| u-u_h |\!|\!|_{\varepsilon,h} \le C\left(\frac{h^2}{\varepsilon}\|f\|_0 + \frac{h}{\varepsilon}\|u\|_3\right). \tag{3.3}$$

Note that if $\varepsilon = 1$ this estimate predicts linear convergence with respect to the mesh parameter h. However, if h is fixed and ε approaches zero, then the estimate for the error behaves like $O(1/\varepsilon)$ (under the assumption that $\|u\|_3$ is uniformly bounded). The following numerical example indicates that this degeneracy is in fact real.

Example 3.1. We consider the problem (1.1) with Ω taken as the unit square and $f = \varepsilon^2\Delta^2 u - \Delta u$, where $u(x) = (\sin(\pi x_1)\sin(\pi x_2))^2$. The domain is triangulated by first dividing it into $h \times h$ squares. Then, each square is divided into two triangles by the diagonal with a negative slope. In Table 1 we have computed the relative error in the energy norm, $|\!|\!| u_h^I - u_h |\!|\!|_{\varepsilon,h} / |\!|\!| u_h^I |\!|\!|_{\varepsilon,h}$, for different values of ε and h. Here u_h^I denote the interpolant of u on M_h defined from the values of u at each vertex and from the value of the normal component of Du on the midpoint of each edge. For a comparison we also consider the case $\varepsilon = 0$, i.e., the Poisson problem with Dirichlet boundary conditions, and the biharmonic problem $\Delta^2 u = f$.

When ε is large, the convergence appears to be linear with respect to h, while the convergence deteriorates as ε approaches zero. $\square$

TABLE 1. The relative error measured by the energy norm

$\varepsilon\backslash h$	2^{-3}	2^{-4}	2^{-5}	2^{-6}
2^{0}	0.3898	0.2008	0.1012	0.0507
2^{-2}	0.4016	0.2085	0.1053	0.0528
2^{-4}	0.5674	0.3262	0.1699	0.0858
2^{-6}	0.8937	0.7499	0.4981	0.2790
2^{-8}	0.9730	0.9934	0.9275	0.7487
2^{-10}	0.9791	1.0214	1.0265	1.0059
Poisson	0.9860	1.0242	1.0348	1.0376
Biharmonic	0.3891	0.2004	0.1009	0.0506

3.1. **The reduced problem.** When ε tends to zero the problem (1.1) formally approaches a Poisson equation with Dirichlet boundary conditions. Below we shall give an analytical argument which shows that for such problems the Morley space will in fact lead to a divergent numerical method. Hence, this suggest once more that the Morley method is not suitable for problems of the form (1.1) when the parameter ε is sufficiently small. We should mention here that the divergence of the Morley method for second order problems has also been discussed in [7].

In order to simplify some calculations below we shall modify the reduced problem slightly. Instead of the pure Dirichlet problem we consider mixed boundary conditions. We assume that $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D and Γ_N are disjoint subsets of $\partial\Omega$ and consider the problem

$$\left\{\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \tfrac{\partial u}{\partial n} &= g \quad \text{on } \Gamma_N. \end{aligned}\right. \tag{3.4}$$

This problem can be considered to be the formal limit of the fourth order problems

$$\left\{\begin{aligned} \varepsilon^2\Delta^2 u - \Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \tfrac{\partial}{\partial n}(u - \varepsilon^2\Delta u) &= g \quad \text{on } \Gamma_N \\ \Delta u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right. \tag{3.5}$$

As above let $\mathcal{T}_h$ be a triangulation of Ω and let M_h be the Morley space corresponding to the boundary conditions of (3.4), i.e., we assume that the functions in M_h are zero on $\mathcal{X}_h \cap \Gamma_D$.

The approximation $u_h \in M_h$ of the solution u of problem (3.4) is determined by the linear system

$$b_h(u_h, w) = (f, w) + \langle g, w\rangle \quad \text{for all } w \in M_h. \tag{3.6}$$

Here

$$\langle g, w\rangle = \int_{\Gamma_N} gw\, ds,$$

where s denotes the arc length along $\partial\Omega$. Furthermore, the exact solution u of (3.4) satisfies

$$b_h(u, w) = (f, w) + \langle g, w\rangle + E_h(u, w) \quad \text{for all } w \in M_h, \tag{3.7}$$

where

$$E_h(u,w) = b_h(u,w) - (f,w) - \langle g, w\rangle.$$

Let $|\!|\!| \cdot |\!|\!|_h = |\!|\!| \cdot |\!|\!|_{0,h}$ be the energy norm for the reduced problem. From (3.6) and (3.7) we obtain the second Strang lemma

$$|\!|\!| u - u_h |\!|\!|_h \leq \inf_{v\in M_h} |\!|\!| u - v |\!|\!|_h + \sup_{w\in M_h} \frac{|E_h(u,w)|}{|\!|\!| w |\!|\!|_h}. \tag{3.8}$$

However, the lower bound

$$|\!|\!| u - u_h |\!|\!|_h \geq \sup_{w\in M_h} \frac{|E_h(u,w)|}{|\!|\!| w |\!|\!|_h} \tag{3.9}$$

is also valid. This follows from the Cauchy–Schwarz inequality since

$$|E_h(u,w)| = |b_h(u-u_h, w)| \leq |\!|\!| u - u_h |\!|\!|_h \, |\!|\!| w |\!|\!|_h.$$

The basic lower bound (3.9) can be used to prove the divergence of the method if we can establish that the right hand side of (3.9) does not tend to zero with h. We will do this below for a suitable choice of the solution u.

