A Saddle Point Approach to the Computation of Harmonic Maps *

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Abstract

In this paper we consider numerical approximations of a constraint minimization problem, where the object function is a quadratic Dirichlet functional for vector fields and where the interior constraint is given by a convex function. The solutions of this problem are usually referred to as harmonic maps. Minimization problems of the form studied here arise for example in liquid crystal and superconductor simulations. The solution is characterized by a nonlinear saddle point problem, and we show that the corresponding linearized problem is well–posed near the exact solution. The main result of this paper is to establish a corresponding result for a proper finite element discretization of the harmonic map problem. Iterative schemes for the discrete nonlinear saddle point problems are investigated. Some mesh independent preconditioners for the iterative methods are also proposed.

Key words: harmonic maps, nonlinear constraints, saddle point problems, error estimates.

1 Introduction

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ we shall consider the problem of finding local minima of a constrained minimization problem of the form:

$$\min_{\mathbf{v}\in\mathbf{H}^1_{\mathbf{g}}(\Omega;\mathcal{M})} \mathcal{E}(\mathbf{v}) = \frac{1}{2}\int_\Omega |\nabla\mathbf{v}|^2 d\mathbf{x}. \tag{1.1}$$

Here $\mathbf{H}^1_{\mathbf{g}}(\Omega;\mathcal{M})$ is the set of vector fields with values in a smooth, compact manifold $\mathcal{M}$ in $\mathbb{R}^d$, with function values and first derivatives in $L^2(\Omega)$, and such that the elements $\mathbf{v}$ of $\mathbf{H}^1_{\mathbf{g}}(\Omega;\mathcal{M})$ satisfies $\mathbf{v}|_{\partial\Omega} = \mathbf{g}$ for fixed vector field $\mathbf{g}$ defined on the boundary $\partial\Omega$. We will further assume that $\mathcal{M}$ is implicitly given on the form

$$\mathcal{M} = \{\mathbf{v}\in\mathbb{R}^d \,|\, F(\mathbf{v}) = 0\,\},$$

where the function $F : \mathbb{R}^d \to \mathbb{R}^k$ is a smooth function, and it will be assumed that the compatibility condition $F(\mathbf{g}) = 0$ holds. More specific assumptions on F and the boundary data $\mathbf{g}$ will be given below. Problems of the form (1.1) arise for example in liquid crystal and superconductor simulations. The solutions of the problem (1.1) are frequently referred as harmonic maps, [3]. In

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the present paper we will restrict our study to the case $d = 2$ and $k = 1$. We will focus on a nonlinear saddle point approach to compute the solutions of the problem (1.1).

For a review of results on the continuous harmonic map problem we refer to [3, 17]. The purpose of the present paper is to discuss a finite element method for approximating the constraint minimization problem (1.1). For the simplest case of (1.1), with interior constraint given by $|\mathbf{v}| = 1$, several numerical approaches have been discussed, cf. for example [1], [2], [9], [10], [12], [13] and [19]. Variants of the projection method are proposed and analyzed in [1], [2] and [9]. However, the standard projection method applies only to the simplest model. Moreover, it was illustrated in [2] that the projection method converges only for very special regular and quasi-uniform triangulations for the discretized harmonic map problem. The relaxation method of [12] is using point relaxation with the constraint required at each grid point. Both convergence analysis and numerical experiments are supplied in [12]. A common approach for constrained minimzation problems, adopted for example in [10], is the penalty method. However, in general it may be hard to design an efficient iterative method for solving the discrete variational problem in this case, since it is difficult to resolve the penalty term accurately.

The main contribution of the present paper is to use a saddle point approach for the construction of numerical methods for the constraint minimization problem (1.1). We shall prove that the corresponding saddle point problem is stable near the exact solution. This is achieved by verifying standard stability conditions for linear saddle point problems. This verification has the extra difficulty that the coercivity condition will not hold in general, but only on the kernel of the linearized constraint. Using the standard stability conditions for the corresponding discrete saddle point problem we will construct finite element methods such that the corresponding discrete solutions admit an optimal error estimate in the energy norm. We will also study Newton's method for the discrete nonlinear saddle point problem, and propose a simple and efficient preconditioner for the linear systems arising during the iterations. Numerical tests will be given to show the efficiency of the proposed method.

