A Robust Scenario MPC Approach for Uncertain Multi-modal Obstacles

Ivo Batkovic, Ugo Rosolia, Mario Zanon, and Paolo Falcone

***Abstract*—Motion planning and control algorithms for autonomous vehicles need to be safe, and consider future movements of other road users to ensure collision-free trajectories. In this paper, we present a control scheme based on Model Predictive Control (MPC) with robust constraint satisfaction where the constraint uncertainty, stemming from the road users' behavior, is multimodal. The method combines ideas from tube-based and scenario-based MPC strategies in order to approximate the expected cost and to guarantee robust state and input constraint satisfaction. In particular, we design a feedback policy that is a function of the disturbance mode and allows the controller to take less conservative actions. The effectiveness of the proposed approach is illustrated through two numerical simulations, where we compare it against a standard robust MPC formulation.**

***Index Terms*—Autonomous vehicles, uncertain systems, predictive control for nonlinear systems**

I. Introduction

AUTONOMOUS driving technologies have shown great potential for safe and efficient driving [1]. While the technology is expected to be first gradually deployed for environments such as highway driving and low-speed parking [2], scenarios such as urban driving pose a greater challenge due to the presence of other road users, e.g., pedestrians, cyclists, and vehicles. To address these challenges, research needs to be focused on (a) deriving prediction methods to model the stochastic behavior of human road users, and (b) control algorithms that ensure safe collision-free trajectories.

To that end, it is of great importance to be able to understand and model the behavior of other road users. Several techniques have been proposed to address (a) in the context of autonomous driving, where in, e.g., [3], the authors presented a pedestrian path prediction study using Gaussian process dynamical models together with trajectory matching. Other approaches consider switching dynamical models for short-term predictions [4], and hybrid models to describe human gaits to switch between different dynamics [5]. More recently, data-driven methods involving Markov processes have been proposed [6], and in [7] environmental context in form of a semantic map was incorporated to guide the predictions. In [8] a graph-based map was leveraged in order to propagate predictions using a feedback controller, while in [9] predictions are propagated using set-based reachability analysis combined

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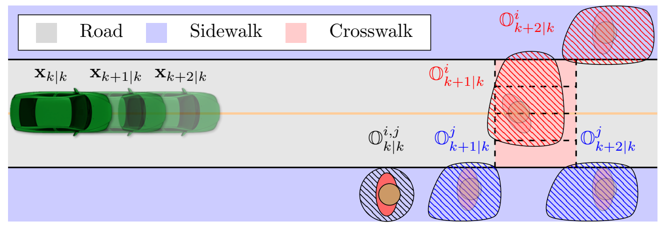

Fig. 1. Example of two prediction modes for an external obstacle.

with the structure of the road topology. However, in order to make use of predictions of the surrounding environment, control and planning algorithms need to be developed with robust safety guarantees against all possible uncertainty realizations. In [10] an optimization-based planning and control framework is proposed that enables safe control and avoids collisions with other traffic participants by assuming availability of vehicle-to-vehicle communication, while [11] proposes a partially observable Markov decision process that estimates the behavior of other road users to form collision avoidance constraints to be used for their controller. The hybrid controller presented in [12] solves the pedestrian interaction problem using gap acceptance to decide whether to yield or not, and in [13] the authors present a planning and control framework that accounts for moving obstacles by explicitly including predicted pedestrian trajectories in an optimization problem in order to be proactive. However, the framework addresses nominal driving scenarios, where an emergency layer is assumed to exist in order to ensure safety. Since safety is paramount in autonomous driving, it is necessary to ensure satisfaction of all constraints, e.g., road boundary limits, actuator limits, and collision avoidance with other road users for all future times.

In this paper, we design a feedback policy that is able to take less conservative actions, for cases when constraints are affected by multimodal uncertainties. This is typically the case for collision avoidance with other road users e.g., a pedestrian might either cross the road or continue walking, as depicted in Figure 1. In particular, we propose a causal feedback policy that is a function of the disturbance mode and prove robust constraint satisfaction by combining ideas introduced in tube-based Model Predictive Control (MPC) [14], [15], [16] and scenario-based MPC [17]. Furthermore, we show, through numerical simulations, the benefits of our control strategy compared to a standard robust tube MPC formulation.

The remainder of the paper is structured as follows. In Section II the optimal control problem is defined and the obstacle model is described in Section III. Section IV proposes a safe, non-conservative control scheme. We illustrate the theoretical developments in Section V with two numerical examples. Finally, we draw conclusions in Section VI.

II. Problem Definition

We consider the following discrete-time system

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k), \tag{1}$$

where at time k the state $\mathbf{x}_k \in \mathbb{X} \subseteq \mathbb{R}^{m_x}$ and control $\mathbf{u}_k \in \mathbb{U} \subseteq \mathbb{R}^{m_u}$. The goal is to design a controller, given an initial state $\mathbf{x}_\mathrm{S} \in \mathbb{S} \subseteq \mathbb{X}$, that minimizes deviations from a reference trajectory $\mathbf{r}_k = (\mathbf{r}^\mathbf{x}_k, \mathbf{r}^\mathbf{u}_k)$, while avoiding collisions with external dynamic obstacles $\mathbf{o}_k \in \mathbb{O} \subseteq \mathrm{R}^{m_o}$. Ideally, one would want to design a controller solving the infinite horizon Optimal Control Problem (OCP), where the future positions of the obstacle are assumed to be known a priori