However, first we will discuss a decomposition of the Morley space M_h. The function space M_h can naturally be written as a sum of two spaces

$$M_h = M_h^v + M_h^e$$

associated with the vertex values and the edge values. More precisely,

$$M_h^v = \{w \in M_h : \int_e \frac{\partial w}{\partial n}\, ds = 0 \quad \text{for all } e \in \mathcal{E}_h\}$$

and

$$M_h^e = \{w \in M_h : w(x) = 0 \quad \text{for all } x \in \mathcal{X}_h\}.$$

For a given $w \in M_h$, let w_h^v be the corresponding interpolant of w on M_h^v. Since this interpolation process preserves constants locally, we have

$$||w - w_h^v||_{L^\infty} \leq ch||Dw||_{L^\infty}, \tag{3.10}$$

where the constant c is independent of h.

Observe that since $\frac{\partial w}{\partial n}$ is linear on each side of an edge we must have that $\frac{\partial w}{\partial n}$ equals zero at the midpoint of each edge if $w \in M_h^v$. Similarly, if $w \in M_h^e$, then w restricted to a side of an edge is a quadratic function which is zero at the two endpoints. We can therefore conclude that the tangential derivative $\frac{\partial w}{\partial s}$ is zero at the midpoint. Also, for any $w \in M_h$ the function Δw is a constant on each triangle. In fact, if $w \in M_h^v$, then $\Delta w = 0$. This follows since

$$\int_T \Delta w\, dx = \int_{\partial T} \frac{\partial w}{\partial n}\, ds = 0.$$

Furthermore, the decomposition of M_h is in fact orthogonal with respect to the bilinear form b_h, i.e.,

$$M_h = M_h^v \oplus M_h^e. \tag{3.11}$$

To see this, note that if $w \in M_h^v$ and $\phi \in M_h^e$, then, since w is harmonic on each T,

$$\int_T Dw \cdot D\phi\, dx = \int_{\partial T} \frac{\partial w}{\partial n} \phi\, ds.$$

However, on each edge the function $\frac{\partial w}{\partial n}\phi$ is a cubic function which is zero at the endpoints and the midpoint. Hence, all the boundary integrals are zero, which implies

$$b_h(w, \phi) = 0.$$

In fact, the decomposition is orthogonal locally on each triangle $T \in \mathcal{T}_h$.

Example 3.2. To construct the counterexample, we let Ω be the unit square. The triangulations $\mathcal{T}_h$ are constructed by dividing Ω into n^2 squares of size $h \times h$, where $h = 1/n$, and then dividing each square into two triangles using the negative sloped diagonals. We assume that Γ_D consists of the intersection of $\partial\Omega$ with the coordinate axis, while we assume Neumann boundary conditions on $x_1 = 1$ and $x_2 = 1$. Hence, the space M_h is assumed to consist of the Morley functions which are zero at the vertices on the coordinate axis.

We assume that the solution is given by $u(x) = x_1 x_2$. Hence, $g = x_1$ on $x_2 = 1$ and $g = x_2$ on $x_1 = 1$. Furthermore u is harmonic (i.e., $f = 0$). Therefore,

$$E_h(u, w) = b_h(u, w) - \langle g, w \rangle. \tag{3.12}$$

Note also that $u \in M_h$. Let u_h^v be the interpolant of u onto M_h^v. Hence,

$$u_h^v(x) = u(x) \quad \text{for all } x \in \mathcal{X}_h \quad \text{and} \quad \int_e \frac{\partial u_h^v}{\partial n}\, ds = 0 \quad \text{for all } e \in \mathcal{E}_h.$$

We shall show that $\lim_{h\to 0} |E_h(u, u_h^v)| / \| u_h^v \|_h$ is strictly positive. Due to the lower bound (3.9) this implies that the method diverges.

Let $u_h^e = u - u_h^v$. Then $u_h^e \in M_h^e$ and u_h^e is exactly the interpolant of u onto M_h^e. Furthermore, from (3.11) we have

$$b_h(u, u) = b_h(u_h^v, u_h^v) + b_h(u_h^e, u_h^e). \tag{3.13}$$

Since $|Du_h^v|^2$ is piecewise quadratic, it follows that for any $T \in \mathcal{T}_h$

$$\int_T |Du_h^v|^2\, dx = \frac{|T|}{3} \sum_{m \in \mathcal{M}(T)} |Du_h^v(m)|^2,$$

where $|T|$ is the area of T and $\mathcal{M}(T)$ denotes the set of the three edge midpoints (cf. [3, page 183]). However, the tangential derivative $\frac{\partial u_h^e}{\partial s}$ at each edge midpoint is zero. Thus $\partial u_h^v / \partial s$ are exactly equal to the corresponding values for u, while the normal component of Du_h^v at each edge midpoint is zero. Therefore, we obtain

$$b_h(u_h^v, u_h^v) = \frac{h^2}{6} \sum_{T \in \mathcal{T}_h} \sum_{m \in M(T)} |\frac{\partial u}{\partial s}(m)|^2.$$