The outline of the paper is as follows. In Section 2, the notations and assumption will be specified. In Section 3, the continuous problem is studied. The problem (1.1) is formally transformed to a saddle point problem, and stability results will be proved for the continuous model. In Section 4 we first describe a finite element discretization for (1.1), and then the discrete stability conditions are established. Using these stability conditions, the existence, local uniqueness and the error estimates are derived in Section 5. Iterative methods are analyzed in Section 6, while numerical experiments are presented in Section 7.

2 Notation and preliminaries

Throughout this paper we will use c and C to denote generic positive constants, not necessarily the same at different occurrences. It is assumed that the constants are independent of the mesh size h which will be introduced later. For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ we use $\mathbf{v} \cdot \mathbf{w}$ to denote the Euclidian inner product, while the notation $\mathbf{A} : \mathbf{B}$ is used to denote the Frobenius inner product of two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d \times d}$. The corresponding norms are given by $|\mathbf{v}|$ and $|\mathbf{A}|$, respectively. For a vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, $\mathbf{v}^\perp = (-v_2, v_1)$ is the vector obtained by a rotation of 90 degrees. For a vector or matrix $\mathbf{A}$, $\mathbf{A}^t$ is the transpose of $\mathbf{A}$.

For $m \geq 0$ we will use $H^m = H^m(K)$ to denote the real valued L^2- based Sobolev spaces on domain $K \subset \mathbb{R}^d$, the corresponding norm by $\| \cdot \|_{m,K}$, and $| \cdot |_{m,K}$ is the semi norm involving only the mth order derivatives. The subspace H_0^m is the closure in H^m of $C_0^\infty(K)$, while H^{-m} is the dual of H_0^m with respect to an extension of the L^2 inner product $\langle \cdot, \cdot \rangle$. The corresponding $L^\infty-$ based Sobolev spaces are denoted $W^{m,\infty}(K)$, with associated norm $\| \cdot \|_{m,\infty,K}$. For all the Sobolev norms, we will omit K in case $K = \Omega$. In general we will use boldface symbols for vector or matrix valued functions. The gradient operator with respect to the spatial variable $\mathbf{x} = (x_1, x_2)$ is denoted $\nabla = (\partial/\partial x_1, \partial/\partial x_2)^t$. Furthermore, the gradient of a vector valued function $\mathbf{v} = (v_1, v_2)^t$, $\nabla \mathbf{v}$, is the matrix valued function obtained by taking the gradient row–wise, i.e. $(\nabla \mathbf{v})_{ij} = \partial v_i / \partial x_j$.

In order to specify the properties of the constraint functional $F : \mathbb{R}^2 \to \mathbb{R}$, defining the constraint manifold $\mathcal{M}$, we will use $\mathbf{D}F$ to denote the gradient of F, i.e. $\mathbf{D}F(\mathbf{v}) = (\partial F/\partial v_1, \partial F/\partial v_2)^t$ and the corresponding Hessian by $\mathbf{D}^2F(\mathbf{v}) = (\partial^2 F/\partial v_i \partial v_j)_{i,j=1}^2$. Throughout this paper we will assume that the constraint functional F satisfies:

(i) F is convex and smooth. Furthermore, there exist constants c_0 and c_1 such that

$$c_0|\mathbf{v}|^2 \leq \mathbf{D}^2F(\xi)\mathbf{v}\cdot\mathbf{v} \leq c_1|\mathbf{v}|^2, \quad \xi, \mathbf{v} \in \mathbb{R}^2. \tag{2.1}$$

(ii) $F(\mathbf{0}) < 0$ and $\mathbf{D}F(\mathbf{0}) = 0$;

(iii) There exists an $\ell > 0$ such that the matrix function $\mathbf{D}^2F$ satisfies

$$|\mathbf{D}^2F(\xi_1) - \mathbf{D}^2F(\xi_2)| \leq \ell|\xi_1 - \xi_2|, \quad \xi_1, \xi_2 \in \mathbb{R}^2. \tag{2.2}$$

The analysis below will still hold if the assumptions (2.1) and (2.2) are only valid for all ξ, ξ_1, ξ_2 in a neighborhood of a continuous solution.

