Progress of Theoretical Physics, Vol. 46, No. 3, September 1971

A Role of the Uncertainty Principle in General Relativity and the Limiting Size of Collapsing Fermion Spheres

Noboru HOKKYO

*Hitachi Atomic Energy Research Laboratory, Ozenji, Kawasaki**)

(Received March 20, 1971)

An attempt is made to modify the Schwarzschild metric by the uncertainty principle in space regions of the linear size of the order of the Planck length $L^*=(\hbar G/c^3)^{1/2}\sim10^{-33}$ cm, and the role of the modified metric in avoiding the unlimited gravitational collapse of superdense Fermion spheres is examined. It is seen that the effect of the uncertainty principle is to introduce, at the center of symmetry of the system, a "repulsive hole" in which the matter is energetically unstable against the escape to outer regions. The radius of this hole for a system of mass M and radius R is seen to be $L^*(R/R_{gr})^{1/2}$, where $R_{gr}=2GM/c^2$ is the gravitational radius associated with the mass M. The limiting radius and mass of the collapsing extreme relativistic Fermi gas are roughly given by $N^{2/5}L^*$ and $N^{2/5}m^*$, respectively, where N is the number of particles in the gas and $m^*=\hbar/cL^*=(\hbar c/G)^{1/2}\sim10^{-5}$ g is the Planck mass.

§ 1. Introduction

The equilibrium of a massive cold ideal Fermi gas in its own gravitational field has been studied by Landau,[1] Oppenheimer and Volkoff[2] and, more recently, by Zel'dovich.[3),4)] It has been shown that there exists no stable equilibrium configuration of the gas for masses greater than a certain critical value, all larger masses tending to collapse. It was further indicated that for any given number N of particles one can obtain a configuration with mass as close to zero as one pleases by prescribing a sufficiently high particle density (of the order of the Planck density $\sim10^{94}$ g/cm^3 for small N).[3),4),5)] Such a configuration cannot go over into the state of the equilibrium (e.g., into the static solutions of Oppenheimer and Volkoff with $N<0.75N_{\odot}$), and can only contract without limit.

In a previous paper[6)] the idea of wavelike geodesics (the idea of using the wavelike test particle to determine the geodesic structure of space-time) was suggested in an attempt to imbed Nambu's mass spectrum or the elementary length of the order of 10^{-13} cm into space-time geometry. In the present paper we apply the idea of using the wavelike test particle to the modification of the Schwarzschild metric at distances of the order of the Planck length $\sim10^{-33}$ cm, and examine how the modified metric changes the classical (unquantized) general relativity picture of the unlimited gravitational collapse of superdense Fermion spheres.

*) Present address: Hitachi Information Systems Laboratory, Totsuka, Yokohama.

§ 2. Modification of the Schwarzschild metric by the uncertainty principle

There have been attempts[7)~13)] to derive the Schwarzschild line element using only three postulates of special relativity, the equivalence principle and the Newton's law of gravitation. From a strict logical stand point the simultaneous use of these postulates is self-contradictory.*) Nevertheless, it is remarkable that the correct form of the line element is obtainable for the specific example of the Schwarzschild field**) from a combination of conceptually inconsistent postulates by the addition of an extra postulate that the purely radial acceleration of a particle is a function only of its distance from the center of gravity.[13),12)]

In order to see qualitatively how the Newton's law modified to the Einstein's law can be further modified to be consistent with the uncertainty principle, we attempt, in this section, to modify the Schwarzschild field from the above three postulates plus the correspondence principle. Now the line element of the Schwarzschild field generated by a spherically symmetric object of mass M in an otherwise empty space is

$$ds^2 = a(r)c^2dt^2 - a^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

with

$$a(r) = 1 - 2GM/c^2r\,. \tag{2}$$

The simplified derivations of this line element are based on the observation that $a^{-1}(r)$ has the form of the square of the Einstein dilatation factor and $a(r)$ has the form of the squared Lorentz contraction factor:

$$a(r) = 1 - v^2(r)/c^2. \tag{3}$$

Here $v(r)$ is the velocity of a test particle (of mass m) freely falling from rest at $r = \infty$ towards M at $r = 0$. According to the equivalence principle a coordinate system freely falling with the test particle is an inertial system, and all observations (e.g., the measurement of line elements of a local inertial frame momentarily at rest with respect to M just as the freely falling system moves past it) in this system are subject to the ordinary rules of special relativity. From (3) and the Newtonian expression for the energy conservation

$$mv^2(r)/2 = GmM/r, \tag{4}$$

we get (2).

