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A ROUTLEY-MEYER SEMANTICS FOR ACKERMANN'S LOGICS OF "STRENGE IMPLIKATION"

Abstract. The aim of this paper is to provide a Routley-Meyer semantics for Ackermann's logics of "strenge Implikation" Π' and Π''. Besides the Disjunctive Syllogism, this semantics validates the rules Necessitation and Assertion. Strong completeness theorems for Π' and Π'' are proved. A brief discussion on Π', Π'' and paraconsistency is included.

Keywords: Ackermann's logics, Routley-Meyer semantics, relevant logics, paraconsistent logics.

1. Introduction

The aim of this paper is to provide a Routley-Meyer semantics for Ackermann's logics of 'strenge Implikation' Π' and Π''. Taking into account that this is just what was claimed to have been achieved in [10], what is then the purpose of this paper? In what follows I shall explain the reasons why I think this work actually contributes something new (knowledge of the Routley-Meyer semantics for relevant logics, especially of that developed in [10] will be presupposed). First of all, let me clearly state how Π', Anderson and Belnap's E and their conservative extensions either with the truth constant or with the falsity constant are axiomatized.

As is known, Ackermann's logics of 'strenge Implikation' Π' and Π'' (cf. [1]) are the origin of relevance logic (cf. [2, 3]).

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The logic Π' can be axiomatized as follows (cf. [1], p. 119).
Axioms:

(A1) $A \to A$

(A2) $(A \to B) \to [(B \to C) \to (A \to C)]$

(A3) $(A \to B) \to [(C \to A) \to (C \to B)]$

(A4) $[A \to (A \to B)] \to (A \to B)$

(A5) $(A \wedge B) \to A$

(A6) $(A \wedge B) \to B$

(A7) $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

(A8) $A \to (A \vee B)$

(A9) $B \to (A \vee B)$

(A10) $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

(A11) $[A \wedge (B \vee C)] \to [B \vee (A \wedge C)]$

(A12) $(A \to B) \to (\neg B \to \neg A)$

(A13) $(A \wedge \neg B) \to \neg(A \to B)$

(A14) $A \to \neg\neg A$

(A15) $\neg\neg A \to A$

Rules of inference:

(α) From A and $A \to B$ to infer B

(β) From A and B to infer $A \wedge B$

(γ) From A and $\neg A \vee B$ to infer B

(δ) From B and $A \to (B \to C)$ to infer $A \to C$

Anderson and Belnap's Logic of Entailment E would then be obtained by dropping (γ) and (δ) and adding the axioms

(a1) $(A \to B) \to [[(A \to B) \to C] \to C]$

(a2) $(\Box A \wedge \Box B) \to \Box(A \wedge B)$

where $\Box A =_{\mathrm{df}} (A \to A) \to A$ (cf. [2, §27.1.1], or [3, §R2]).

On the other hand, once the falsity constant F is added to the propositional language, the logic Π'' (the label is due to the authors of [3], cf. pp. 135–136) is axiomatized (cf. [1], p. 124) by adding to Π' the axioms

(A16) $(A \to F) \to \neg A$

(A17) $(A \wedge \neg A) \to F$

together with the rule

(ε) If $A \to B$ and $[(A \to B) \wedge C] \to F$ are provable, then $C \to F$ is also provable.

Remark 1. Notice that while Ackermann formulates (α), (β), (γ) and (δ) as rules of inference, he formulates (ε) only as a rule of proof. We shall return to this important question in the following section.

Consider now the following conservative extension of E (cf. [3], §R2). The propositional truth constant t is added to the propositional language, and, then, E is supplemented with the axioms

(a3) $t \to (A \to A)$

(a4) $(t \to A) \to A$

In [10], it is proved that this extension of E (also labelled E by the authors of [10]) is equivalent "as to theorems" to Π''. In particular, in the cited paper, it is proved that the rule *Necessitation*, to wit,

(NR) $\vdash A \;\Rightarrow\; \vdash t \to A$

as well as (γ) (taken only as a rule of proof) are admissible rules of E, and, consequently, that so is (δ). However, the authors of [10] remark:

> Thus, normal E.m.s and indeed E.m.s are characteristic structures for Π' and Π'', i.e., they verify all and only theorems: however, these structures validate neither the rule of necessity, nor (δ). In fact, it is impossible to design the base T of E.m.s so that these rules are verified in T. Suppose that the non-theorem $\bar{A} \vee .t \to A$ i.e., $A \supset \Box A$ does not belong to T. Since all theorems of Π belong to T, $\bar{A} \vee A \in T$. Hence $A \in T$ or $\bar{A} \in T$. But if $A \in T$ and Necessitation is verified $t \to A \in T$, whence $\bar{A} \vee .t \to A \in T$, contradicting assumptions. [10, p. 416]

(E.m.s abbreviates E-model structures; Π is an equivalent ("as to theorems") formulation of Π'': in fact, the conservative extension of E considered above.)

Of course, the authors of [10] are here referring to the canonical model. But it is easy to to see that neither (NR) nor (δ) are in general validated in their models, as it is shown below. (Cf. the comment following Definition 3 in the following section.)

Proof. (NR): Suppose $\vDash A$, and, for *reductio*, $\nvDash t \to A$. Then, $a \vDash t$, $a \nvDash A$ for some $a \in K$ in some model. So, Pb for some $b \leq a$. But as validity is defined in respect of O, not necessarily $b \vDash A$. Consequently, $\vDash A$ and $a \nvDash A$ do not contradict each other.

(δ): Suppose $\vDash B$, and, for reductio, $a \vDash A \to (B \to C)$, $a \nvDash A \to C$ for some $a \in K$ in some model. Then, $b \vDash A$, $c \nvDash C$ for b, $c \in K$ such that $Rabc$. Then, $c \vDash B \to C$. As $Rcxc$, for some Px (cf. P5 in [10]), $x \nvDash B$. But again, as validity is defined in respect of O, $x \nvDash B$ and $\vDash B$ do not contradict each other. ⊣

It would only be natural to conclude that it suffices to define validity in respect of all members in P in order to validate (NR) and (δ). If this change would do is a question that cannot be discussed here properly. Let me just point out that if it is incorporated, then one of the aims pursued by the authors of [10] is not fulfilled, namely, that of giving a semantics for E, because this modification of the models being introduced, a semantics for the conservative extension of E noted above is almost close at hand, but a semantics for E would depend on the admissibility of (δ) in this logic, as this rule is not derivable in E (cf. Appendix). And, although this rule seems to be admissible in E, I have not found a proof of this fact in the literature.

Anyway, the aim of this paper is to provide model structures that, in addition to validate (γ) (i.e., Disjunctive Syllogism) also validate (NR) and (δ), thus complying with Ackermman's intentions, as we shall see in Section 2. Concerning the models defined in [10], the authors of that paper remark ([10], p. 407):

> The point of property P, from the logical engineering standpoint, is to get round the problem raised by the fact that it is impossible to obtain [...] a prime regular theory T which validates the rule of necessitation [...] Admittedly, the presence of P, and its apparent

> uneliminability, makes the semantics of E more cumbersome and less attractive than that of some of its relevant rivals such as R and T.

However, the property P is not necessary in the models for Π' defined in this paper.