$$\begin{aligned} J^\star_{0\to\infty}(x_\mathrm{S}) = \min_{\mathbf{u}} \ & \sum_{k=0}^{\infty} \ell(\mathbf{x}_k - \mathbf{r}^\mathbf{x}_k, \mathbf{u}_k - \mathbf{r}^\mathbf{u}_k) && \text{(2a)} \\ \text{s.t.} \ & \mathbf{x}_0 = \mathbf{x}_\mathrm{S} && \text{(2b)} \\ & \mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k), \qquad \forall k \geq 0, && \text{(2c)} \\ & h(\mathbf{x}_k, \pi(\mathbf{x}_k, \mathbf{o}_k)) \leq 0, \qquad \forall k \geq 0, && \text{(2d)} \\ & g(\mathbf{x}_k, \pi(\mathbf{x}_k, \mathbf{o}_k), \mathbf{o}_k) \leq 0, \quad \forall k \geq 0, && \text{(2e)} \end{aligned}$$

where constraints (2b) and (2c) represent the initial state and system dynamics, and constraint (2d) includes, e.g., state and control limits. Finally, constraint (2e) ensures collision avoidance with obstacles $\mathbf{o}_k$ for all future positions and times $k \geq 0$. We assume that the stage cost $\ell(\cdot,\cdot)$ is continuous positive-definite and $\ell(0, 0) = 0$.

Our goal is to guarantee the safety of the controller for all $\mathbf{x}_\mathrm{S}$ in an initial set $\mathbb{S}$, which we define formally as follows.

Definition 1 (Safety)**.** *A controller is safe in the set* $\mathbb{S}$*, if it generates control inputs* $\mathbf{U} = \{\mathbf{u}_0, ..., \mathbf{u}_\infty\}$ *and state trajectories* $\mathbf{X} = \{\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_\infty\}$*, such that* $h(\mathbf{x}_k, \mathbf{u}_k) \leq 0$ *and* $g(\mathbf{x}_k, \mathbf{u}_k, \mathbf{o}_k) \leq 0$*,* $\forall \mathbf{x}_0 \in \mathbb{S}$*,* $\forall k \geq 0$*,* $\forall \mathbf{o}_k \in \mathbb{O}$*.*

Solving problem (2) is challenging for the following reasons: (a) the exact knowledge of the future states $\mathbf{o}_k$ is generally impossible to know a priori; (b) solving an infinite-horizon OCP is computationally challenging, even though the exact solution can be computed for special cases, e.g., for linear systems, convex cost and constraints [18]; and (c) we want to optimize over feedback policies, which depend on the various configurations of $\mathbf{o}_k$ and system states $\mathbf{x}_k$.

In the following we first address issue (a) by defining a prediction model at time k, allowing the prediction of $\mathbf{o}_k$ for future times. Then, to tackle (b) we approximate the infinite horizon OCP (2) by a model predictive control strategy which defines a feasible, i.e., safe, receding horizon feedback policy. Finally, to overcome (c) we fix the structure of the feedback policy by solving only over a subset of feedback policies.

III. Environment Prediction Model

Since the full information of the future obstacles' states $\mathbf{o}_k$ are not known a priori, we use the notation $\mathbf{o}_{n|k}$ to denote the predicted obstacle state at time n given the information available at time k. We assume that we are given a multimodal prediction model for the obstacle. In particular, at each time k for each mode $i \in \{1, \dots, L\}$ the state of the obstacle is given by

$$\mathbf{o}^i_{n+1|k} = \omega(\mathbf{o}^i_{n|k}, \mathbf{w}^i_{n|k}), \tag{3}$$

where the random variable $\mathbf{w}^i_{n|k}$ is the uncertainty that drives the dynamics for mode i and has a compact support $\mathbb{W}^i_{n|k} \subset \mathbb{R}^{m_w}$, i.e., $\mathbf{w}^i_{n|k} \in \mathbb{W}^i_{n|k}$. We denote β_i as the probability associated with prediction mode i. Note that the different modes of our prediction model may be used to describe different intentions, e.g., a pedestrian can either choose to cross the road or continue walking, as depicted in Figure 1. In order to introduce safe constraint satisfaction, we need to predict the set of possible values that $\mathbf{o}_{n|k}$ can take. For each mode i, we denote such uncertainty set as $\mathbb{O}^i_{n|k}$ and $\mathbb{O}_{n|k} := \bigcup_{i=1}^{L} \mathbb{O}^i_{n|k}$. The following assumption guarantees that $\mathbf{o}_n \in \mathbb{O}_{n|k}, \ \forall \ n \geq k$.

Assumption 1. *The uncertainty sets* $\mathbb{O}^i_{n|k}$ *are outer approximations of the robust reachable sets given by* (3)*, i.e.,*

$$\mathbb{O}^i_{n+1|k} \supseteq \{\omega(\mathbf{o}^i_{n|k}, \mathbf{w}^i_{n|k}) \,|\, \mathbf{o}^i_{n|k} \in \mathbb{O}^i_{n|k}, \forall \mathbf{w}^i_{n|k} \in \mathbb{W}^i_{n|k}\},$$

for $\mathbb{O}^i_{k|k} = \mathbf{o}_k$*,* $\forall i \in \{1, ..., L\}$*.*

Note that for each mode i, the sets $\mathbb{O}^i_{n|k}$ can group different realizations of the same intention, e.g., pedestrians who cross the road might do so in multiple ways (faster, slightly on one side, etc.). Also, when the sets of two modes are disjoint it is possible to infer which mode cannot be realized anymore. This is formalized in the following proposition and allows us to define a causal feedback policy in Section IV-B, which is a function of the prediction mode.