For the boundary function $\mathbf{g}$ of (1.1) we assume that it has been extended into the interior of Ω such that $\mathbf{g} \in \mathbf{H}^1(\Omega)$. Corresponding to $\mathbf{g}$, we let

$$\mathbf{H}^1_{\mathbf{g}}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} = \mathbf{g} \text{ on } \partial\Omega\}.$$

If $\mathbf{v} : \Omega \to \mathbb{R}^2$ is a smooth vector field then it follows from the chain rule that

$$\nabla F(\mathbf{v}) = (\nabla \mathbf{v})^t \mathbf{D}F(\mathbf{v}), \tag{2.3}$$

where the product on the right hand side is the ordinary matrix–vector product. Furthermore, we have

$$\nabla \mathbf{D}F(\mathbf{v}) = \mathbf{D}^2F(\mathbf{v})\nabla\mathbf{v}. \tag{2.4}$$

From assumption (i)-(ii) and the Taylor expansion we obtain the following estimate:

$$2c_1^{-1}|F(\mathbf{0}) \leq |\mathbf{v}(\mathbf{x})|^2 \leq 2c_0^{-1}|F(\mathbf{0})|, \quad \mathbf{x} \in \Omega,$$

for any $\mathbf{v}$ satisfying $F(\mathbf{v}) \equiv 0$ on Ω. Similarly, we derive

$$|\mathbf{D}F(\mathbf{v})| \geq c_0|\mathbf{v}|$$

for any $\mathbf{v}$, and hence $|\mathbf{D}F(\mathbf{v}(\mathbf{x}))| > 0$ if $\mathbf{v}(\mathbf{x}) \in \mathcal{M}$.

Let us note that the interior constraint in (1.1), given by $\mathbf{v}(\mathbf{x}) \in \mathcal{M}$, implies that a local minimum of (1.1) satisfies $\mathbf{u} \in \mathbf{H}^1_{\mathbf{g}}(\Omega) \cap \mathbf{L}^\infty(\Omega)$. In fact, if the boundary $\partial\Omega$ and the boundary data $\mathbf{g}$ are sufficiently regular, and $\mathcal{M}$ is the unit circle $\mathbf{S}^1$, then there is a unique smooth solution of (1.1), cf. [3, Theorem 12].

We will consider the more general problem of approximating any critical point of the functional $\mathcal{E}$ over $\mathbf{H}^1_{\mathbf{g}}(\Omega; \mathcal{M})$. A vector field $\mathbf{u} \in \mathbf{H}^1_{\mathbf{g}}(\Omega; \mathcal{M})$ is such a critical point if it satisfies

$$\langle \nabla\mathbf{u}, \nabla\mathbf{v}\rangle = 0 \tag{2.5}$$

for any $\mathbf{v}$ in the tangent space of $\mathbf{H}^1_{\mathbf{g}}(\Omega; \mathcal{M})$ at $\mathbf{u}$, i.e. for any $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ such that $\mathbf{D}F(\mathbf{u}) \cdot \mathbf{v} \equiv 0$. In the saddle point approach which we shall consider here we will view the critical points $\mathbf{u}$ as elements of the larger space $\mathbf{H}^1_{\mathbf{g}}(\Omega)$. Assume that $\mathbf{u}$ has the extra regularity property that

$$\mathbf{u} \in \mathbf{H}^1_{\mathbf{g}}(\Omega) \cap \mathbf{W}^{1,\infty}(\Omega). \tag{2.6}$$

Then any such $\mathbf{u}$ is a critical point if and only if there is a $\lambda \in L^2(\Omega)$ such that the pair $(\mathbf{u}, \lambda)$ satisfies the first order conditions

$$\begin{aligned} \langle \nabla\mathbf{u}, \nabla\mathbf{v}\rangle + \langle \mathbf{D}F(\mathbf{u})\cdot\mathbf{v}, \lambda\rangle &= 0, \quad \mathbf{v} \in \mathbf{H}^1_0(\Omega), \\ \langle F(\mathbf{u}), \mu\rangle &= 0, \quad \mu \in L^2(\Omega). \end{aligned} \tag{2.7}$$

To see this assume that $\mathbf{u}$ is a critical point satisfying (2.6), and let $\mathbf{z} = \mathbf{D}F(\mathbf{u})/|\mathbf{D}F(\mathbf{u})|$. For any $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ let $\mathbf{v}_\tau = \mathbf{v} - (\mathbf{v}\cdot\mathbf{z})\mathbf{z}$. As a consequence $\mathbf{D}F(\mathbf{u})\cdot\mathbf{v}_\tau = 0$, and by (2.5),

$$0 = \langle \nabla\mathbf{u}, \nabla\mathbf{v}_\tau\rangle = \langle \nabla\mathbf{u}, \nabla\mathbf{v}\rangle - \langle \nabla\mathbf{u}, \nabla(\mathbf{v}\cdot\mathbf{z})\mathbf{z}\rangle.$$