The structure of the paper is as follows. In Section 2 some proof-theoretical questions about Π' are remarked. In Section 3, a semantics for Π' is provided, and strong soundness is proved. In sections 4–6 Π' is proved deductively complete, i.e, strongly complete. In Section 7, some proof-theoretical facts about Π'' are discussed. In Section 8, a semantics for Π'' is defined and strong soundness and completeness are proved. Finally, in Section 9, we briefly discuss Ackermann's logics in relation to paraconsistency. As pointed out above, knowledge of the Routley-Meyer semantics for relevant logics is assumed. Finally, let me state clearly that if the present contribution has any value, it wholly depends on the work by Routley and Meyer on the semantics named after them.

2. The logic Π'. Some proof-theoretical remarks

Ackermman remarks ([1], pp. 119–120):

> Für die Gültigkeit von (δ), im Gegenstaz zu (α), (β) und (γ) is wesentlich, daß $\mathfrak{B}$ eine logische Identität ist. Die Regel (δ) könte also nicht in dieser Form bestehen bleiben falls nicht logische Zusatzaxiome neben (1)–(15) [cf. §1 above] auftreten.[1]

Regarding (α), (β) and (γ), he notes ([1], p. 125):

> Die allein übrig bleibenden Regeln (α), (β), (γ) sind so gefaßt, daß sie auch bei Hinzufüngun von nicht logischen Zusatzaxiomen gültig bleiben.[2]

It is then clear that in Ackermman's opinion, whereas (α), (β) and (γ) are valid ("gültig") when applied to no matter which premises, (δ) is valid only if B is a theorem ("eine logische Identität"). In other words,

[1] "In contrast to (α), (β) and (γ), in order for (δ) to be valid, it is essential that $\mathfrak{B}$ is a logical identity. Therefore, rule (δ) does not hold in this form if in addition to (1)–(15), non-logical axioms are introduced."

[2] "The remainig rules (α), (β), (γ) are formulated so that they are still valid when non-logical axioms are introduced."

(α), (β) and (γ) are valid rules of inference, in general, but (δ) is valid only in the following form:

(δ') If B is provable, then from $A \to (B \to C)$ to infer $A \to C$

However, one presumably wants Π' to be applied, that is to say, one may want to build theories upon Π', or, in other words, to add "nicht logische Axiome" to it. Therefore, we shall axiomatize Π' with (δ') instead of (δ), hoping not betraying Ackermann in doing so.

Be it as it may, we note the following:

Proposition 1. *Given Routley and Meyer's basic positive logic* B_+ (cf. [8] or [9]) *the rule* (δ') *and the following rule Assertion*

(Asser) $\vdash A \Rightarrow \vdash (A \to B) \to B$

are equivalent.

Proof. (a) Suppose $\vdash B$ and $A \to (B \to C)$. By (Asser), $\vdash (B \to C) \to C$. By Prefixing, $\vdash [A \to (B \to C)] \to (A \to C)$. So, $A \to C$. (b) Suppose $\vdash B$. By (A1) (cf. §1), $\vdash (B \to C) \to (B \to C)$. So, $\vdash (B \to C) \to C$ by (δ'). ⊣

Consequently, (δ') can be replaced by (Asser) in Π' and, therefore, in Π', (α), (β) and (γ) are *rules of inference* (they may be applied to whichever premises) but (Asser) has to be understood as a *rule of proof* (it can only be applied to theorems).

Let us now define the following relation of (syntactical) consequence in a classical way.

Definition 1. Let Γ be a set of wff and A be a wff. Then, $\Gamma \vdash_{\Pi'} A$ iff there is a finite sequence of wff $B_1, \ldots, B_m$ such that B_m is A and for each i ($1 \leq i \leq m$), one of the following is the case: (a) $B_i \in \Gamma$ (b) B_i is an axiom (c) B_i is the result of applying (α), (β), (γ) or (δ') to two previous formulas in the sequence.

The concept "Theorem of Π'" is then defined as a special case of Definition 1:

Definition 2. $\vdash_{\Pi'} A$ (i.e. A is a theorem of Π') iff $\varnothing \vdash_{\Pi'} A$ ($\varnothing$ is the empty set of premises).

Next, we note for future references some theorems and rules of Π'. In fact, T1–T7 below are provable in Meyer and Routley's B (cf. [4]). But T8 (γ') depends on (γ) as a rule of proof (or as an admissible rule).

(T1) $\vdash A \to B \Rightarrow \vdash \neg B \to \neg A$ (A12)

(T2) $\vdash \neg A \to B \Rightarrow \vdash \neg B \to A$ (T1), (A15)

(T3) $\vdash A \to \neg B \Rightarrow \vdash B \to \neg A$ (T1), (A14)

(T4) $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$ (T1), (T2)

(T5) $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ (T1), (T3)

(T6) $\neg(A \wedge \neg A)$ (A1), (A13), (T3)

(T7) $\neg A \vee A$ (A15), (T4), (T6)

(T8 γ') $(\vdash A \ \& \vdash \neg(A \wedge B)) \Rightarrow \vdash \neg B$ (γ), (T4)

Then, the following rules are provable for any set of wff Γ and wff A, B:[3]

(VEQ) $\vdash_{\Pi'} A \Rightarrow \Gamma \vdash_{\Pi'} A$

(ECQ) $A \wedge \neg A \vdash_{\Pi'} B$

(EFQ) $\vdash_{\Pi'} A \Rightarrow \neg A \vdash_{\Pi'} B$

Proof. For (VEQ): By definitions 1 and 2.

For (ECQ):

1. $A \wedge \neg A$ Hyp.
2. A (A5), 1
3. $\neg A$ (A6), 1
4. $\neg A \vee B$ (A8), 3
5. B (γ), 2, 4[4]

[3] (VEQ) = "*Verum e quodlibet*", (ECQ) = "*E contradictione quodlibet*", (EFQ) = "*E falso quodlibet*". Notice that these rules are, of course, satisfied by Ackermann's set of matrices (cf. [1], pp. 127–128; cf. also [3], Ch. VIII).

[4] This proof is, in fact, the famous "Lewis' proof" or "Lewis' argument" (cf. [4], p. 250). Notice that (γ) is used as a rule of inference.

For (EFQ):

1. $\vdash_{\Pi'} A$ Hyp.
2. $\neg A$ Hyp.
3. $A \wedge \neg A$ (β), 1, 2
4. B (ECQ), 3

$\dashv$

In the following section, a Routley-Meyer semantics for Π' is defined.

3. Semantics for Π'. Strong soundness

First, the notion of a Π'-model is defined.