Proposition 1 (Mode Distinction)**.** *Given a sequence of predicted obstacle states* $[\mathbf{o}^p_{k|k}, \dots, \mathbf{o}^p_{n|k}]$*,* $p \in \{i, j\}$*, one can infer whether* $p = i$ *or* $p = j$ *if* $n \geq \bar{n}_{ij}$*, with*

$$\bar{n}_{ij} := \min_{n} \ n, \ \text{s.t.} \ \mathbb{O}^i_{n|k} \cap \mathbb{O}^j_{n|k} = \emptyset, \qquad \forall \ n \geq k. \tag{4}$$

Proof. Definition (4) implies that $\mathbb{O}^i_{\bar{n}_{ij}|k} \cap \mathbb{O}^j_{\bar{n}_{ij}|k} = \emptyset$. Therefore, $\mathbf{o}^p_{\bar{n}_{ij}|k} \in \mathbb{O}^i_{\bar{n}_{ij}|k}$ if and only if the mode $p = i$. □

Remark 1. *The computation of the reachable set is not trivial for nonlinear systems. Standard strategies tackle this issue by computing overapproximations* $\mathbb{O}_{n|k}$*, see, e.g., [19], [20], [21], [22]. The computation of* $\bar{n}_{ij}$ *requires one to perform set intersection, which can be done as in, e.g., [23], [24]. Unfortunately, such computations are unavoidable if safety is sought. Note, however, that computationally efficient approaches have been proposed in the literature. Since this is outside the scope of this paper, future research will adapt the computationally inexpensive methods of [8], [13] such that they satisfy Assumption 1.*

Remark 2. *We assume that the probabilities* β_i *are given. In practice they can be estimated or learned from sensor data.*

IV. Controller Design

In this section, we first show that a feasible policy for problem (2) can be synthesized using robust tube MPC strategies [15]. However, in case the predicted states $\mathbf{o}_{n|k}$ are multimodal, the robust tube approach can be conservative. Therefore, in the second part of this section we present a control strategy which leverages Proposition 1: we use a causal feedback policy which considers different scenarios as a function of the mode of the prediction model. As we will show in Section V, the proposed strategy allows us to achieve better performance on average compared to standard robust tube MPC, while having a comparable computational cost.

A. Tube MPC approximation

A simple, yet often conservative, solution is to approximate problem (2) with a standard tube MPC formulation that optimizes over a sequence of controls rather than policies. More formally, given the measured state $\mathbf{x}_k$, we solve the following finite time optimal control problem

$$\min_{\mathbf{x},\mathbf{u}} \sum_{n=k}^{k+N-1} q(\mathbf{x}_{n|k}-\mathbf{r}^{\mathbf{x}}_{n|k}, \mathbf{u}_{n|k}-\mathbf{r}^{\mathbf{u}}_{n|k}) + p(\mathbf{x}_{k+N|k}-\mathbf{r}^{\mathbf{x}}_{k+N|k}) \quad (5a)$$

$$\text{s.t.} \quad \mathbf{x}_{k|k} = \mathbf{x}_k, \quad (5b)$$

$$\mathbf{x}_{n+1|k} = f(\mathbf{x}_{n|k}, \mathbf{u}_{n|k}), \quad (5c)$$

$$h(\mathbf{x}_{n|k}, \mathbf{u}_{n|k}) \le 0, \quad (5d)$$

$$g(\mathbf{x}_{n|k}, \mathbf{u}_{n|k}, \mathbf{o}_{n|k}) \le 0, \quad \forall \mathbf{o}_{n|k} \in \mathbb{O}_{n|k}, \quad (5e)$$

$$[\mathbf{x}^\top_{k+N|k},\ \mathbf{o}^\top_{k+N|k}]^\top \in \mathcal{X}^{\mathrm{f}}_{\mathbf{r}}, \quad \forall \mathbf{o}_{k+N|k} \in \mathbb{O}_{k+N|k}. \quad (5f)$$

where N is the prediction horizon, and q and p are the stage and terminal costs, and constraint (5f) is a terminal set. We denote the optimal sequence of problem (5) to be $\mathbb{U}^\star_k = \{\mathbf{u}^\star_{k|k}, \ldots, \mathbf{u}^\star_{k+N-1|k}\}$ at time k, with the policy

$$\pi^{\text{tube}}(\mathbf{x}_k) := \mathbf{u}^*_{k|k}. \quad (6)$$

Problem (5) is then solved at time $k+1$ based on the new state $\mathbf{x}_{k+1} = f(\mathbf{x}_k, \pi^{\text{tube}}(\mathbf{x}_k))$, yielding a receding horizon control strategy. While problem (5) is formulated to guarantee robust constraint satisfaction [16], it can be conservative for scenarios where the predictions stemming from $\mathbf{o}_k$ become multimodal. Therefore, to reduce conservativeness we introduce our proposed control strategy in the following section.

Remark 3. *Since satisfying the constraints with probability* 1 *might be conservative, if a certain degree of risk can be accepted one can define sets $\mathbb{W}_{n|k}$ such that they correspond to the desired probability of constraint satisfaction, i.e., in the presence of probability information of the uncertainties, one can formulate* (5) *as a chance-constrained (risk bounded) problem which results in a less conservative solution [25].*

B. Scenario Expectation Strategy

As discussed previously, the solution to problem (5) is robust for all possible realizations of the obstacle position. We propose to leverage the result from Proposition 1 to design causal time-varying feedback policies, which are a function of the obstacle mode i. Note that this design of the feedback policy follows a similar principle in [17]. Furthermore, this strategy allows us to approximate the expected cost, which depends on the random position of the obstacle, thus allowing for improvements of the closed-loop performance, as we will show in Section V.