In analogy to the construction of early quantum mechanics from the correspondence

---

*) In short, the equivalence principle implies a curved space-time. This is inconsistent with special relativity which deals with the flat Minkovsky space-time, and with Newtonian gravitation which is itself inconsistent with special relativity.[11)]

**) The Einstein gravitational field is not a vector field (like Newtonian field), but has a tensor character. For the specific case of the spherically symmetric system, however, the exterior field is completely determined by four components of the metric tensor. This point was stressed by Sommerfeld.[7)] principle, we modify the Newtonian expression (4) so as to be consistent with the uncertainty principle:

$$mv^2(r)/2c^2 = GmM/rc^2 - m(\hbar/mc)^2/2r^2. \tag{5}$$

Here $m(\hbar/m)^2/2r^2 = (\hbar/r)^2/2m$ is the quantum mechanical repulsive potential associated with the momentum uncertainty $\hbar/r$. Using (5) in (3), we obtain

$$a(r) = 1 - 2GM/rc^2 + (\hbar/mc)^2/r^2. \tag{6}$$

Here the usual difficulty arises that the quantized metric depends on the mass m of the test particle.[14] This violates the equivalence principle stating that all masses fall with the same acceleration in a given gravitational field. Let us therefore restrict the mass m to the Planck value:[6]

$$m^* = (\hbar c/G)^{1/2} \sim 10^{-5}\ \mathrm{g}. \tag{7}$$

With m equal to m^* we find

$$a(r) = 1 - 2GM/rc^2 + L^{*2}/r^2, \tag{8}$$

where

$$L^* = (G\hbar/c^3)^{1/2} \sim 10^{-33}\ \mathrm{cm} \tag{9}$$

is the Planck length. We notice that the expression (8) has the form of the Reissner-Nordström metric[15] generated by a spherically symmetric electrified matter of mass M and charge e_0:

$$a(r) = 1 - 2GM/rc^2 + Ge_0^2/r^2c^4. \tag{10}$$

The Reissner-Nordström field is known as the only static electromagnetic vacuum field which is asymptotically flat and possesses nonsingular event horizon. Comparing (8) with (10) we find that the quantity

$$(L^{*2}c^4/G)^{1/2} = (\hbar c)^{1/2} \tag{11}$$

plays the role of the charge e_0. This is suggestive of the quantum-gravitational origin of the unrenormalized electric charge $(\hbar c)^{1/2} = \sqrt{137}e$.[16] Further, if we put $M = m^*$ we get

$$a(r) = 1 - 2L^*/r + L^{*2}/r^2 = (1 - L^*/r)^2. \tag{12}$$

This has the form of the metric considered by Papapetrou,[17] Bonner,[18] Arnowitt, Deser and Misner,[19] and in particular by Markov[20] as the geometry characterizing the external field of a static charged dust in which the gravitational attraction is balanced by the electrostatic repulsion. We may regard (12) as the geometry of the gravitational Bohr atom consisting of a pair of Planck masses.

§ 3. The role of the uncertainty principle in avoiding the unlimited gravitational collapse of Fermion spheres

In order to see how the quantum principle affects the usual conclusion of the unlimited gravitational collapse of superdense Fermion spheres, let us first modify the well-known internal solution of Einstein field equations for a motionless spherically symmetric matter characterized by the mass density ρ and the pressure P:*)

$$ds^2 = e^\nu c^2 dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi), \tag{13}$$

$$-\nu(r) = \int_r^\infty [(8\pi G/c^4)(\rho c^2 + P) re^\lambda - d\lambda/dr] dr, \tag{14}$$

$$e^{-\lambda(r)} = 1 - (2G/rc^2)\int_0^r 4\pi\rho r^2 dr + L^{*2}/r^2. \tag{15}$$

Here the usual expression for $e^{-\lambda}$ is modified by the addition of the quantum mechanical potential L^{*2}/r^2 arising from the use of the wavelike test particle of mass m^*.