However, since $u(x) = x_1 x_2$, we can verify that

$$\begin{aligned}
&\lim_{h\to 0} \frac{h^2}{6} \sum_{T \in \mathcal{T}_h} \sum_{m \in M(T)} |\frac{\partial u}{\partial s}(m)|^2 \\
&\quad = \lim_{h \to 0} \{ \frac{h^2}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} (ih - jh)^2 + \frac{h^2}{3} \sum_{i=1}^{n} \sum_{j=1}^{n} ((ih)^2 + (jh)^2) \}.
\end{aligned}$$

From this expression we obtain that

$$\lim_{h \to 0} b_h(u_h^v, u_h^v) = 1/4. \tag{3.14}$$

It is also straightforward to check that

$$b_h(u,u) = \int_\Omega |\nabla u|^2 \, dx = 2/3,$$

and hence (3.13) implies

$$\lim_{h\to 0} b_h(u_h^e, u_h^e) = 5/12. \tag{3.15}$$

From (3.12) we obtain

$$\begin{aligned} E_h(u, u_h^v) &= b_h(u, u_h^v) - \langle g, u_h^v\rangle = b_h(u_h^v, u_h^v) - \langle g, u_h^v\rangle \\ &= b(u,u) - \langle g, u_h^v\rangle - b_h(u_h^e, u_h^e) \\ &= \langle g, u - u_h^v\rangle - b_h(u_h^e, u_h^e). \end{aligned} \tag{3.16}$$

Hence, we derive from (3.10) and (3.16) that

$$\lim_{h\to 0} E_h(u, u_h^v) = -\lim_{h\to 0} b_h(u_h^e, u_h^e) = -5/12.$$

By using (3.14) this implies that

$$\lim_{h\to 0} \frac{|E_h(u, u_h^v)|}{\|u_h^v\|_h} = \lim_{h\to 0} \frac{b_h(u_h^e, u_h^e)}{b_h(u_h^v, u_h^v)^{1/2}} = 5/6.$$

The divergence of the method is therefore a consequence of the basic lower bound (3.9). □

4. Modifications of the Morley element

As we have seen in the discussion above, the Morley method is not well suited for solving problems of the form (1.1) when the positive parameter ε is small. The purpose of this section is to propose an alternative nonconforming finite element space which will lead to a numerical method that is robust with respect to the parameter ε. For a review of other conforming and nonconforming finite element methods for fourth order problems we refer to [3, Chapter 6].

Let $T \subset \mathbb{R}^2$ be a triangle and consider the polynomial space on T given by

$$W(T) = \{ w \in \mathbb{P}_4 : w|_e \in \mathbb{P}_2 \quad \forall e \in \mathcal{E}(T)\}.$$

Here $\mathbb{P}_k$ denotes the set of polynomials of degree k and $\mathcal{E}(T)$ denotes the set of the edges of T. It is a consequence of Lemma 4.1 that the space $W(T)$ can alternatively be defined as all functions

$$w = q + pb, \tag{4.1}$$

where $q \in \mathbb{P}_2$, $p \in \mathbb{P}_1$ and b is the cubic bubble function. We recall that the cubic bubble function b is defined by $b = \lambda_1\lambda_2\lambda_3$, where $\lambda_i(x)$ are the barycentric coordinates of x with respect to the three corners $\mathcal{X}(T)$ of T. Associated with an $x_i \in \mathcal{X}(T)$, the function $\lambda_i \in \mathbb{P}_1$ is uniquely determined by

$$\lambda_i(x_i) = 1, \quad \lambda_i(x) = 0 \quad \text{for } x \in \mathcal{X}(T), x \neq x_i.$$

Furthermore,

$$\sum_{i=1}^{3} \lambda_i(x) \equiv 1.$$

A basis for the space $W(T)$, which is useful below, can be derived from the following result.

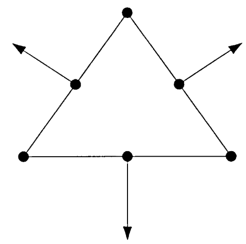

FIGURE 2. The nine degrees of freedom of the modified Morley element

Lemma 4.1. *The space $W(T)$ is a linear space of dimension nine. Furthermore, an element $w \in W(T)$ is uniquely determined by the following degrees of freedom:*

- *the values of w at the corners and edge midpoints;*
- $\int_e \frac{\partial w}{\partial n}\, ds \quad$ *for all $e \in \mathcal{E}(T)$.*

Proof. Since any function of the form (4.1) is in $W(T)$ we must have $\dim W(T) \geq 9$. Furthermore, it is consequence of the standard Langrangian basis for $\mathbb{P}_4(T)$ (cf. for example [2, Chapter 3]) that if $w \in \mathbb{P}_4(T)$, with $w|_{\partial T} \equiv 0$, then

$$w = pb, \tag{4.2}$$

where $p \in \mathbb{P}_1$ and b is the cubic bubble function.

Assume that $w \in W(T)$ is such that the nine degrees of freedom specified in Lemma 4.1 are all zero. The proof will be completed if we can show that $w \equiv 0$. Since $w|_e \in \mathbb{P}_2$, with three roots, we must have $w|_{\partial T} = 0$. Therefore, w is of the form (4.2).