At predicted time n, we allow the controller to pick different control actions if the prediction mode can be inferred from the obstacle's positions. In the following, we denote by $\mathbf{u}^i_{n|k}$ the control input for mode i at time n predicted at time k, and we constrain the predicted inputs of different modes to be equal when the mode cannot be distinguished from past obstacle positions, i.e., we set $\mathbf{u}^i_{n|k} = \mathbf{u}^j_{n|k}$ when $n < \bar{n}_{ij}$, for $\bar{n}_{ij}$ defined from Proposition 1. The resulting finite time optimal control problem is then formulated as

$$\min_{\substack{\mathbf{x}^i,\mathbf{u}^i \\ \forall i \in \mathbb{I}_1^L}} \sum_{i=1}^{L} \beta_i \Big(\sum_{n=k}^{k+N-1} q(\mathbf{x}^i_{n|k}-\mathbf{r}^{\mathbf{x},i}_k, \mathbf{u}^i_{n|k}-\mathbf{r}^{\mathbf{u},i}_k) + p(\mathbf{x}^i_{k+N|k}-\mathbf{r}^{\mathbf{x},i}_{k+N|k}) \Big) \quad (7a)$$

$$\text{s.t.} \quad \mathbf{x}^i_{k|k} = \mathbf{x}_k, \quad \forall i \in \mathbb{I}_1^L, \quad (7b)$$

$$\mathbf{x}^i_{n+1|k} = f(\mathbf{x}^i_{n|k}, \mathbf{u}^i_{n|k}), \quad \forall i \in \mathbb{I}_1^L, \quad (7c)$$

$$h(\mathbf{x}^i_{n|k}, \mathbf{u}^i_{n|k}) \le 0, \quad \forall i \in \mathbb{I}_1^L, \quad (7d)$$

$$g(\mathbf{x}^i_{n|k}, \mathbf{u}^i_{n|k}, \mathbf{o}^i_{n|k}) \le 0, \quad \forall \mathbf{o}^i_{n|k} \in \mathbb{O}^i_{n|k}, \forall i \in \mathbb{I}_1^L, \quad (7e)$$

$$\mathbf{u}^i_{n|k} = \mathbf{u}^j_{n|k}, \quad \forall n < \bar{n}_{ij},\ \forall i,j \in \mathbb{I}_1^L, \quad (7f)$$

$$[\mathbf{x}^{i\top}_{k+N|k},\ \mathbf{o}^{i\top}_{k+N|k}]^\top \in \mathcal{X}^{\mathrm{f}}_{\mathbf{r}}, \quad \forall i \in \mathbb{I}_1^L, \quad (7g)$$

$$(7c) - (7f) \quad \forall n \in \mathbb{I}_k^{k+N-1},$$

where we use $\mathbb{I}_a^b := \{a, a+1, ..., b\}$ as a short-hand notation. We denote the optimal solution to problem (7) as $\mathbb{U}^\star_k = \{\mathbf{u}^{1\star}_{k|k}, \ldots, \mathbf{u}^{1\star}_{k+N|k}, \ldots, \mathbf{u}^{L\star}_{k|k}, \ldots, \mathbf{u}^{L\star}_{k+N|k}\}$, and define the feedback policy at time k as

$$\pi^{\text{scenario}}(\mathbf{x}_k, \mathbf{o}_k) := \mathbf{u}^{1\star}_{k|k}. \quad (8)$$

At time $k+1$ we solve (7) with initial state $\mathbf{x}_{k+1} = f(\mathbf{x}_k, \pi^{\text{scenario}}(\mathbf{x}_k, \mathbf{o}_k))$ in a receding horizon fashion. By Assumption 1, $\mathbb{O}^i_{k|k} = \boldsymbol{o}_k,\ \forall i \in \mathbb{I}_1^L$, therefore $\mathbf{u}^{i\star}_{k|k} = \mathbf{u}^{j\star}_{k|k}$, $\forall i,j \in \mathbb{I}_1^L$ and the policy (8) is well-defined.

Note that this formulation does not optimize over L separate trajectories for L scenarios. Instead, it groups the L scenarios together in the same problem, where the control input ties together the scenarios and trajectories through constraint (7f). Since β_i denotes the probability associated with prediction mode i, i.e., $\sum_{i=1}^{L} \beta_i = 1$, the objective (7a) therefore becomes an approximation of the expected cost. Consequently, problem (7) minimizes the expected cost, and on average performs better than problem (5). Since all possible modes are accounted for, this cost reduction still preserves safety: if, e.g., a pedestrian crossing is very unlikely, the vehicle will make sure to always be able to stop without slowing down more than necessary. By Proposition 1, collision avoidance constraints related to some mode are removed from the problem only once the obstacle cannot be in that mode anymore. E.g., once a pedestrian starts crossing it cannot keep walking on the sidewalk it was on.

Note that problem (7) will in general not be solved in its form, but rather reformulated as a robust MPC problem through constraint tightening [16]. This essentially requires knowledge on the global maximizer of g, which can be hard to compute in the general case. However, in some cases this computation can be made very efficient. Notable examples include the accurate pedestrian models proposed in [8], [13].

Remark 4. *While problem* (7) *can become computationally intractable in the presence of many obstacles, the problem can be massively simplified by carefully identifying the most restricting mode. Note that modes can be used to construct piecewise linear approximations of nonlinear dynamics* (3) *and, therefore, make the worst-case computation computationally tractable, as in [8], [13]. These topics are the subject of ongoing research.*

Remark 5. *The solution to problem* (7) *coincides with problem* (5) *whenever* $\bar{n}_{ij} = N$.

Problem (7) establishes the first step towards the desired results: it proposes the policy (8) that is an approximation to the optimal policy from OCP (2), and is on average less conservative, in terms of cost, than the standard tube MPC formulation (5). However, since the policy in OCP (2) is by definition safe, i.e., recursively feasible, we use standard approaches in MPC [18], [26], [27], which assume the existence of a robust invariant terminal set. Hence, the following assumptions are introduced to prove safety also for policy (8).

Assumption 2. $\forall i \in \{1, \ldots, L\}$ *there exists a* $j \in \{1, \ldots, L\}$ *such that* $\mathbb{O}^i_{n|k} \subseteq \mathbb{O}^j_{n|k-1}$*, for all* $n \geq k$*. Furthermore,* $\boldsymbol{o}_k \in \mathbb{O}_{k|k-1}\ \forall\, k \geq 0$.