Definition 3. A Π'-model is a structure $\langle K, O, R, *, \vDash \rangle$, where O is a subset of K, R is a ternary relation on K, and $*$ is a unary operation on K subject to the following definitions and postulates for all a, b, c, $d \in K$:

(d1) $a \leq b =_{\text{df}} (\exists x \in O)Rxab$

(d1′) $a = b =_{\text{df}} (a \leq b \ \ \& \ \ b \leq a)$

(d2) $R^2abcd =_{\text{df}} (\exists x \in K)(Rabx \ \ \& \ \ Rxcd)$

(P1) $a \leq a$

(P2) $(a \leq b \ \ \& \ \ Rbcd) \Rightarrow Racd$

(P3) $R^2abcd \Rightarrow (\exists x \in K)(Racx \ \ \& \ \ Rbxd)$

(P4) $R^2abcd \Rightarrow (\exists x \in K)(Rbcx \ \ \& \ \ Raxd)$

(P5) $Rabc \Rightarrow R^2abbc$

(P6) $(\exists x \in O)Raxa$

(P7) $Rabc \Rightarrow Rac{*}b{*}$

(P8) $Ra{*}aa{*}$

(P9) $a \leq a{*}{*}$

(P10) $a{*}{*} \leq a$

(P11) $a \in O \Rightarrow a \leq a{*}$

Finally, $\vDash$ is a relation from K to the formulas of the propositional language such that the following conditions are satisfied for any propositional variable p, wff A, B and $a \in K$:

(i) $(a \leq b \ \ \& \ \ a \vDash p) \Rightarrow b \vDash p$

(ii) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$

(iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv) $a \vDash A \to B$ iff for all $b, c \in K$, $(Rabc \ \ \& \ \ b \vDash A) \ \Rightarrow \ c \vDash B$

(v) $a \vDash \neg A$ iff $a* \nvDash A$

The essential differences between the models defined in [10] and the ones here presented are:

1. O is not a subset of K but a selected member of this set.
2. A unary relation, P, is introduced governed by the postulates

   - $\exists x(Px \ \ \& \ \ Raxa)$
   - $\forall x(Px \ \ \& \ \ Rxab) \Rightarrow a \leq b$

   The truth constant t is then interpreted according to the clause

   - $a \vDash t$ iff for some $b \in K$, Pb and $b \leq a$

3. Our postulates P6 and P11 are deleted.

Now, consider the following definition of (semantical) consequence:

Definition 4. Let Γ be the set of wff and A a wff. Then, $\Gamma \vDash_{\Pi'} A$ iff for all $a \in O$ in all models, if $a \vDash \Gamma$, then $a \vDash A$ ($a \vDash \Gamma$ iff $a \vDash B$ for every $B \in \Gamma$).

As a special case of Definition 4, we define the concept of validity in Π' in this way:

Definition 5. $\vDash_{\Pi'} A$ (A is Π'-valid) iff $a \vDash A$ for all $a \in O$ in all models.

Next, some auxiliary lemmas and propositions are introduced in order to prove strong soundness.

Proposition 2. *The following postulates hold in all* Π'*-models*:

(P12) $a \leq b \Rightarrow b* \leq a*$

(P13) $a = a**$

(P14) $Raa*a$

(P15) $a \in O \Rightarrow a* \leq a$

(P16) $a \in O \Rightarrow a = a*$

(P17) $(\exists x \in O)Raa*x$

Proof. For (P12): by (P7), (d1). For (P13): by (P9), (P10), (d1′). For (P14): by (P8), (P13). For (P15): by (P14), (d1). For (P16): by (P11), (P15), (d1′). For (P17): by (P6), (P7), (P16). $\dashv$

Lemma 1. *For any wff* A *and* $a, b \in K$, $(a \leq b \ \& \ a \vDash A) \Rightarrow b \vDash A$.

Proof. Induction on the length of A. The conditional case is proved by (P2), and the negation case by (P12). $\dashv$

Lemma 2. *For any wff* A, B, $\vDash_{\Pi'} A \to B$ *iff* $a \vDash A \Rightarrow a \vDash B$ *for all* $a \in K$ *in all models.*

Proof. By Lemma 1 and (P1) (with (d1)). $\dashv$

Lemmas 1 and 2 are also useful in subsequent sections. Concerning their proofs, see, e.g., [9].

Proposition 3. *Let* M *be any model and* $a \in K$*. Then, for every wff* A, $a* \vDash \neg A$ *iff* $a \nvDash A$.

Proof. By clause (v) and (P13). $\dashv$

Lemma 3. (A1)–(A15) *are* Π'*-valid.*

Proof. (A1)–(A15) are proved valid as in e.g., the semantics for E (cf. for example, [9]). In particular, (A1)–(A4) and (A12)–(A15) are proved valid with (P1), (P3), (P4), (P5), (P7), (P8), (P9) and (P10), respectively. $\dashv$

Lemma 4. *Let Γ be a set of wff and A, B wff. Then*

(i) $(\Gamma \vDash_{\Pi'} A \to B \ \ \& \ \ \Gamma \vDash_{\Pi'} A) \Rightarrow \Gamma \vDash_{\Pi'} B$,

(ii) $(\Gamma \vDash_{\Pi'} A \ \ \& \ \ \Gamma \vDash_{\Pi'} B) \Rightarrow \Gamma \vDash_{\Pi'} A \wedge B$,

(iii) $(\Gamma \vDash_{\Pi'} A \ \ \& \ \ \Gamma \vDash_{\Pi'} \neg A \vee B) \Rightarrow \Gamma \vDash_{\Pi'} B$,

(iv) $(\vDash_{\Pi'} B \ \ \& \ \ \Gamma \vDash_{\Pi'} A \to (B \to C)) \Rightarrow \Gamma \vDash_{\Pi'} A \to C$.

Proof. Let M be any model and $a \in O$.

(i) Suppose $a \vDash \Gamma$. Then, $a \vDash A \to B$ and $a \vDash A$. As $a \in O$, for every $x, y \in K$, $(x \leq y \ \ \& \ \ x \vDash A) \Rightarrow y \vDash B$ (clause (iv), Definition 3). By (P1), $a \leq a$. So, $a \vDash B$, as was to be proved.

(ii) It is trivial.

(iii) Suppose $a \vDash \Gamma$. Then, $a \vDash A$, $a \vDash \neg A \vee B$. By Proposition 3, $a* \nvDash \neg A$. So, $a \nvDash \neg A$ by (P11) and Lemma 1. Consequently, $a \vDash B$.

(iv) Suppose $\vDash_{\Pi'} B$ and $a \vDash \Gamma$. Then, $a \vDash A \to (B \to C)$. Suppose for reductio $a \nvDash A \to C$. Then, $b \vDash A$, $c \nvDash C$ for $b, c \in K$ such that $Rabc$. So, $c \vDash B \to C$ ($a \vDash A \to (B \to C)$, $b \vDash A$, $Rabc$). By (P6), $Rcxc$ for some $x \in O$. As B is valid, $x \vDash B$. Consequently, $c \vDash C$ ($c \vDash B \to C$, $Rcxc$, $x \vDash B$), a contradiction. ⊣

Then, we immediately have:

Theorem 1 (Strong soundness of Π'). *If $\Gamma \vdash_{\Pi'} A$, then $\Gamma \vDash_{\Pi'} A$.*

Proof. Induction on the length of the proof $\Gamma \vdash_{\Pi'} A$: by Lemmas 3 and 4. ⊣

And, finally, by Theorem 1 we have the following corollary:

Corollary 1 (Soundness of Π'). *If $\vdash_{\Pi'} A$, then $\vDash_{\Pi'} A$.*

4. Completeness of Π' I. The canonical model. Preliminary propositions and lemma

We begin by recalling some definitions.

Definition 6. (i) A Π'-*theory* is a set of formulas closed under adjunction and provable Π'-entailment. That is, a is a Π'-theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem of Π' and $A \in a$, then $B \in a$.

(ii) A theory a is *prime* iff whenever $A \vee B \in a$, then $A \in a$ or $B \in a$.

(iii) A theory is *regular* iff all theorems of Π' belong to it.

(iv) A theory a is *w-inconsistent* (inconsistent in a weak sense) iff $\neg A \in a$, A being a theorem of Π'. Then, a is w-consistent (consistent in a weak sense) iff a is not w-inconsistent.