Let e be a fixed edge of T. Hence,

$$w = pb = p\lambda_+\lambda_-\lambda_e,$$

where $\lambda_e \in \mathbb{P}_1$ is the barycentric coordinate function such that $\lambda_e \equiv 0$ on e, and λ_+ and λ_- are the two other barycentric coordinates. Note that the "quadratic bubble," $b_e = \lambda_+\lambda_-$, is strictly positive in the interior of e. Furthermore,

$$(Dw)|_e = (pb_e D\lambda_e)|_e.$$

Note also that $\partial\lambda_e/\partial n < 0$, where n is the outward unit normal on e. Therefore, the condition

$$\int_e \frac{\partial w}{\partial n}\, ds = \int_e pb_e \frac{\partial \lambda_e}{\partial n}\, ds = 0$$

implies that p must have a root in the interior of e. Hence, $p \in \mathbb{P}_1$ has a root in the interior of all three edges. This implies that $p \equiv 0$, or equivalently, $w \equiv 0$. $\square$

As before let $\mathcal{T}_h$ be a quasi-uniform and shape-regular family of triangulations of Ω. The new finite element space W_h, associated with the triangulation $\mathcal{T}_h$, will consist of continuous functions which are zero at the boundary, i.e., $W_h \subset C^0(\Omega)$. In addition

- $w|_T \in W(T) \quad$ for all $T \in \mathcal{T}_h$,
- $\int_e \frac{\partial w}{\partial n}\, ds$ is continuous for all interior edges and zero for boundary edges.

It follows from Lemma 4.1 that any function $w \in W_h$ is uniquely determined by the values of w at all interior vertices and edge midpoints, and by the mean value of $\partial w/\partial n$ for all interior edges (see Figure 2).

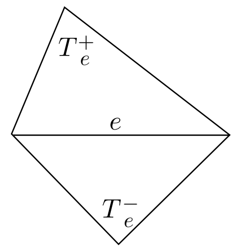

FIGURE 3. $\Omega_e = T_e^+ \cup T_e^-$

These degrees of freedom also defines a local interpolation operator $I_h : H^2 \mapsto W_h$. Furthermore, since this operator preserves quadratics locally, it follows from a standard scaling argument, using the Bramble–Hilbert lemma, that there is a constant c independent of h such that

$$\sum_{T \in \mathcal{T}_h} \|v - I_h v\|_{j,T} \le c h^{k-j} |v|_k \quad \text{for } v \in H_0^2 \cap H^k, \tag{4.3}$$

where $j = 0, 1, 2$ and $k = 2, 3$. In fact, if $\hat{T}$ is a reference triangle and $\hat{I} : H^2(\hat{T}) \mapsto W(\hat{T})$ is the interpolation operator with respect to $\hat{T}$, then for all $v \in H^2(\hat{T})$

$$\|\hat{I}v\|_{1,\hat{T}} \le c_1(\|v\|_{L^\infty(\hat{T})} + \|\frac{\partial v}{\partial n}\|_{0,\partial\hat{T}}) \le c_2 \|v\|_{1,\hat{T}}^{1/2} \|v\|_{2,\hat{T}}^{1/2},$$

where the constants c_1 and c_2 only depend on $\hat{T}$. Here we have used the standard trace inequality

$$\|v\|_{0,\partial\hat{T}} \le c \|v\|_{0,\hat{T}}^{1/2} \|v\|_{1,\hat{T}}^{1/2}$$

(cf. [5, Theorem 1.5.1.10]). Hence, again from a Bramble–Hilbert argument, we obtain

$$\|v - I_h v\|_1 \le c h^{1/2} |v|_1^{1/2} |v|_2^{1/2} \quad \text{for } v \in H_0^2. \tag{4.4}$$

Compared to the estimate (4.3), with $j = 1$, the new estimate (4.4) predicts a lower order convergence with respect to h, but the dependence on the function v is weaker. This will be useful below.

Since the elements of W_h are continuous functions which vanish on the boundary, the inclusion $W_h \subset H_0^1$ holds. However, W_h is not a subspace of H^2. Therefore, the space W_h again leads to a nonconforming finite element method for the fourth order problem (1.1). If $w \in W_h$ and e is an interior edge, we let $[\partial w/\partial n]_e$ denote the jump of the normal derivative on e. We let Ω_e denote the union of the two triangles T_e^+ and T_e^- which have e as common edge (see Figure 3). Observe that if $w \in W_h$ and $|w|_{2,T_e^+} + |w|_{2,T_e^-} = 0$, then w is linear on Ω_e. Furthermore, if w is linear on Ω_e, then $[\partial w/\partial n]_e \equiv 0$. Also, the continuity requirement on W_h implies that for any $w \in W_h$ the function $[\partial w/\partial n]_e$ is a cubic polynomial such that

$$\int_e [\frac{\partial w}{\partial n}]_e \, ds = 0. \tag{4.5}$$

Hence, by a standard scaling argument (see, for example, [2, Section 8.3]), this implies that there is a constant c independent of h such that

$$\int_e \phi[\frac{\partial w}{\partial n}]_e \, ds \le ch|\phi|_{1,\Omega_e}(|w|_{2,T_e^+} + |w|_{2,T_e^-}) \tag{4.6}$$

for all $\phi \in H^1$ and $w \in W_h$.