Assumption 3. *There exists a robust invariant set* $\mathcal{X}_{\text{safe}}$*, i.e., if* $[\mathbf{x}^\top_{k+N|k}, \mathbf{o}^\top_{k+N|k}]^\top \in \mathcal{X}_{\text{safe}}$*, then* $\exists\, \mathbf{u}$ *such that* $[f(\mathbf{x}_{k+N|k}, \mathbf{u})^\top, \mathbf{o}^\top_{k+N+1|k}]^\top \in \mathcal{X}_{\text{safe}},\ \forall\, \mathbf{o}_{k+N+1|k} \in \mathbb{O}_{k+N+1|k}$*, and* $g(\mathbf{x}_{k+N|k}, \mathbf{u}, \mathbf{o}_{k+N|k}) \leq 0$.

This assumption postulates the existence of a safe set ensuring constraint satisfaction. While this might seem strong, it is often implicitly used in many situations in the form

$$\begin{aligned}\mathcal{X}_{\text{safe}} := \{\mathbf{x}_{n|k},\ \mathbf{o}_{n|k} \mid \mathbf{x}_{n|k} = f(\mathbf{x}_{n|k}, \mathbf{u}_{n|k}),\\ h(\mathbf{x}_{n|k}, \mathbf{u}_{n|k}) \leq 0,\ g(\mathbf{x}_{n|k}, \mathbf{u}_{n|k}, \mathbf{o}_{n|k}) \leq 0\}.\end{aligned} \tag{9}$$

Note that the prediction model can be constructed to avoid unrealistic behavior that would cause the controller not to be feasible. For instance, the prediction model can be constructed to include the following cases: (a) a parked vehicle is considered safe and not responsible for collisions with other road users; (b) a robotic joint moving in a human-robot environment is considered safe if it does not move; and (c) an electric circuit which is switched off is safe.

Proposition 2 (Recursive feasibility)**.** *Suppose that Assumptions 2 and 3 hold, and that problem* (7)*, formulated with* $\mathcal{X}^{\text{f}}_{\mathbf{r}} = \mathcal{X}_{\text{safe}}$*, is feasible for the initial state* $\mathbf{x}_0$*. Then system* (1) *in closed loop with the solution of* (7) *applied in receding horizon is recursively feasible.*

Proof. The proof follows from Assumptions 2-3, Proposition 1 and standard MPC arguments [17], [18]. Assume that problem (7) is feasible at time k, and let

$$\mathbb{U}^\star_k = \{\mathbf{u}^{1\star}_{k|k}, \ldots, \mathbf{u}^{1\star}_{k+N|k}, \ldots, \mathbf{u}^{L\star}_{k|k}, \ldots, \mathbf{u}^{L\star}_{k+N|k}\} \tag{10}$$

be the optimal solution. Now, notice that by Assumption 2 $\forall i \in \{1, \ldots, L\}$ there exists a $j_i \in \{1, \ldots, L\}$ such that

$$\mathbb{O}^i_{n|k+1} \subseteq \mathbb{O}^{j_i}_{n|k},\ \forall n \geq k+1. \tag{11}$$

Therefore, by Assumption 3, we have that for some $u^{j_i}_c \in \mathbb{U}$ the following candidate input sequence

$$\{\mathbf{u}^{j_1\star}_{k+1|k}, ..., \mathbf{u}^{j_1\star}_{k+N|k}, u^{j_1}_c, ..., \mathbf{u}^{j_L\star}_{k+1|k}, ..., \mathbf{u}^{j_L\star}_{k+N|k}, u^{j_L}_c\} \tag{12}$$

is a feasible solution to problem (7) at time $k+1$. We have just shown that problem (7) is feasible at time $k+1$ if problem (7) is feasible at time k. Finally, we have by assumption that Problem (7) is feasible at time $k = 0$, therefore we conclude by induction that problem (7) is feasible at time $k \geq 0$. ☐

While we only focus on safety, i.e., recursive feasibility of problem (7), it seems possible to also prove some form of stability, similarly to [28].

We will show next in numerical simulations that this strategy performs better than standard robust MPC on average.

V. Simulations

In this section we propose two examples in order to illustrate the performance of our proposed strategy. We define the state and controls to be $\mathbf{x} = [p,\ \dot{p}]^\top$, $\mathbf{u} = a$, where p and $\dot{p}$ denote a scalar position and velocity, and a the acceleration. The constraints are $\dot{p} \geq 0$ and $|a| \leq 5$ together with the collision avoidance constraint $g(\mathbf{x}, \mathbf{u}, \mathbf{o}) \leq 0$. We are given the reference $\mathbf{r}^{\mathbf{x}}(t) = (t\dot{p}^{\text{ref}}, \dot{p}^{\text{ref}})$, $\mathbf{r}^{\mathbf{u}}(t) = 0$, with constant velocity reference $\dot{p}^{\text{ref}} = 5$m/s. All examples are framed in a standard tracking MPC formulation with stage and terminal cost functions defined as $q(\mathbf{x}, \mathbf{u}) := \mathbf{x}^\top Q\mathbf{x} + \mathbf{u}^\top R\mathbf{u}$, and $p(\mathbf{x}) := \mathbf{x}^\top P\mathbf{x}$, where $Q = \text{diag}(0, 10)$, $R = 1$, and P is the LQR cost-to-go matrix. We use sampling time $t_{\text{s}} = 0.1$s and prediction horizon $N = 90$. The safe set is defined as per (9) using the approach of [28] and we solve the MPC problem using CasADi [29] and IPOPT [30]. We select simple but accurate pedestrian models [8] in order to ease the interpretation of the results while considering realistic situations.