Remark 2. The concept of *w-inconsistency* is introduced in [7]. But it is, of course, no novelty. For example, in defining the canonical model for E extended with the truth constant t and axioms (a3) and (a4) (cf. §1), the authors of [10] remark "T is consistent provided T does not contain the negation of some theorem of E" (see [10], p. 412).

Next, the canonical model is defined.

Definition 7. Let K^T be the set of all Π'-theories and R^T be defined on K^T as follows: for all a, b, $c \in K^T$ and wff A, B, $R^T abc$ iff $(A \to B \in a \ \ \& \ \ A \in b) \Rightarrow B \in c$. Now, let K^C be the set of all prime Π'-theories, and O^C the set of all regular, w-consistent, prime Π'-theories. On the other hand, let R^C be the restriction of R^T to K^C and $*^C$ be defined on K^C as follows: for any $a \in K^C$, $a*^C = \{A \mid \neg A \notin a\}$. Finally, $\vDash^C$ is defined as follows: for any $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the Π'-canonical model is the structure $\langle K^C, O^C, R^C, *^C, \vDash^C\rangle$.

Remark 3. What distinguishes the Π'-canonical model from those for standard relevant logics is just one important fact: in the latter members of O^C need not be consistent in any sense of the term.

In what follows, some preliminary propositions and lemmas are proved.

Proposition 4. *$*^C$ is an operation on K^C.*

Proof. By (T1), (T4) and (T5) (cf. [9]). $\dashv$

Proposition 5. (i) *$\neg A \in a*^C$ iff $A \notin a$,*

(ii) *$a*^C*^C = a$,*

(iii) *$a \subseteq b \Leftrightarrow b*^C \subseteq a*^C$.*

Proof. (i) by (A14), (A15); (ii) and (iii), by (i) and definitions. $\dashv$

Proposition 6. (i) *Let a be a Π'-theory. Then, a is w-consistent iff for no theorem of Π' of the form $\neg A$, $A \in a$.*

(ii) *Let a be a Π'-theory. Then, if a contains some contradiction, a is w-inconsistent.*

(iii) *Let a be a regular Π'-theory. Then, if a is w-inconsistent, a contains some contradiction.*

(iv) *Let a be a regular Π'-theory. Then, a is w-consistent iff a does not contain some contradiction.*

Proof. (i) Immediate by (A14).

(ii) If a contains some contradiction, then a is w-inconsistent by (i); cf. (T6).

(iii) Let a be a regular Π'-theory. Then, if $\neg B \in a$, B being a theorem of Π', a contains the contradiction $B \wedge \neg B$.

(iv) By (ii) and (iii). ⊣

In consequence, notice that if regularity is not present, w-consistency and consistency (understood in the customary sense of the term) are not necessarily equivalent in the case of Π'-theories. We shall return to this important question later (cf. Remark 5).

We end this section with the following lemma.

Lemma 5. *Let a, $b \in K^T$ and $c \in K^C$. Moreover, let $R^T abc$. Then, there are x, $y \in K^C$ such that $R^T xbc$ and $R^T ayc$.*

Proof. As in, e.g., the semantics for E (cf. [9]). ⊣

5. Completeness of Π' II. Regularity and w-consistency

We begin by proving a proposition on the relationship between regularity and w-consistency in the context of K^C.

Proposition 7. (i) *If a is a regular member in K^C, then $a*^C$ is w-consistent. Moreover, $a*^C \subseteq a$.*

(ii) *If a is a w-consistent member in K^C, then $a*^C$ is regular. Moreover, $a \subseteq a*^C$.*

Proof. (i) If $a*^C$ is w-inconsistent, then it follows from Proposition 5i that a is not regular. Suppose, on the other hand, that for some wff A, $A \in a*^C$, but $A \notin a$. Then, $\neg A \in a*^C$ (Proposition 5i). So, $A \wedge \neg A \in a*^C$, contradicting the w-consistency of $a*^C$ (Proposition 6ii).

(ii) If $a*^C$ is not regular, then it follows that a is w-inconsistent from Definition 7. On the other hand, suppose that for some wff A, $A \in a$ but $A \notin a*^C$. By Definition 7, $\neg A \in a$, contradicting the w-consistency of a (Proposition 6ii). ⊣

The following corollary follows from Proposition 7.

Corollary 2. (i) *$*^C$ is an operation on O^C.*

(ii) *If $a \in O^C$, then $a*^C = a$.*

We now introduce a lemma on the extension of w-consistent theories to prime w-consistent theories:

Lemma 6. *Let a be a w-consistent element in K^T. Then, there is some w-consistent x in K^C such that $a \subseteq x$.*

Proof. Define from a a maximal w-consistent theory x such that $a \subseteq x$. If x is not prime, then there are wff A, B such that $A \vee B \in x$, $A \notin x$, $B \notin x$. Define the set $[x, A] = \{C \mid \exists D[D \in x \ \ \& \vdash_{\Pi'} (A \wedge D) \to C]\}$. Define $[x, B]$ similarly. It is easy to prove that $[x, A]$ and $[x, B]$ are theories strictly including x. By the maximality of x, they are w-inconsistent. That is, $\neg C \in [x, A]$, $\neg D \in [x, B]$ for some theorems C, D. By definitions, we have $\vdash_{\Pi'} (A \wedge E) \to \neg C$, $\vdash_{\Pi'} (B \wedge E') \to \neg D$ for some E, $E' \in x$, whence by basic theorems of B_+ (cf. Proposition 1 in Section 2), $\neg C \vee \neg D \in x$, and, by (T4), $\neg(C \wedge D) \in x$. But, by Adjunction, the rule (β), $\vdash_{\Pi'} C \wedge D$. Therefore, if x is not prime, it is w-inconsistent, which is impossible. ⊣

Remark 4. Notice that this Lemma is provable in any extension of B_+ in which $(\neg A \vee \neg B) \to \neg(A \wedge B)$ is a theorem.

A second primeness lemma is:

Lemma 7. *Let $a \in K^T$ and A be a wff such that $A \notin a$. Then, there is some x in K^C such that $a \subseteq x$ and $A \notin x$.*

Proof. Similar to that of Lemma 6 (cf., e.g., [9]). ⊣

Next, a lemma on which the proof of the fundamental lemma in the following section leans.

LEMMA 8. *Let a be a w-consistent member in K^T. Then, there is a regular, w-consistent element in K^T such that $a \subseteq x$.*

PROOF. We can assume that a is non-null: if it is null, then, the proof of Lemma 8 is trivial because Π' is clearly a regular, w-consistent theory. Consider now the set $x = \{B \mid \exists A, C[C \in a \ \& \vdash_{\Pi'} A \ \& \vdash_{\Pi'} (A \wedge C) \to B]\}$. It is easy to prove that x is a Π'-theory. Moreover, it is proved:

1. $a \subseteq x$: Let $A \in a$ and B be a theorem of Π'. As $\vdash_{\Pi'} (B \wedge A) \to A$ (A6), $A \in x$.

2. x is regular: Let $A \in a$ and B be a theorem of Π'. By (A5), $B \in x$.