The estimate (4.6) can be generalized such that it is also valid for boundary edges if w is taken to be zero outside Ω. Below we shall need this estimate when $\phi = \Delta\psi - \partial^2\psi/\partial s^2$, where s is a unit tangent vector on e. In this case we obtain from (4.6) that

$$\int_e (\Delta\psi - \partial^2\psi/\partial s^2)[\frac{\partial w}{\partial n}]_e \, ds \le ch|\psi|_{3,\Omega_e}(|w|_{2,T_e^+} + |w|_{2,T_e^-})$$

for all $\psi \in H^3$ and $w \in W_h$. By summing this estimate over all edges we derive

$$\sum_{e\in\mathcal{E}_h} \int_e (\Delta\psi - \partial^2\psi/\partial s^2)[\frac{\partial w}{\partial n}]_e \, ds \le c\frac{h}{\varepsilon}|\psi|_3 |\!|\!| w |\!|\!|_{\varepsilon,h} \tag{4.7}$$

for all $\psi \in H^3, w \in W_h$.

As above, we can also obtain a variant of this estimate which is lower order with respect to h, but which requires a weaker dependence on the function u. If $\hat{T}$ is a reference triangle, and $\hat{e}$ is an edge of $\hat{T}$, then

$$\|\phi\|_{0,\hat{e}} \le c\|\phi\|_{0,\hat{T}}^{1/2}\|\phi\|_{1,\hat{T}}^{1/2}.$$

Therefore, the estimate (4.6) can be replaced by

$$\int_e \phi[\frac{\partial w}{\partial n}]_e \, ds \le ch^{1/2}\|\phi\|_{0,\Omega_e}^{1/2}|\phi|_{1,\Omega_e}^{1/2}(|w|_{2,T_e^+} + |w|_{2,T_e^-})$$

for all $\phi \in H^1, w \in W_h$. This again leads to the bound

$$\sum_{e\in\mathcal{E}_h} \int_e (\Delta\psi - \partial^2\psi/\partial s^2)[\frac{\partial w}{\partial n}]_e \, ds \le c\frac{h^{1/2}}{\varepsilon}|\psi|_2^{1/2}|\psi|_3^{1/2} |\!|\!| w |\!|\!|_{\varepsilon,h} \tag{4.8}$$

for all $\psi \in H^3, w \in W_h$, as an alternative to (4.7).

The finite element solution $u_h \in W_h$ is defined as the solution of

$$\varepsilon^2 a_h(u_h, w) + b(u_h, w) = (f, w), \quad \text{for all } w \in W_h. \tag{4.9}$$

The following theorem shows that for any fixed $\varepsilon \in (0, 1]$ the new nonconforming finite element method converges linearly with respect to h. Furthermore, this convergence is uniform with respect to ε if the quantity $|u|_2 + \varepsilon|u|_3$ is uniformly bounded.

Theorem 4.2. *Assume that the weak solution u of* (1.1) *is in $H_0^2 \cap H^3$ for a given $f \in L^2$. Furthermore, let $u_h \in W_h$ be the corresponding solution of* (4.9). *Then there is a constant c, independent of ε and h, such that*

$$|\!|\!| u - u_h |\!|\!|_{\varepsilon,h} \le c \begin{cases} (h^2 + \varepsilon h)|u|_3\,, \\ h(|u|_2 + \varepsilon|u|_3). \end{cases}$$

Proof. The second Strang lemma (3.2) is still valid, i.e.

$$|\!|\!| u - u_h |\!|\!|_{\varepsilon,h} \le \inf_{v\in W_h} |\!|\!| u - v |\!|\!|_{\varepsilon,h} + \sup_{w\in W_h} \frac{|E_{\varepsilon,h}(u, w)|}{|\!|\!| w |\!|\!|_{\varepsilon,h}}, \tag{4.10}$$

where $E_{\varepsilon,h}(u,w)$ is given by

$$E_{\varepsilon,h}(u,w) = \varepsilon^2 a_h(u,w) + b(u,w) - (f,w).$$

Furthermore, the interpolation estimate (4.3) implies that

$$\inf_{v \in W_h} \|\!|u - v|\!\|_{\varepsilon,h} \le c \begin{cases} (h^2 + \varepsilon h)|u|_3\,, \\ h(|u|_2 + \varepsilon |u|_3). \end{cases} \tag{4.11}$$

Hence, it remains to estimate $E_{\varepsilon,h}(u,w)$.