A. Uncertain Static Obstacle

To illustrate the closed-loop behavior for our proposed method, we consider a scenario where an obstacle $\mathbf{o}_k := p^{\text{obs}}$ is positioned at $p^{\text{obs}} = 20$m. We assume that the obstacle may disappear after 6s, hence we consider only two modes $i \in \{1, 2\}$. The constraint function g can then be defined as

$$g(\mathbf{x}^i_{n|k}, \mathbf{u}^i_{n|k}, \mathbf{o}^i_{n|k}) = p^i_{n|k} - \mathbf{o}^{\text{i}}_{n|k} \leq 0,$$

where the true obstacle dynamics are $\boldsymbol{o}_k = 20$ for $t \leq 6$s and

$$\boldsymbol{o}_k = \begin{cases} 20 & \text{with probability } \gamma, \\ \infty & \text{with probability } 1-\gamma, \end{cases} \qquad t > 6\text{s},$$

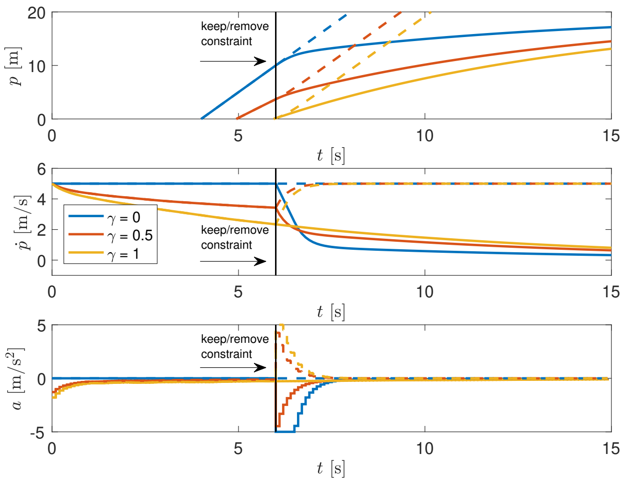

Fig. 2. Closed loop trajectories for our proposed scenario MPC strategy, with an obstacle at $p = 20$. Different lines show the behavior for varying prediction probabilities γ. The dashed lines show the closed loop behavior when the constraint is lifted after $t = 6$s.

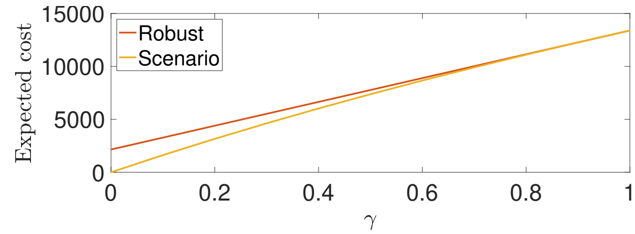

Fig. 3. Expected cost for robust tube MPC and the proposed strategy.

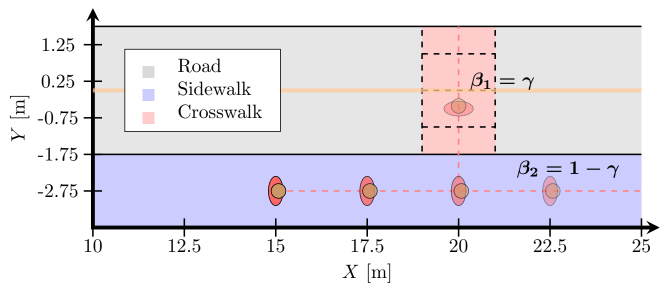

Fig. 4. Simplified pedestrian scenario. The pedestrian is predicted to move along the dashed lines and turn left with probability $\beta_1 = \gamma$ and continue straight with probability $\beta_2 = 1 - \gamma$.

i.e., $\beta_1 = \gamma$, $\beta_2 = 1 - \gamma$. For $t \leq 6$s we define the sets

$$\mathbb{O}^1_{n|k} = \{20\},\ \forall n \geq 0, \quad \mathbb{O}^2_{n|k} = \begin{cases} \{20\}, & \text{if } nt_s \leq 6 \\ \{\infty\} & \text{otherwise} \end{cases},$$

and, for $t > 6$s, the sets

$$\mathbb{O}^1_{n|k} = \mathbb{O}^2_{n|k} = \begin{cases} \{20\} & \text{If mode 1 is active} \\ \{\infty\} & \text{If mode 2 is active} \end{cases}.$$

For $t > 6$s the obstacle position $\boldsymbol{o}_k$ allows us to infer if the obstacle is in mode 1 or 2.

Figure 2 shows closed-loop trajectories for the initial state $\mathbf{x}_0 = [-20, 5]^\top$. The effect of having $\gamma = 0$, i.e., assuming that there is zero probability for the constraint to be present in mode $i = 1$ after 6s, results in having closed-loop trajectories that become very aggressive and only slow down after the mode can be distinguished. Alternatively, for $\gamma = 1$, the closed loop trajectories become instead conservative and approach the obstacle p^{obs} smoothly. Note, that for $\gamma = 1$, the proposed solution for the setting coincides with a standard robust MPC strategy, i.e., the controller defined in Section IV-A. For $\gamma = 0.5$, the controller tries to balance the aggressiveness and conservatism from the other two behaviors, hence, keeping a faster velocity compared to the case when $\gamma = 1$, but also decelerating earlier compared to when $\gamma = 0$.

The dashed lines in Figure 2 show the closed loop behavior after $t \geq 6$ when mode $i = 2$ is active, i.e., the obstacle disappeared. Since the controller with $\gamma = 0$ is optimistic and will only decelerate when truly necessary, it does not change its velocity. However, for $\gamma > 0$ the controller becomes more conservative, and therefore approach the constraint at a reduced velocity. This is also visible from Figure 3 that compares the expected cost for our proposed approach with a standard robust tube MPC. We discuss the expected cost results in more detail for the example in the following section.