3. x is w-consistent: Suppose $\neg A \in x$, A being a theorem of Π'. By definition of x, $\vdash_{\Pi'} (C \wedge B) \to \neg A$ for some $B \in a$ and theorem C of Π'. By (T3), $\vdash_{\Pi'} A \to \neg(C \wedge B)$. So, $\vdash_{\Pi'} \neg(C \wedge B)$, by the rule ($\alpha$). Therefore, $\vdash \neg B$ by γ'. But, then, a is w-inconsistent (cf. Proposition 6i), contradicting the hypothesis. $\dashv$

Remark 5. Notice that if consistency is understood in the standard sense (i.e., absence of contradiction), Lemma 8 is not provable in the present context: suppose that a contains no contradictions, define x as above and suppose that x contains a contradiction for reductio. Similarly as in the proof of Lemma 8, we can show that B is in a, $\neg B$ being a theorem. But a is not necessarily regular. Therefore, we cannot conclude $B \wedge \neg B \in a$, and, consequently, the supposition that x contains a contradiction is not absurd. On the other hand, note the essential rule γ' plays in the proof of Lemma 8.

Now, Lemma 8 is used in the proof of all lemmas that follow. And as was just remarked, in order to prove this lemma, consistency cannot be understood as the absence of any contradiction but as the absence of the negation of any theorem (equivalently: absence of the argument of any negation theorem (cf. Proposition 6i). In this sense, we say that Π' is *adequate to w-consistency*, but it is *not adequate to consistency as the absence of any contradiction.*

Finally, we state a useful and significant lemma.

LEMMA 9. *Let a be a w-consistent element in K^C. Then, there is some x in O^C such that $x \subseteq a*^C$.*

Proof. Assume the hypothesis of the lemma. By Lemma 8, there is some regular w-consistent theory y such that $a \subseteq y$. By Lemma 6, there is a prime (regular) w-consistent Π'-theory z (i.e., some z in O^C) such that $y \subseteq z$. So, $a \subseteq z$. By Proposition 5iii, $z{*}^C \subseteq a{*}^C$. Moreover, by Corollary 2i, $z{*}^C \in O^C$. Therefore, $z{*}^C$is the required x. ⊣

We remark the following corollary of Lemma 9.

Corollary 3. *Let a be a regular element in K^C. Then, there is some $x \in O^C$ such that $x \subseteq a$.*

Proof. Assume the hypothesis of the lemma. By Proposition 7i, $a{*}^C$ is a w-consistent element in K^C. By Lemma 9, $x \subseteq a{*}^C{*}^C$ for some $x \in O^C$. So, by Proposition 5ii, $x \subseteq a$. ⊣

6. Completeness of Π'. The fundamental lemma. The canonical model is in fact a model. Strong completeness

First, we prove the fundamental lemma, which reads:

Lemma 10. *Let a be a regular member in K^T and A be a wff such that $A \notin a$. Then, there is some x in O^C such that $A \notin x$.*

Proof. By Lemma 7, there is some (regular) element y in K^C such that $a \subseteq y$ and $A \notin y$. If y is w-consistent, then it is the required x. If y is not w-consistent, then $y{*}^C$ is w-consistent anyway because y is regular (Proposition 7i). Moreover, $\neg A \in y{*}^C$ (Proposition 5i). Then, by Lemma 8, there is a regular, w-consistent z in K^T such that $y{*}^C \subseteq z$. By Lemma 6, there is some prime (regular) w-consistent member u in K^T (i.e., some $u \in O^C$) such that $z \subseteq u$. As $\neg A \in u$ ($\neg A \in y{*}^C$), $A \notin u$: otherwise, u would be w-inconsistent (cf. Proposition 6ii). Consequently, if y is not w-consistent, u is the required x. ⊣

Next, two preliminary lemmas in order to prove that the canonical model is in fact a model.

Lemma 11. *For any $a, b \in K^C$, $a \leq^C b$ iff $a \subseteq b$.*

Proof. From left to right, it is immediate. So, suppose $a \subseteq b$. Clearly, $R^T\Pi'aa$. Then, by Lemma 5, there is some (regular) y in K^C such that $\Pi' \subseteq y$ and $R^C yaa$. By Proposition 7i, $y{*}^C$ is a w-consistent member in

K^C. Therefore, by Lemma 9, there is some x in O^C such that $x \subseteq y{*}^C{*}^C$, i.e., $x \subseteq y$ by Proposition 5ii. Then, obviously, $R^C xaa$, and $R^C xab$, by hypothesis. Finally, $a \leq^C b$ by (d1). $\dashv$

Lemma 12. *For any $a \in K^C$ there is some $x \in O^C$ such that $R^C axa$.*

Proof. Let $a \in K^C$. Consider the set $y = \{A \mid\vdash_{\Pi'} A\}$. It is obvious that y is a regular element in K^T. Next, we prove that $R^T aya$ holds. Suppose for wff A, B, $A \to B \in a$, $A \in y$. Clearly, $\vdash_{\Pi'} A$. Then, $\vdash_{\Pi'} (A \to B) \to B$ by Asser (cf. Proposition 1 in Section 2) and, so, $B \in a$, as it was required. Now, by Lemma 5, there is some (regular) z in K^C such that $R^C aza$. By Corollary 3, there is some x in O^C such that $x \subseteq z$. Clearly, $R^C axa$, as it was to be proved. $\dashv$

Then, we can prove:

Lemma 13. *The Π' canonical model is indeed a Π'-model.*

We have to prove:

(1) The set O^C is not empty.

(2) Clauses (i)–(v) are satisfied by the canonical model.

(3) Postulates (P1)–(P11) hold in the canonical model.

Case (1) is immediate by Lemma 10 (Π' is a regular w-consistent theory), and (2) is proved as, e.g., in the semantics for E (see, e.g., [9]). So, let us prove (3). The fundamental fact is stated in Lemma 11. With the assistance of this lemma, (P1)–(P5) and (P7)–(P10) are proved as in the semantics for E (cf., e.g [9]). Next, (P6) is immediate by Lemma 12, and, finally, (P11) follows by Proposition 7ii and Lemma 11.

In what follows, we proceed into proving strong completeness. First, we define the set of consequences of a set of wff Γ (in symbols, conΓ) in a classical way:

Definition 8. $\mathrm{con}\Gamma =_{\mathrm{df}} \{A \mid \Gamma \vdash_{\Pi'} A\}$.

Then, we immediately have

Proposition 8. *For any set of wff Γ, conΓ is a regular Π'-theory.*

And finally,

Theorem 2 (Strong completeness of Π'). *If* $\Gamma \vDash_{\Pi''} A$, *then* $\Gamma \vdash_{\Pi'} A$.

Proof. Suppose $\Gamma \nvdash_{\Pi'} A$. Then, $A \notin \mathrm{con}\Gamma$. By Lemma 7, there is some x in K^C such that $\mathrm{con}\Gamma \subseteq x$ and $A \notin x$. It is clear that x contains no contradictions (otherwise, $A \in x$ by ECQ —cf. §2). On the other hand, x is regular ($\mathrm{con}\Gamma$ is regular by Proposition 8). Consequently, x is w-consistent (cf. Proposition 6iv) and so, $x \in O^C$. Now, as $\Gamma \subseteq \mathrm{con}\Gamma$, $\Gamma \subseteq x$. By Lemma 13, the canonical model is a model, so by Definition 7, $x \vDash^C \Gamma$ (i.e., for every $B \in \Gamma$, $x \vDash^C B$) and $x \nvDash^C A$. Therefore, $\Gamma \nvDash^C A$ (cf. Definition 4), as was to be proved. $\dashv$

As a corollary, by Theorem 2, we have

Corollary 4 (Completeness of Π'). *If* $\vDash_{\Pi''} A$, *then* $\vdash_{\Pi'} A$.