Let us next consider an example of a spherical distribution of an extreme relativistic Fermi gas for which the particle density n and the mass-energy density ρ($\gg 10^{15}$ g/cm^3 for neutrons) are related by the equation of state:[1)]

$$\rho = (3\hbar/4c)(3\pi^2)^{1/3} n^{4/3}. \tag{16}$$

The equilibrium of such a system in its own gravitational field has been studied by Landau,[1)] by Oppenheimer and Volkoff[2)] and, more recently, by Zel'dovich.[3)] It has been indicated that for any given number of particles one can obtain a configuration with mass as close to zero as one pleases by prescribing a sufficiently high density of particles.[3),4),5)] In order to see how our modified metric (15) changes the situation, let us choose the same distribution of ρ as Zel'dovich's:

$$\begin{aligned} \rho &= a/r^2 \quad \text{for } r<R, \\ &= 0 \quad \text{for } r>R, \end{aligned} \tag{17}$$

where a is an arbitrary constant. For the gravitational mass M (measured by an external observer) and the proper total number N of particles of the Fermi gas we have

$$M = 4\pi \int_0^R \rho r^2 dr = 4\pi a R \tag{18}$$

and

$$N = \int_0^R n dV = 4\pi \int_0^R n(r) e^{\lambda/2} r^2 dr, \tag{19}$$

where $dV = 4\pi[\exp(\lambda/2)] r^2 dr$ is the invariant volume element. If we denote by m the mass of individual particles, then

$$mN = 4\pi \int mne^{\lambda/2} r^2 dr \tag{20}$$

is the total proper mass, and the quantity

*) In the crudest approximation in the analysis of the dynamics of the gravitational collapse, we can neglect the effect of the pressure and put $P=0$.[21)]

$$\Delta M = mN - M = 4\pi \int_0^R (mne^{\lambda/2} - \rho) r^2 dr \tag{21}$$

expresses the mass defect.

With the distribution (17) the expression for $e^{-\lambda}$ becomes

$$e^{-\lambda} = 1 - 8\pi Ga/c^2 + L^{*2}/r^2, \tag{22}$$

so that

$$N = 4\pi \int_0^R (1 - 8\pi Ga/c^2 + L^{*2}/r^2)^{-1/2} nr^2 dr\,. \tag{23}$$

The Planck's constant h enters two places in (23); in the particle density $n \sim (c\rho/\hbar)^{3/4} \sim (ca/r^2\hbar)^{3/4}$ of the gravitating matter through the equation of state (16) and in the term L^{*2}/r^2 arising from the use of the wavelike test particle of mass m^*. In the absence of the term L^{*2}/r^2, the expression (23) is readily integrated to give Zel'dovich's results:

$$N \sim (ca/\hbar)^{3/4} (1 - 8\pi Ga/c^2)^{-1/2} R^{3/2}, \tag{24}$$

$$R \sim (\hbar/ca)^{1/2} N^{2/3} (1 - 8\pi Ga/c^2)^{1/3} \tag{25}$$

and

$$M \sim (a\hbar/c)^{1/2} N^{2/3} (1 - 8\pi Ga/c^2)^{1/3}. \tag{26}$$

On the basis of the expression (26), Zel'dovich concludes that $M \to 0$ as $a \to c^2/8\pi G$, whatever the value of N. Such a state obviously cannot go over into the state of equilibrium (e.g., into the static solution of Oppenheimer and Volkoff with $N < 0.75 N_\odot$), and can only contract without limit. In order to reduce ordinary matter, e.g., neutrons, to such a state, it is necessary to spend an enormous amount of energy to compress the matter to desired densities $a \sim c^2/G$. The energy barrier that separates the equilibrium state with $M < mN$ from the collapsing state ($M \to 0$) is estimated from (26) as[3]

$$M_{\max} \sim N^{2/3} (a\hbar/c)^{1/2} \sim N^{2/3} (\hbar c/G)^{1/2}. \tag{27}$$

Markov[5] rewrites (27) as

$$M_{\max} \sim N^{2/3} m^* \tag{28}$$

and points out that for a system consisting of a small number (say 2) of particles of mass m^* ("maximons"),*) there does not exist, within the framework of the classical general relativity, a density barrier for a transition into a collapsing state.