Since $u \in H^3$ it follows from the weak formulation (2.3) and the identity (2.4) that

$$\int_\Omega (-\varepsilon^2 D(\Delta u) + Du) \cdot Dw \, dx = (f,w) \quad \forall w \in H^1_0.$$

In particular, this identity holds for $w \in W_h$. The consistency error $E_{\varepsilon,h}(u,w)$ can therefore be expressed as

$$E_{\varepsilon,h}(u,w) = \varepsilon^2 \sum_{T \in \mathcal{T}_h} \int_T (D^2 u : D^2 w + D\Delta u \cdot Dw) \, dx. \tag{4.12}$$

The tensor $D^2 u$ can be written as

$$D^2 u = (\Delta u)\mathbb{I} + C^2 u,$$

where $\mathbb{I}$ is the identity tensor and

$$C^2 u = -\operatorname{curl}\operatorname{curl} u = \begin{pmatrix} -\frac{\partial^2 u}{\partial x_2^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u}{\partial x_1 \partial x_2} & -\frac{\partial^2 u}{\partial x_1^2} \end{pmatrix}.$$

Furthermore,

$$(\Delta u)\mathbb{I} : D^2 w = (\Delta u)\Delta w$$

and therefore

$$\int_T (\Delta u)\mathbb{I} : D^2 w \, dx = \int_{\partial T} \Delta u \frac{\partial w}{\partial n} \, ds - \int_T D(\Delta u) \cdot Dw \, dx. \tag{4.13}$$

On the other hand, since each row of $C^2 u$ is divergence free, we have

$$\operatorname{div}(C^2 u \cdot Dw) = C^2 u : D^2 w,$$

and from this we obtain

$$\int_T C^2 u : D^2 w \, dx = \int_{\partial T} n \cdot C^2 u \cdot Dw \, ds.$$

However, by combining this identity with (4.12) and (4.13), and by using the fact that the tangential component of Dw is continuous on each edge, we obtain

$$\begin{aligned} E_{\varepsilon,h}(u,w) &= \varepsilon^2 \sum_{e \in \mathcal{E}_h} \int_e (\Delta u + n \cdot C^2 u \cdot n)[\frac{\partial w}{\partial n}]_e \, ds \\ &= \varepsilon^2 \sum_{e \in \mathcal{E}_h} \int_e (\Delta u - \frac{\partial^2 u}{\partial s^2})[\frac{\partial w}{\partial n}]_e \, ds. \end{aligned} \tag{4.14}$$

It therefore follows from (4.7) that

$$E_{\varepsilon,h}(u,w) \le ch\varepsilon |u|_3 \|\!|w|\!\|_{\varepsilon,h},$$

and together with (4.10) and (4.11) this implies the desired estimates. $\square$

TABLE 2. The relative error measured by the energy norm

$\varepsilon\backslash h$	2^{-3}	2^{-4}	2^{-5}	2^{-6}
2^{0}	0.3359	0.1790	0.09108	0.0457
2^{-2}	0.3016	0.1589	0.08061	0.0405
2^{-4}	0.1519	0.07627	0.03819	0.0190
2^{-6}	0.0564	0.0229	0.0107	0.0052
2^{-8}	0.0416	0.0113	0.0036	0.0014
2^{-10}	0.0406	0.0103	0.0026	0.0007
Poisson	0.0409	0.0102	0.0026	0.0006
Biharmonic	0.3386	0.1806	0.0919	0.0462

Note that in the limit as ε tends to zero the first estimate in Theorem 4.2 gives the bound

$$\|u - u_h\|_1 \le ch^2|u|_3.$$

In fact, since W_h is a subset of H_0^1, this estimate follows directly from (4.3) for the reduced problem

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \quad u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.15}$$

However, if $f \in L^2$ and Ω is convex, then we can only expect that the solution of the reduced problem (4.15) is in H^2. The second estimate in Theorem 4.2 is consistent with the proper result for the reduced problem in this case.

Example 4.1. We will redo the computations we did in Example 3.1, the only difference being that we use the new finite element space W_h instead of the Morley space M_h. The results for the relative error are given in Table 2. These results clearly demonstrate the improved behavior of the modified method when ε is close to zero. In particular, when ε is large the convergence appears to be linear with respect to h, while we observe nearly quadratic convergence when ε is small. This is in fact consistent with Theorem 4.2, since $|u|_3$ is independent of ε. $\square$

Let us compare the dimension of the space W_h with the dimension of the Morley space M_h. Let $|\mathcal{X}_h|$ be the number of interior vertices in the triangulation $\mathcal{T}_h$ and let $|\mathcal{E}_h|$ be the corresponding number of interior edges. It follows from the discussion above that

$$\dim(W_h) = |\mathcal{X}_h| + 2|\mathcal{E}_h|,$$

while the dimension of the Morley space M_h is $|\mathcal{X}_h|+|\mathcal{E}_h|$. Hence, since $|\mathcal{E}_h| \approx 3|\mathcal{X}_h|$, the replacement of M_h by W_h leads to an increase in the number of unknowns of approximately 75 percent. Below we shall briefly discuss an alternative nonconforming finite element space $\tilde{W}_h$. This space has a similar robustness property as W_h with respect to parameter ε, and the dimension of the space is $3|\mathcal{X}_h| + |\mathcal{E}_h|$, which represents an increase of approximately 50 percent as compared to the Morley space. On the other hand, the sparsity structure of the space W_h is more favorable than the sparsity structure of the new space.

If T is a triangle, let

$$\tilde{W}(T) = \{w \in \mathbb{P}_4 : w|_e \in \mathbb{P}_3 \quad \forall e \in \mathcal{E}(T)\}.$$

It can easily be verified, using arguments as above, that $\tilde{W}(T)$ can be equivalently defined as all functions

$$w = q + pb, \tag{4.16}$$

where $q \in \mathbb{P}_3$, $p \in \mathbb{P}_1$ and b is the cubic bubble function. Hence, $\tilde{W}(T)$ is a linear space of dimension twelve. The basis given below will be used to define a proper finite element space.