7. The logic Π''. Some proof-theoretical remarks

As remarked in Section 1, once the falsity constant F is added to the propositional language, the logic Π'' is axiomatized by adding to Π'

(A16) $\quad (A \to F) \to \neg A$

(A17) $\quad (A \wedge \neg A) \to F$

and the following rule

(ε) $\quad (\vdash A \to B \ \ \& \vdash [(A \to B) \wedge C] \to F) \ \Rightarrow \vdash C \to F$

Ackermann proves that the following rule

(θ) $\quad \vdash A \ \Rightarrow \vdash \neg A \to F$

is a derived rule of Π'' ([1], p. 125). Moreover, (δ) and (γ) are also derivable (from (A1)–(A17), (α), (β) and (ε)) in the form ([1], pp. 125–126):

(δ'') $\quad (\vdash B \ \ \& \vdash A \to (B \to C)) \ \Rightarrow \vdash A \to C$

(γ'') $\quad (\vdash A \ \ \& \vdash \neg A \vee B) \ \Rightarrow \vdash B$

That is, provided "Treten keine nicht logischen Zusatzaxiome auf" ("Non logical axioms are introduced"); see [1], p. 126.

On the other hand, notice that ε is formulated as a rule of proof ("Ist $\mathfrak{A} \to \mathfrak{B}$ und $(\mathfrak{A} \to \mathfrak{B}) \wedge \mathfrak{C} \to \curlywedge$ beweisbar, so auch $\mathfrak{C} \to \curlywedge$" ("If $\mathfrak{A} \to \mathfrak{B}$ and $(\mathfrak{A} \to \mathfrak{B}) \wedge \mathfrak{C} \to \curlywedge$ are provable, then so is $\mathfrak{C} \to \curlywedge$"); see [1], p. 124. Concerning this rule, when comparing it with (δ), Ackermann remarks:

> Es is naheliegend zu fragen, weshalb ich nicht (ε) in der einfacheren Form des Schlusses von $\mathfrak{A}$ und $(\mathfrak{A} \wedge \mathfrak{B}) \to \curlywedge$ auf $\mathfrak{B} \to \curlywedge$ eingefürht habe. Der Grund ist der, daß wir alle Regeln so formulieren wollen, daß sie auch bei Hinzunahme von nicht-logischen Zusatzaxiomen richtige Ergebnisse liefern (Die davon eine Ausnahme machende Regel (δ) wird gleich entfernt werden). Das ist aber bei der obigen Regel nicht der Fall. Sei $\mathfrak{A}$ eine aus den Zusatzaxiomen beweisbaren Formel, die keine logische Identität darstellt. Aus $\mathfrak{A} \wedge \bar{\mathfrak{A}} \to \curlywedge$ würden wir nach der Regel $\bar{\mathfrak{A}} \to \curlywedge$, d.h, $\mathfrak{A}$ is notwending, erhalten, was natürlich nicht sein darf.[5] [1, pp. 124–125]

But ε in the form

$$(\varepsilon') \qquad (\vdash A \ \ \& \quad \vdash (A \wedge B) \to F) \ \Rightarrow \vdash B \to F$$

is, of course, derivable, as shown below:

1.	$\vdash A$	Hyp.
2.	$\vdash (A \wedge B) \to F$	Hyp.
3.	$\vdash \neg(A \wedge B)$	(A16), 2
4.	$\vdash \neg B$	(γ'), 1, 3
5.	$\vdash \neg\neg B \to F$	(θ), 4
6.	$\vdash B \to F$	(A14), 1

[5] Ackermann uses the symbol $\curlywedge$ for the constant F. "It is only natural to ask why I have not formulated (ε) in the simpler form: from $\mathfrak{A}$ and $(\mathfrak{A} \wedge \mathfrak{B}) \to \curlywedge$ to infer $\mathfrak{B} \to \curlywedge$. The reason is that we want the rules to be formulated so that they always provide correct results if non logical axioms are introduced—and this would not happen with the rule (δ). But this is not the case with the above rule: let $\mathfrak{A}$ be a non-logical axiom. According to the aforementioned rule, from $\mathfrak{A} \wedge \bar{\mathfrak{A}} \to \curlywedge$ follows that $\bar{\mathfrak{A}} \to \curlywedge$, i.e., $\mathfrak{A}$ is necessary, which, of course, is not admissible."

Therefore, (ε) can be replaced by the more general (ε'). Notice that if (ε) were formulated as a a rule of inference, it would give us such an undesirable result as that remarked by Ackermann in the quote above.

Now, consider the following:

Proposition 9 (cf. Proposition 1 in §2). *Let* $\mathrm{B}_{+,\neg F}$ *be any extension of Routley and Meyer's basic positive logic* B_+, *where the following are provable:*

(α1) $(\neg A \to B) \to (\neg B \to A)$

(α2) $(A \to B) \to (\neg B \to \neg A)$

(α3) $\neg F$

(α4) $\vdash A \Rightarrow \vdash \neg A \to F$

Then, the rule

(Asser) $\vdash A \Rightarrow \vdash (A \to B) \to B$

and (A16) *are equivalent.*

Proof. For (Asser):

1. $\vdash A$	Hyp.
2. $\vdash \neg A \to F$	(α4), 1
3. $\vdash \neg F \to A$	(α1), 2
4. $\vdash (A \to B) \to (\neg F \to B)$	Suffixing, 3
5. $\vdash \neg\neg B \to B$	Immediate from (α1)
6. $\vdash (\neg B \to F) \to B$	(A16), 5
7. $\vdash (\neg F \to B) \to B$	(α1), 6
8. $\vdash (A \to B) \to B$	4, 7

For (A16):

1. $\vdash (\neg F \to \neg A) \to \neg A$	(Asser), (α3)
2. $\vdash (A \to F) \to \neg A$	(α2), 1 $\dashv$

Then, the following is a more economical axiomatization of Π''. Consider:

(A18) $\neg F$

PROPOSITION 10. *The logic* Π''b *is axiomatized by* (A1)–(A15), (A18) *and* (α), (β), (γ), (δ') *and*

(θ) $\vdash A \Rightarrow \vdash \neg A \to F$

as rules of inference. Then, Π'' *and* Π''b *are deductively equivalent.*

PROOF. 1. $\Pi''\text{b} \subseteq \Pi''$. It suffices to prove (A18), which is immediate by (A1) and (A16).

2. $\Pi'' \subseteq \Pi''\text{b}$. We have to show that (A16), (A17) and (ε) are provable in Π''b. Now, it follows from Proposition 9 that (A16) is provable; (A17) is immediate by (T6) (cf. Section 2) and (θ). Finally, (ε) is a special case of (ε'), which has been proved above with (A16), (γ') and (θ). ⊣

We end this section with the following remark:

Remark 6. Another possibility of axiomatizing Π'' is worth noting: (A18) is changed for (A16). Then, the rule (δ') may be dropped.

8. Semantics for Π''. Strong soundness and completeness

We now define a semantics for Π''.

DEFINITION 9. A Π''-model is a structure $\langle K, O, S, R, *, \vDash\rangle$, where S is a subset of K such that $O \subseteq S$, and K, O, R, $*$, $\vDash$ are defined, similarly, as in Π' models except for the following postulate and clauses:

(P18) $a \in S \;\Rightarrow\; (\exists x \in O) x \leq a*$

(vi) $a \vDash F$ iff $a \notin S$

(vii) $(a \leq b \;\&\; a \vDash F) \;\Rightarrow\; a \vDash A$

Then, consequence relations $\vdash_{\Pi''}$, $\vDash_{\Pi''}$ are defined, similarly, as those of Π', and so are the accompanying concepts of *Theorem of* Π'' and Π''-*validity* (cf. Definitions 1, 2, 4, 5).