B. Uncertain Dynamic Obstacle

To compare the overall closed-loop performance, we use a more involved scenario, where we consider the setting with a moving obstacle displayed in Figure 4, where the reference $\mathbf{r}^{\mathbf{x}}(t)$ is shown by the orange line centered around $y = 0$.

The pedestrian states are modeled as $\mathbf{o}_k := [x_k,\ y_k]^\top$, with x_k and y_k being the positions in the global frame shown in Figure 4. We assume that as the pedestrian reaches $x = 20$m, it might turn and cross the road, or continue walking along the straight line $y = -2.75$m. Therefore, for a measurement $\mathbf{o}_k$, we predict the two modes of the pedestrian using

$$\mathbf{o}^i_{k+1} = \omega(\mathbf{o}^i_k, \mathbf{w}_k) = \begin{bmatrix} x^i_{n|k} \\ y^i_{n|k} \end{bmatrix} + \begin{bmatrix} t_s \cos(\mathbf{w}^i(x^i_k)) \\ t_s \sin(\mathbf{w}^i(x^i_k)) \end{bmatrix}, \tag{13}$$

where $i \in \{1, 2\}$ and

$$\mathbf{w}^1(x_k) = \begin{cases} 0 & \text{if } x_k < 20 \\ \frac{\pi}{2} & \text{otherwise} \end{cases}, \qquad \mathbf{w}^2(x_k) = 0. \tag{14}$$

For $x_k < 20$ we define two sets $\mathbb{O}^1_{n|k}$ and $\mathbb{O}^2_{n|k}$, which contain all possible evolutions of the obstacle. Note that this model is a piecewise linear approximation of the nonlinear model (13), whose accuracy has been demonstrated in [8]. When $x_k \geq 20$, the active mode becomes known and $\mathbb{O}^1_{n|k} = \mathbb{O}^2_{n|k}$ contains the only active mode. The constraint function g is then set as

$$g(\mathbf{x}^i_{n|k}, \mathbf{u}^i_{n|k}, \mathbf{o}^i_{n|k}) := \begin{cases} p^i_{n|k} - x^i_{n|k} - r & \text{if } |y^i_{n|k}| \leq \Delta + r, \\ -\infty & \text{otherwise}, \end{cases}$$

where $r = 1$m is an increased safety margin around the pedestrian, and $\Delta = 1.5$m is a distance threshold from the road deciding when the pedestrian should be considered for collision avoidance. Similarly to Section V-A, we set the probabilities of the two modes as $\beta_1 = \gamma$ and $\beta_2 = 1 - \gamma$.

We run simulations where we sweep $\gamma \in [0, 1]$ and compute the expected cost of the closed loop trajectories, using the stage cost $q(\mathbf{x} - \mathbf{r}^{\mathbf{x}}, \mathbf{u} - \mathbf{r}^{\mathbf{u}})$. We compare the expected cost for our proposed scenario MPC, a tube MPC, i.e., problem (5), and a

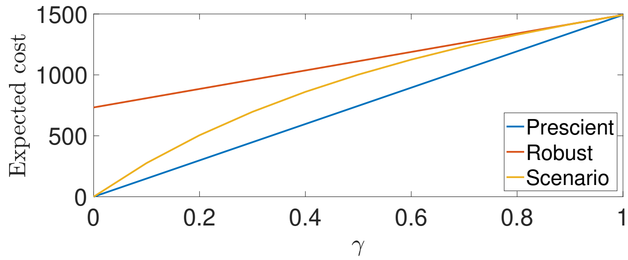

Fig. 5. Expected cost for a prescient MPC, robust tube MPC, and our proposed scenario MPC for different crossing probabilities γ.

prescient MPC, i.e., a controller that knows all constraints g a priori and provides a lower bound on the expected cost.

Figure 5 displays the expected cost for the three different methods with initial state $\mathbf{x}_0 = [-20, 5]^\top$, and initial obstacle state $\mathbf{o}_0 = [15, -2.75]^\top$. The prescient controller results in the lowest expected cost, since it knows beforehand whether a constraint will be present or not in the future. For low crossing probabilities γ, it is visible that the scenario MPC results in a much lower expected cost than the conservative robust MPC. In particular, for very small probabilities γ, the cost for scenario MPC is close to the optimal prescient MPC cost. However, for higher values of γ, the robust MPC is not much more conservative than the scenario MPC. Finally, for $\gamma = 1$, the pedestrian always crosses, and the solutions and expected cost coincide for all controllers.

VI. Conclusions

In this paper, we introduced an MPC scheme that guarantees collision avoidance in the presence of obstacles. In particular, we designed a feedback policy as function of the obstacle mode that on average performs at least as good as a standard robust MPC, and plans a sequence of control actions that guarantee constraint satisfaction regardless of the obstacle trajectory. We evaluated the proposed control scheme in simulations, and showed that our strategy outperforms a standard robust MPC formulation in terms of expected cost. Future work will focus on finding suitable approximations to contain the increasing computational complexity for multi-obstacle settings.

References

[1] J. Ziegler, P. Bender, M. Schreiber, H. Lategahn, T. Strauss, C. Stiller, T. Dang, U. Franke, N. Appenrodt, C. G. Keller *et al.*, "Making bertha drive - an autonomous journey on a historic route," *IEEE Intelligent Transportation Systems Magazine*, vol. 6, no. 2, pp. 8–20, 2014.

[2] J. Becker, M.-B. A. Colas, S. Nordbruch, and M. Fausten, "Boschs vision and roadmap toward fully autonomous driving," in *Road vehicle automation*. Springer, 2014, pp. 49–59.

[3] C. G. Keller and D. M. Gavrila, "Will the pedestrian cross? a study on pedestrian path prediction," *IEEE Transactions on Intelligent Transportation Systems*, vol. 15, no. 2, pp. 494–506, 2013.