Lemma 4.3. *The space $\tilde{W}(T)$ is a linear space of dimension twelve. Furthermore, an element $w \in \tilde{W}(T)$ is uniquely determined by the following degrees of freedom:*

- *the values of w and Dw at the corners;*
- *$\frac{\partial w}{\partial n}$ at the midpoint m_e of e $\forall e \in \mathcal{E}(T)$.*

Proof. Since $\dim \tilde{W}(T) = 12$ it is enough to show that the twelve degrees of freedom determine elements of $\tilde{W}(T)$ uniquely. Assume $w \in \tilde{W}(T)$ such that the twelve degrees of freedom are all zero. Since $w|_e \in \mathbb{P}_3$, with a double zero at each endpoint we must have $w|_{\partial T} = 0$. From (4.16) we obtain that w is of the form

$$w = pb,$$

where $p \in \mathbb{P}_1$ and b is the cubic bubble function. Note that

$$(Dw)|_e = (pD(b))|_e$$

for each edge e, and that

$$(Db) \cdot n)(m_e) \neq 0.$$

Therefore, the three conditions $(\partial w/\partial n)(m_e) = 0$ imply that $p(m_e) = 0$. Hence, $p \equiv 0$, or equivalently $w \equiv 0$. □

As above the polynomial space $\tilde{W}(T)$ can be used to define a finite element space $\tilde{W}_h$ consisting of functions which are locally in $\tilde{W}(T)$, with w and Dw continuous at each vertex, and with $\partial w/\partial n$ continuous at the midpoint of each edge. If $w \in \tilde{W}_h$ and $e \in \mathcal{E}_h$, then $[\partial w/\partial n]_e$ is a cubic function which has a root at the two endpoints and the midpoint. Therefore, the property (4.5) holds for all functions in $\tilde{W}_h$. Hence, by arguing almost exactly as above, we can also derive a result like Theorem 4.2 in this case.

5. The influence of boundary layers

Throughout this section we assume that the domain Ω is a convex, bounded polygonal domain in $\mathbb{R}^2$. As in most of the previous section we consider the nonconforming finite element method (4.9), derived from the space W_h, approximating the singular perturbation problem (1.1).

As we observed above, Theorem 4.2 ensures linear convergence with respect to h, uniformly in ε, as long as the seminorm $|u|_2 + \varepsilon|u|_3$ is uniformly bounded. However, by studying the effect of boundary layers in the one dimensional analogs of (1.1), one will quickly be convinced that the best one can hope for is that this quantity behaves like $O(\varepsilon^{-1/2})$ as ε tends to zero. In fact, these one dimensional analogs admit solutions of the form

$$u(x) = \varepsilon e^{-x/\varepsilon} - p(x), \tag{5.1}$$

where p is a cubic polynomial which is bounded independently of ε, chosen such that the Dirichlet boundary conditions hold, and $f = p''$.

In the lemma below we state some useful estimates in the two dimensional case. Here, and in the rest of this section, $u = u^\varepsilon \in H_0^2 \cap H^3$ denotes the weak solution of problem (1.1), while $u^0 \in H_0^1 \cap H^2$ is the corresponding solution of the reduced problem (4.15).

Lemma 5.1. *There is a constant c, independent of ε and f, such that*

$$|u|_2 + \varepsilon |u|_3 \le c\,\varepsilon^{-1/2}\|f\|_0 \qquad \text{and} \qquad |u - u^0|_1 \le c\,\varepsilon^{1/2}\|f\|_0$$

for all $f \in L^2$.

Proof. We will establish the bounds above by energy arguments. Throughout the proof c will denote a generic constant, independent of ε and f, and not necessarily the same at different occurences. Let us first recall the standard estimates for convex domains. It follows from [5, Theorem 3.2.1.2] that,

$$\|u^0\|_2 \le c\|f\|_0 \tag{5.2}$$

and, since $\Delta^2 u = \varepsilon^{-2}\Delta(u - u^0)$, it is a consequence of [5, Corollary 7.3.2.5] that

$$\|u\|_3 \le c\,\varepsilon^{-2}\|\Delta(u - u^0)\|_{-1} \le c\,\varepsilon^{-2}|(u - u^0)|_1. \tag{5.3}$$

Furthermore, from the weak formulations of the problems (1.1) and (4.15), and the fact that $u \in H_0^2 \cap H^3$, we derive that

$$\varepsilon^2(\Delta u, \Delta v) + (D(u - u^0), Dv) = \varepsilon^2 \int_{\partial\Omega} (\Delta u)\frac{\partial v}{\partial n}\,ds$$

for all $v \in H_0^1 \cap H^2$. In particular, by choosing $v = u - u^0$ we obtain

$$\varepsilon^2\|\Delta u\|_0^2 + |u - u^0|_1^2 \le -\varepsilon^2 \int_{\partial\Omega} (\Delta u)\frac{\partial u^0}{\partial n}\,ds - \varepsilon^2(\Delta u, f). \tag{5.4}$$