Notice now that by clause (vii), analogues of Lemmas 1 and 2 (cf. Section 2) for Π'' are immediate. Then, the following two lemmas are proved:

LEMMA 14. (A1)–(A17) *are* Π''-*valid.*

Proof. (A1)–(A15) are proved valid, similarly, as in the case of Π' (cf. Lemma 3). So, we prove that (A16) and (A17) are valid (we use Lemma 2, for Π'').

1. (A16) is Π''-valid: Suppose that for some wff A and $a \in K$ in some model, $a \vDash A \to F$ and $a \nvDash \neg A$. Then, $a* \vDash A$. By (P17) (cf. Proposition 2), $Raa*x$ for some $x \in O$. Then, $x \vDash F$. But, as $x \in S$, $x \nvDash F$, a contradiction.

2. (A17) is Π''-valid: Suppose that for some wff A and $a \in K$ in some model, $a \vDash A \wedge \neg A$ and $a \nvDash F$. Then, $a* \nvDash A$, $a* \nvDash \neg A$. As $a \in S$, $x \leq a*$ for some $x \in O$ (P18). So, $x \nvDash A$, $x \nvDash \neg A$ (Lemma 1, $x \leq a*$), then $x* \vDash \neg A$ (Proposition 3). But by (P16) and Lemma 1, $x \vDash \neg A$, a contradiction. ⊣

Lemma 15. *For any wff A, B, C,*

$$(\vDash_{\Pi''} A \to B \ \ \& \vDash_{\Pi''} [(A \to B) \wedge C] \to F) \ \Rightarrow \vDash_{\Pi''} C \to F$$

Proof. Suppose for wff A, B, C,

1. $\vDash_{\Pi''} A \to B$ Hyp.
2. $\vDash_{\Pi''} [(A \to B) \wedge C] \to F$ Hyp.
3. $\nvDash_{\Pi''} C \to F$ Hyp.

From 3, it follows that for some $a \in K$ (Lemma 2)

4. $a \vDash C, a \nvDash F$

whence for some $x \in O$ (clause (vi) and (P18)),

5. $x \leq a*$ 4

As $x \nvDash F$ ($x \in S$),

6. $x \nvDash A \to B$ or $x \nvDash C$ 2

But as $x \in O$

7. $x \vDash A \to B$ 1

So,

8. $x \nvDash C$ 6, 7

On the other hand,

9. $a* \nvDash \neg C$ 4

10. $x \nvDash \neg C$ 5, 9

or, equivalently,

11. $x* \vDash C$ 10

12. $x \vDash C$ (P16), 11

But 8 and 12 contradict each other. ⊣

Then, it is proved:

Theorem 3 (Strong soundness of Π''). *If $\Gamma \vdash_{\Pi''} A$, then $\Gamma \vDash_{\Pi''} A$.*

Proof. Induction on the length of the proof that $\Gamma \vdash_{\Pi''} A$: by Lemmas 4, 14 and 15. ⊣

And, by Theorem 3, as a corollary,

Corollary 5 (Soundness of Π''). *If $\vdash_{\Pi''} A$, then $\vDash_{\Pi''} A$.*

Next, we proceed into proving strong completeness.
We begin by defining the canonical model.

Definition 10. The Π''-canonical model is the structure $\langle K^C, O^C, S^C, R^C, *^C, \vDash^C \rangle$, where S^C is the set of all prime w-consistent Π''-theories, and K^C, O^C, R^C, $*^C$, $\vDash^C$ are defined in a similar way to which they were defined in Definition 7, its items being now referred to Π''-theories, of course.

Then, it is proved:

Proposition 11. *Let $a \in K^T$. Then, a is w-consistent iff $F \notin a$.*

Proof. 1. If $F \in a$, then a is w-inconsistent, by (A18) (cf. Proposition 9).

2. If a is w-inconsistent, i.e., if $\neg A \in a$, A being a theorem, then $\neg A \to F$ is a theorem by rule (θ). So, $F \in a$. ⊣

Next, it is proved:

Lemma 16. *The* Π''*-canonical model is indeed a* Π''*-model.*

Proof. It is clear that we just have to prove that clauses (vi) and (vii) are satisfied by the canonical model, that (P18) is canonically valid, and that $O^C \subseteq S^C$. Now, clause (vi) follows by Proposition 11; clause (vii), by Lemma 11, and (P18), by Lemma 9. Finally, it is clear that $O^C \subseteq S^C$. ⊣

Then, similarly as in the case of Theorem 2, we have:

Theorem 4 (Strong completeness of Π''). *If* $\Gamma \vDash_{\Pi''} A$*, then* $\Gamma \vdash_{\Pi''} A$*.*

And as a corollary, we have:

Corollary 6 (Completeness of Π''). *If* $\vDash_{\Pi''} A$*, then* $\vdash_{\Pi''} A$*.*

We end the section with a remark.

Remark 7. It has doubtless been remarked, along the completeness proof of Π', the clear-cut division between members in O^C that are w-consistent but not necessarily regular, and those regular but not necessarily w-consistent. Now, once matters are translated into the canonical model, in order to interpret F, we need the former set, which accounts for the somehow clumsy specification that O is a subset of S in Π''-models. Nevertheless, Π''-models could be defined more clearly (and maybe more enlightening) as follows:

A Π''-model is a structure $\langle K, O, S, U, R, *, \vDash\rangle$, where O and S are subsets of K; $U = S \cap O$ and $U \neq \emptyset$. Then, R, K, and $*$ are as in Definition 9, except that (d1) is replaced by

(d1b) $\quad a \leq b =_{\text{df}} (\exists x \in U)Rxab$

(P6), by

(P6b) $\quad (\exists x \in U)Raxa$

and (P11) by

(P11b) $\quad a \in O \;\Rightarrow\; a* \leq a$

(P11b$'$) $\quad a \in S \Rightarrow a \leq a*$

Validity is then defined in respect of all members of U. Canonically, O^C is the set of all prime, regular Π''-theories; and S^C, the set of all prime w-consistent theories.

Notice, however, that it has not been necessary to isolate the set S from O in order to prove the completeness of Π'. Therefore, we have preferred to define Π'-models and Π''-models as in Definitions 3 and 9.

Nevertheless, notice, on the other hand, that this new definition of Π''-models makes still more clear what has doubtless been observed by the reader by now: the completeness proof of Π'' shows that it is a conservative extension of Π'.

9. Relevance and paraconsistency

The concept of paraconsistency is clearly rendered as follows:

> Let $\vDash$ be a relation of logical consequence, defined either semantically or proof theoretically. Let us say that $\vDash$ is *explosive* iff for every formula A and B, $\{A, \neg A\} \vDash B$ [...]. A logic is said to be *paraconsistent* iff its relation of logical consequence is not explosive. [6]

As is known, Π' and Π'' are relevant logics, but when interpreted as in this paper, they are not paraconsistent logics: we have seen in Section 2 that rule (ECQ) $A \wedge \neg A \vdash B$ is a rule of Π' and Π''. Nevertheless, it is, of course, possible to define consequence relations so that Π' and Π'' are paraconsistent in addition to being relevant. The idea is essentially to consider γ only as a *rule of proof*, not as a rule of inference, as in Section 2. With this proviso, the concept of *Theorem of* Π' can be understood as in Definition 2; and the notion of Π'*-valid wff*, as in Definition 4. In the case of Π'', these notions are understood in a similar way (cf. Section 8). Then, soundness and theorem-completeness can easily be proved independently of Theorems 1–4:

Theorem 5 (Soundness and completeness of Π' and Π'').