However, by using (2.3) the constraint implies that $(\nabla\mathbf{u})^t\mathbf{z} = 0$ and therefore the final inner product above can be rewritten as

$$\langle \nabla\mathbf{u}, \nabla(\mathbf{v}\cdot\mathbf{z})\mathbf{z}\rangle = \langle \nabla\mathbf{u} : \nabla\mathbf{z}, \mathbf{v}\cdot\mathbf{z}\rangle.$$

Hence the system (2.7) is satisfied with

$$\lambda = -\nabla\mathbf{u} : \nabla\mathbf{z}/|\mathbf{D}F(\mathbf{u})| = -\nabla\mathbf{u} : \nabla\mathbf{D}F(\mathbf{u})/|\mathbf{D}F(\mathbf{u})|^2, \tag{2.8}$$

where the last identity again is a consequence of the constraint. Note that it follows from (2.6) that the multiplier λ is actually in $L^\infty(\Omega)$.

The variational problem (2.7) is the Euler-Lagrangian equation for the constrained minimization problem (1.1), and the system is a weak formulation of the problem

$$\begin{aligned} -\Delta\mathbf{u} + \lambda\mathbf{D}F(\mathbf{u}) &= 0, \quad \text{in } \Omega, \\ F(\mathbf{u}) &= 0, \quad \text{in } \Omega. \end{aligned} \tag{2.9}$$

In the simplest case when $\mathcal{M} = \mathbf{S}^1$, we have $\lambda = -|\nabla\mathbf{u}|^2$ and

$$-\Delta\mathbf{u} - |\nabla\mathbf{u}|^2\mathbf{u} = 0, \qquad \text{in } \Omega, \quad \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega.$$

This equation is frequently referred as *the harmonic map equation* [3].

We would like to point out a relationship between the saddle point approach and the penalty method. In the commonly used penalty approach, c.f. [10], one is seeking a minimizer of the following regularized problem:

$$\min_{\mathbf{v}\in\mathbf{H}^1_\mathbf{g}(\Omega)} \mathcal{E}(\mathbf{v}) + \frac{1}{2\epsilon}\int_\Omega |F(\mathbf{v})|^2 d\mathbf{x},$$

where the penalty parameter $\epsilon > 0$ is small. Formally, the necessary equilibrium condition for this problem is that

$$\int_\Omega \nabla\mathbf{u}^\epsilon\cdot\nabla\mathbf{v}d\mathbf{x} + \frac{1}{\epsilon}\int_\Omega F(\mathbf{u}^\epsilon)\mathbf{D}F(\mathbf{u}^\epsilon)\cdot\mathbf{v}d\mathbf{x} = 0, \quad \mathbf{v}\in\mathbf{H}_0^1(\Omega).$$

A difficulty with this approach is that the penalty parameter ϵ needs to be chosen sufficiently small in order to resolve the constraint, and usually it also needs to be related to the discretization parameter. However, for small penalty parameters, numerical instabilities may occur.

In order to see the relation between the penalty method and the saddle point system (2.7) we introduce $\lambda^\epsilon = \frac{1}{\epsilon}F(\mathbf{u}^\epsilon)$. The above system then reduces to

$$\begin{aligned} \langle\nabla\mathbf{u}^\epsilon, \nabla\mathbf{v}\rangle + \langle\mathbf{D}F(\mathbf{u}^\epsilon)\cdot\mathbf{v}, \lambda^\epsilon\rangle &= 0, \quad \mathbf{v}\in\mathbf{H}_0^1(\Omega), \\ \langle F(\mathbf{u}^\epsilon), \mu\rangle - \epsilon\langle\lambda^\epsilon, \mu\rangle &= 0, \quad \mu\in L^2(\Omega). \end{aligned}$$

If $\epsilon \to 0$, we see that the above system formally converges to the saddle point system (2.7), i.e. the saddle point approach can be regarded as the limit case of the penalty system. The advantage of the saddle point approach is that the standard mixed finite element theory, cf. [5], tells us how to choose the finite element spaces properly to avoid possible instabilities. Furthermore, there is no need to choose a penalty parameter.