[4] N. Schneider and D. M. Gavrila, "Pedestrian path prediction with recursive bayesian filters: A comparative study," in *German Conference on Pattern Recognition*. Springer, 2013, pp. 174–183.

[5] A. Bissacco and S. Soatto, "Hybrid dynamical models of human motion for the recognition of human gaits," *International journal of computer vision*, vol. 85, no. 1, pp. 101–114, 2009.

[6] A. Toytziaridis, P. Falcone, and J. Sjöberg, "A data-driven markovian framework for multi-agent pedestrian collision risk prediction," in *2019 IEEE Intelligent Transportation Systems Conference (ITSC)*. IEEE, 2019, pp. 777–782.

[7] V. Karasev, A. Ayvaci, B. Heisele, and S. Soatto, "Intent-aware long-term prediction of pedestrian motion," in *IEEE International Conference on Robotics and Automation*, 2016, pp. 2543–2549.

[8] I. Batkovic, M. Zanon, N. Lubbe, and P. Falcone, "A computationally efficient model for pedestrian motion prediction," in *2018 European Control Conference (ECC)*, June 2018, pp. 374–379.

[9] M. Koschi, C. Pek, M. Beikirch, and M. Althoff, "Set-based prediction of pedestrians in urban environments considering formalized traffic rules," in *2018 21st International Conference on Intelligent Transportation Systems (ITSC)*. IEEE, 2018, pp. 2704–2711.

[10] T. Weiskircher, Q. Wang, and B. Ayalew, "Predictive guidance and control framework for (semi-) autonomous vehicles in public traffic," *IEEE Transactions on control systems technology*, vol. 25, no. 6, pp. 2034–2046, 2017.

[11] B. Zhou, W. Schwarting, D. Rus, and J. Alonso-Mora, "Joint multi-policy behavior estimation and receding-horizon trajectory planning for automated urban driving," in *2018 IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2018, pp. 2388–2394.

[12] N. R. Kapania, V. Govindarajan, F. Borrelli, and J. C. Gerdes, "A hybrid control design for autonomous vehicles at uncontrolled crosswalks," in *2019 IEEE Intelligent Vehicles Symposium (IV)*. IEEE, 2019, pp. 1604–1611.

[13] I. Batkovic, M. Zanon, M. Ali, and P. Falcone, "Real-time constrained trajectory planning and vehicle control for proactive autonomous driving with road users," in *2019 18th European Control Conference (ECC)*. IEEE, 2019, pp. 256–262.

[14] A. Bemporad and M. Morari, "Robust model predictive control: A survey," in *Robustness in identification and control*. Springer, 1999, pp. 207–226.

[15] W. Langson, I. Chryssochoos, S. V. Rakovi, and D. Mayne, "Robust model predictive control using tubes," *Automatica*, vol. 40, pp. 125–133, 01 2004.

[16] L. Chisci, J. A. Rossiter, and G. Zappa, "Systems with persistent disturbances: predictive control with restricted constraints," *Automatica*, vol. 37, no. 7, pp. 1019–1028, 2001.

[17] P. O. Scokaert and D. Mayne, "Min-max feedback model predictive control for constrained linear systems," *IEEE Transactions on Automatic control*, vol. 43, no. 8, pp. 1136–1142, 1998.

[18] F. Borrelli, A. Bemporad, and M. Morari, *Predictive control for linear and hybrid systems*. Cambridge University Press, 2017.

[19] M. Althoff and J. M. Dolan, "Online verification of automated road vehicles using reachability analysis," *IEEE Transactions on Robotics*, vol. 30, no. 4, pp. 903–918, 2014.

[20] E. Asarin, T. Dang, and A. Girard, "Reachability analysis of nonlinear systems using conservative approximation," in *International Workshop on Hybrid Systems: Computation and Control*. Springer, 2003, pp. 20–35.

[21] M. Althoff, O. Stursberg, and M. Buss, "Reachability analysis of nonlinear systems with uncertain parameters using conservative linearization," in *2008 47th IEEE Conference on Decision and Control*. IEEE, 2008, pp. 4042–4048.

[22] J. K. Scott and P. I. Barton, "Bounds on the reachable sets of nonlinear control systems," *Automatica*, vol. 49, no. 1, pp. 93–100, 2013.

[23] M. I. Shamos and D. Hoey, "Geometric intersection problems," in *17th Annual Symposium on Foundations of Computer Science (sfcs 1976)*. IEEE, 1976, pp. 208–215.

[24] D. P. Dobkin and D. G. Kirkpatrick, "Fast detection of polyhedral intersection," *Theoretical Computer Science*, vol. 27, no. 3, pp. 241–253, 1983.

[25] A. Wang, X. Huang, A. Jasour, and B. Williams, "Fast risk assessment for autonomous vehicles using learned models of agent futures," *arXiv preprint arXiv:2005.13458*, 2020.

[26] E. C. Kerrigan, "Robust constraint satisfaction: Invariant sets and predictive control," Ph.D. dissertation, University of Cambridge, 2001.

[27] S. Yu, C. Maier, H. Chen, and F. Allgöwer, "Tube mpc scheme based on robust control invariant set with application to lipschitz nonlinear systems," *Systems & Control Letters*, vol. 62, no. 2, pp. 194–200, 2013.

[28] I. Batkovic, M. Ali, P. Falcone, and M. Zanon, "Safe trajectory tracking in uncertain environments," *Submitted to IEEE Transactions on Automatic Control. Preprint https://arxiv.org/abs/2001.11602*, 2020.

[29] J. A. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl, "Casadi: a software framework for nonlinear optimization and optimal control," *Mathematical Programming Computation*, vol. 11, no. 1, pp. 1–36, 2019.

[30] A. Wächter and L. T. Biegler, "On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming," *Mathematical programming*, vol. 106, no. 1, pp. 25–57, 2006.