(i) $\vdash_{\Pi'} A$ *iff* $\vDash_{\Pi'} A$.

(ii) $\vdash_{\Pi''} A$ *iff* $\vDash_{\Pi''} A$.

Proof. (i) It has been shown that (A1)–(A15) are Π'-valid and, in fact, that (α), (β), (γ) and (δ') preserve Π'-validity (cf. Section 3).

Suppose $\nvdash_{\Pi'} A$, i.e., $A \notin \Pi'$. By Lemma 10, there is some x in O^C such that $\Pi' \subseteq x$ and $A \notin x$. As the Π'-canonical model is indeed a model (Lemma 13), $x \nvDash^C A$, and consequently, $\nvDash_{\Pi'} A$.

The proof of the case (ii) is similar. ⊣

Consider now the following definition of semantical consequence.

Definition 11. Let Γ be a set of wff and A a wff. Then, $\Gamma \vDash^2_{\Pi'} A$ ($\Gamma \vDash^2_{\Pi''} A$) iff for all $a \in K$ in all Π'-models (respectively, all Π''-models), if $a \vDash \Gamma$, then $a \vDash A$ ($a \vDash \Gamma$ iff $a \vDash B$ for every $B \in \Gamma$.)

Remark 8. Let Γ be the set $\{B_1, \ldots, B_m\}$. By Lemma 2, it is clear that the following is obtained: $\Gamma \vDash^2_{\Pi'} A$ ($\Gamma \vDash^2_{\Pi''} A$) iff $\vDash_{\Pi'} (B_1 \wedge \cdots \wedge B_m) \to A$ ($\vDash_{\Pi''} (B_1 \wedge \cdots \wedge B_m) \to A$). Then, it is easy to prove that the rule ECQ fails semantically, as it is shown in what follows.

Let p_i be the i-th variable. Consider the set $y = \{B \mid \vdash_{\Pi''} (p_i \wedge \neg p_i) \to B\}$. It is easy to prove that y is a Π''-theory. Now, by using Ackermann's matrices ([1], pp. 127–128), it is shown that $(p_i \wedge \neg p_i) \to (p_i \to p_i)$ is not a theorem of Π''. So, $p_i \to p_i \notin y$. Now, by Lemma 7, y is extended to a prime theory x such that $y \subseteq x$ and $p_i \to p_i \notin x$. By Lemma 13, $x \vDash p_i \wedge \neg p_i$ but $x \nvDash p_i \to p_i$. Consequently, rule (ECQ) $A \wedge \neg A \vDash B$ fails.

Next, let us specify to which proof-theoretical consequence relation the semantical relation introduced in Definition 12 corresponds.

Consider the following rules

(Π'-entailment) Given $\vdash_{\Pi'} A \to B$, from A to infer B

(Π''-entailment) Given $\vdash_{\Pi''} A \to B$, from A to infer B

Now, let us define:

Definition 12. Let Γ be a set of wff and A be a wff. Then, $\Gamma \vdash^2_{\Pi'} A$ ($\Gamma \vdash^2_{\Pi''} A$) iff there is a finite sequence of wff $B_1, \ldots, B_m$ such that B_m is A and for each i ($1 \leq i \leq m$), one of the following is the case: (a) $B_i \in \Gamma$, (b) B_i is obtained from previous formulas by β, (c) B_i is obtained from previous formulas by Π'-entailment (respectively, by Π''-entailment).

Remark 9. Notice that, given that $[A \wedge (A \to B)] \to B$ is a theorem of Π', the rule of inference (α) can be applied both in $\vdash^2_{\Pi'}$-proofs and $\vdash^2_{\Pi''}$-proofs.

Definition 13. For any set of wff $\mathit{\Gamma}$,

(i) $\mathrm{con}\Pi'\mathit{\Gamma} =_{\mathrm{df}} \{A \mid \mathit{\Gamma} \vDash^2_{\Pi'} A\}$,

(ii) $\mathrm{con}\Pi''\mathit{\Gamma} =_{\mathrm{df}} \{A \mid \mathit{\Gamma} \vDash^2_{\Pi''} A\}$.

Then, we immediately have:

Proposition 12. *For any set of wff $\mathit{\Gamma}$,*

(i) $\mathrm{con}\Pi'\mathit{\Gamma}$ *is a* Π'*-theory.*

(ii) $\mathrm{con}\Pi''\mathit{\Gamma}$ *is a* Π''*-theory.*

And, unsurprisingly enough, we obtain:

Theorem 6. *For any set of wff $\mathit{\Gamma}$ and wff A,*

(i) $\mathit{\Gamma} \vdash^2_{\Pi'} A$ *iff* $\mathit{\Gamma} \vDash^2_{\Pi'} A$.

(ii) $\mathit{\Gamma} \vdash^2_{\Pi''} A$ *iff* $\mathit{\Gamma} \vDash^2_{\Pi''} A$.

Proof. (i) If $\mathit{\Gamma} \vdash^2_{\Pi'} A$, then $\mathit{\Gamma} \vDash^2_{\Pi'} A$. Induction on the length of the proof $\mathit{\Gamma} \vdash^2_{\Pi'} A$. If $A \in \mathit{\Gamma}$ or A is by (β), the proof is trivial. Suppose then, that A is by Π'-entailment. Then, $\mathit{\Gamma} \vdash^2_{\Pi'} C$, $C \to A$ being a theorem of Π'. Now, let $a \vDash \mathit{\Gamma}$ for $a \in K$ in an arbitrary model. By hypothesis, $\mathit{\Gamma} \vDash^2 C$. So, $a \vDash C$. On the other hand, $\vDash_{\Pi'} C \to A$ by Theorem 5. Then, by Lemma 2, $a \vDash A$, as was to be proved.

If $\mathit{\Gamma} \vDash^2_{\Pi'} A$ then $\mathit{\Gamma} \vdash^2_{\Pi'} A$. The proof is similar to that of Theorem 2 (but notice that now it suffices to build up a member in K^C).

The proof of the case (ii) is similar. ⊣

We end this paper with a couple of observations.

As we have seen, if (γ) is restricted to a rule of proof, Π' and Π'' are paraconsistent logics, besides being relevant. In this case, Π'' is equivalent “as to theorems” to the extension of E described in Section 1. But E, formulated as in [2, §27.1.1] or in [3, §R2], and Π' are not, however, equivalent logics: (Asser) is not a derived rule of E, and to our knowledge, to date, it has not been shown that it is an admissible rule of this logic.

10. Appendix

Consider the following set of matrices where designated values are starred:

$\to$	0	1	2	3	$\neg$
0	2	2	2	2	3
*1	0	2	2	2	2
*2	0	0	2	2	1
*3	0	0	0	2	0

$\wedge$	0	1	2	3
0	0	0	0	0
*1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
*1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

This set satisfies the axioms and rules of E as axiomatized in [2, §27.1.1] or [3, §R2] (cf. Section 1), but falsifies the rule Assertion

(Asser) $\quad \vdash A \;\Rightarrow \vdash (A \to B) \to B$

when $A = B = 1$, or the rule (δ'), when $A = 2$, $B = C = 1$.

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