However,

$$\varepsilon^2|(\Delta u, f)| \le \frac{\varepsilon^2}{2}(\|\Delta u\|_0^2 + \|f\|_0^2). \tag{5.5}$$

Furthermore, standard trace inequalities and (5.2) imply

$$\int_{\partial\Omega} |\frac{\partial u^0}{\partial n}|^2\,ds \le c\,\|f\|_0^2$$

and

$$\int_{\partial\Omega} |\Delta u|^2\,ds \le c\,\|\Delta u\|_0\,\|\Delta u\|_1.$$

Hence, from the arithmetic geometric mean inequality we obtain that for any $\delta > 0$ there is a constant c_δ such that

$$\varepsilon^2 | \int_{\partial\Omega} (\Delta u)\frac{\partial u^0}{\partial n}\,ds| \le c_\delta \varepsilon \|f\|_0^2 + \delta\varepsilon^3\|\Delta u\|_0\|u\|_3. \tag{5.6}$$

However, from (5.3) we derive

$$\begin{aligned} \varepsilon^3\|\Delta u\|_0\|u\|_3 &\le \tfrac{1}{2}(\|\varepsilon^2\Delta u\|_0^2 + \varepsilon^4\|u\|_3^2) \\ &\le \tfrac{1}{2}\varepsilon^2\|\Delta u\|_0^2 + c\,|u - u^0|_1^2. \end{aligned} \tag{5.7}$$

The inequalities (5.4)–(5.7) lead to the bound

$$\varepsilon^2\|\Delta u\|_0^2 + |u - u^0|_1^2 \le c\,\varepsilon\|f\|_0^2,$$

and together with (5.3) this implies the desired estimates. ∎

The regularity result given in the previous lemma leads to the following uniform convergence property for the nonconforming finite element method (4.9).

Theorem 5.2. *Assume that $f \in L^2$ and $u \in H_0^2 \cap H^3$ is the corresponding weak solution of* (1.1). *Furthermore, let $u_h \in W_h$ be the solution of* (4.9). *Then there is a constant c, independent of ε, h and f, such that*

$$|||u - u_h|||_{\varepsilon,h} \le ch^{1/2}\|f\|_0.$$

Proof. As in the proof of Theorem 4.2 we start with the basic estmate (4.10), i.e., the second Strang lemma. Here, and below, c will denote a constant independent of ε, h and f. We first show that

$$\inf_{v\in W_h} |||u - v|||_{\varepsilon,h} \le |||u - I_h u|||_{\varepsilon,h} \le ch^{1/2}\|f\|_0. \tag{5.8}$$

From (4.3) and Lemma 5.1 we obtain

$$\begin{aligned}\varepsilon\|u - I_h u\|_2 &\le c\,\varepsilon\|u\|_2^{1/2}\|u - I_h u\|_2^{1/2}\\ &\le c\,\varepsilon h^{1/2}\|u\|_2^{1/2}|u|_3^{1/2}\\ &\le ch^{1/2}\|f\|_0.\end{aligned}$$

In order to estimate the H^1-part of the energy norm we use the triangle inequality to obtain

$$\|u - I_h u\|_1 \le \|u - u^0 - I_h(u - u^0)\|_1 + \|u^0 - I_h u^0\|_1.$$

From (4.4), (5.2) and Lemma 5.1 it follows that

$$\begin{aligned}\|u - u^0 - I_h(u - u^0)\|_1 &\le ch^{1/2}\|u - u^0\|_1^{1/2}\|u - u^0\|_2^{1/2}\\ &\le ch^{1/2}\|f\|_0,\end{aligned}$$

while (4.3) and (5.2) gives

$$\|u^0 - I_h u^0\|_1 \le ch\|u^0\|_2 \le ch\|f\|_0.$$

Hence, the interpolation estimate (5.8) is established.

Furthermore, it follows from (4.8) and (4.14) that the consistency error, $E_{\varepsilon,h}(u, w)$, is bounded by

$$E_{\varepsilon,h}(u, w) \le c\,\varepsilon h^{1/2}|u|_2^{1/2}|u|_3^{1/2}|||w|||_{\varepsilon,h}$$

for any $w \in W_h$. Hence, Lemma 5.1 implies that

$$E_{\varepsilon,h}(u, w) \le ch^{1/2}\|f\|_0|||w|||_{\varepsilon,h},$$

and together with (4.10) and (5.8) this completes the proof. □

Remark. It follows from (5.1) that the Sobolev norm $\|u\|_s$ will blow up as ε tends to zero for any $s > 3/2$. Therefore, since the energy norm $|||\cdot|||_{\varepsilon,h}$ bounds the H^1-norm, the uniform estimate given in Theorem 5.2 seems to be the best we can possibly obtain for any finite element method, even if we use a more complex conforming method.

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Department of Mathematics, University of Bergen, Johannes Brunsgt. 12, 5007 Bergen, Norway

E-mail address: Trygve.Nilssen@mi.uib.no

Department of Mathematics, University of Bergen, Johannes Brunsgt. 12, 5007 Bergen, Norway

E-mail address: Xue-Cheng.Tai@mi.uib.no

Department of Informatics and Department of Mathematics, University of Oslo, P.O. Box 1080 Blindern, 0316 Oslo, Norway

E-mail address: ragnar@ifi.uio.no