Now we shall see that the presence of the term L^{*2}/r^2 in the expression (22) for $e^{-\lambda}$ changes the situation in the limit of $a \to c^2/8\pi G$. In fact, from (23)

*) Markov[5] advanced a hypothesis that "maximons" play the role of the structural units (such as quarks) of elementary particles and that, in the first stage of the development of the universe, gravitational collapse of these quasiparticles had the character of quantum transitions into states having discrete mass values.

we have in case of $a=c^2/8\pi G$

$$N \sim (c^3/\hbar G)^{3/4}(R/L^*) \sim R^{5/2}L^{*-5/2}, \tag{29}$$

$$R \sim N^{2/5}L^* \tag{30}$$

and

$$M \sim N^{2/5}m^*. \tag{31}$$

It is thus seen that, after an initial rise from $M=0$ with the increase in a from $a=0$, M reaches a maximum $M_{\max} \sim N^{2/3}m^*$ at $R=R_{\max} \sim N^{2/3}L^*$, then decreases with the further increase in a until the minimum value $M_{\min} \sim N^{2/5}m^*$ is reached at $R_{\min} \sim N^{2/5}L^*$ when $a=c^2/8\pi G$. That further contraction of the system is not possible may be understood, if we notice the presence of a small central region ("repulsive hole") of radius $r_0=L^*(c^2/8\pi Ga)^{1/2}=L^*(Rc^2/2MG)^{1/2}$ at the center of symmetry of the system for which $e^{-\lambda/2}>1$ (c.f. (22)). Since the gravitational mass defect of the matter contained in this region is negative:

$$\Delta M = 4\pi \int_0^{r_0} (mne^{\lambda/2} - \rho) r^2 dr < 0 \tag{32}$$

(note that $\rho > mn$), the matter is energetically unstable against the escape to outer regions and the central region tends to go over into a state of infinitesimally small density.

The author is indebted to Professor K. Aizu for valuable suggestions.

References

1) L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, London, 1958).
2) J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55** (1939), 374.
3) Ya. B. Zel'dovich, Soviet Phys.—JETP **15** (1962), 446.
4) For a review: Ya. B. Zel'dovich and I. D. Novikov, Soviet Phys.—Uspekhi, MAY-JUNE (1965), 763.
See also J. A. Wheeler, *Relativity, Groups and Topology*, ed. C. M. De Witt and B. D. De Witt (Gordon and Breach, New York, 1964).
5) M. A. Markov, Soviet Phys.—JETP **24** (1967), 584.
6) N. Hokkyo, Prog. Theor. Phys. **44** (1970), 801.
7) W. Lenz, in A. Sommerfeld, *Electrodynamics* (Academic Press, New York, 1952).
8) N. L. Balazs, Z. Phys. **154** (1959), 264.
9) L. I. Schiff, Am. J. Phys. **28** (1960), 340.
10) K. E. Eriksson and Yngström, Phys. Letters **5** (1963), 119, 327.
11) A. Schild, Am. J. Phys. **28** (1960), 778.
12) W. M. Sacks and J. A. Ball, Am. J. Phys. **36** (1968), 240.
13) F. R. Tangherlini, Nouvo Cim. **25** (1962), 1081.
14) D. Greenberger, Ann. of Phys. **47** (1968), 116.
15) Discussed by V. L. Ginzburg and L. M. Ozernoi, in Soviet Phys.—JETP **20** (1965).
A. G. Doroshkevich, Ya. B. Zel'dovich and I. D. Novikov, Soviet Phys.—JETP **22** (1966), 122.
V. de la Cruz, J. E. Chese and W. Israel, Phys. Rev. Letters **24** (1970), 423.
16) J. A. Wheeler, Ann. of Phys. **2** (1957), 604.
17) A. Papapetrou, Proc. Irish Acad. **A51** (1948), 191.
18) W. Bonner, Z. Phys. **160** (1960), 59.
19) R. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. **120** (1960), 313; Ann of Phys. **33** (1965), 88.
20) M. A. Markov, JINR-E2-4880 (1969);-5271 (1970).
21) J. R. Oppenheimer and H. Synder, Phys. Rev. **56** (1939), 374.