3 Stability of the linearized problem

Throughout this paper we will assume that the pair $(\mathbf{u}, \lambda)$ is a solution of (2.7) with the additional regularity property that

$$\mathbf{u} \in \mathbf{H}^1_{\mathbf{g}}(\Omega) \cap \mathbf{W}^{1,\infty}(\Omega), \quad \lambda \in L^\infty(\Omega). \tag{3.1}$$

In particular, $\mathbf{u}$ and λ are related by (2.8). For the analysis below it will be useful to consider linearization of the saddle point system (2.7). More precisely, we consider systems of the form:

Find $(\mathbf{v}, \mu) \in \mathbf{H}^1_0(\Omega) \times H^{-1}(\Omega)$ such that

$$\begin{aligned} a(\mathbf{u}, \lambda; \mathbf{v}, \hat{\mathbf{v}}) + \langle \mathbf{D}F(\mathbf{u}) \cdot \hat{\mathbf{v}}, \mu \rangle &= \langle \mathbf{f}, \mathbf{v} \rangle, \quad \hat{\mathbf{v}} \in \mathbf{H}^1_0(\Omega), \\ \langle \mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}, \hat{\mu} \rangle &= \langle \sigma, \mu \rangle, \quad \hat{\mu} \in H^{-1}(\Omega), \end{aligned} \tag{3.2}$$

where $(\mathbf{u}, \lambda)$ is the exact solution of (2.9) satisfying (3.1), and the bilinear form $a(\mathbf{u}, \lambda; \cdot, \cdot)$ is given by

$$a(\mathbf{u}, \lambda; \mathbf{v}, \hat{\mathbf{v}}) = \langle \nabla \mathbf{v}, \nabla \hat{\mathbf{v}} \rangle + \langle \mathbf{D}^2 F(\mathbf{u}) \mathbf{v} \cdot \hat{\mathbf{v}}, \lambda \rangle.$$

Here $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ and $\sigma \in H^1_0(\Omega)$ represents data. Our goal is to show that this linear system is well–posed by verifying the standard stability conditions for saddle points systems, cf. [4] or [5]. It should be noted that the bilinear form $a(\mathbf{u}, \lambda; \cdot, \cdot)$ is in general not coercive on $\mathbf{H}^1_0(\Omega)$. For example, in the simplest case, when $\mathcal{M} = \mathbf{S}^1$, we have

$$a(\mathbf{u}, \lambda; \mathbf{v}, \mathbf{v}) = \int_\Omega (|\nabla \mathbf{v}|^2 - |\nabla \mathbf{u}|^2 |\mathbf{v}|^2) \, d\mathbf{x}.$$

However, as we shall show below, the bilinear form $a(\mathbf{u}, \lambda; \cdot, \cdot)$ is coercive on the proper subspace of $\mathbf{H}^1_0(\Omega)$.

Associated with the solution $(\mathbf{u}, \lambda)$ satisfying (3.1), we define

$$\mathbf{Z}_{\mathbf{u}} = \{\mathbf{v} \in \mathbf{H}^1_0(\Omega) : (\mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}, \hat{\mu}) = 0, \quad \hat{\mu} \in L^2(\Omega)\}.$$

We shall also frequently use the following estimate often

$$a(\mathbf{u}, \lambda; \mathbf{v}, \hat{\mathbf{v}}) \le C(\mathbf{u}, \lambda) |\mathbf{v}|_1 |\hat{\mathbf{v}}|_1 \ \mathbf{v}, \hat{\mathbf{v}} \in \mathbf{H}^1_0(\Omega) \tag{3.3}$$

where the constant $C(\mathbf{u}, \lambda)$ depends on the norms of $\mathbf{u}$ and λ indicated by (3.1).

A key property for the analysis below is that the bilinear form $a(\mathbf{u}, \lambda; \cdot, \cdot)$ is coercive on the linearized constraint space $\mathbf{Z}_{\mathbf{u}}$. This is stated in the following theorem.

Theorem 3.1 *Let* $(\mathbf{u}, \lambda)$ *satisfy* (3.1) *and be related by (2.8). Then there is a positive constant* β_1*, depending on* $\mathbf{u}$*, such that*

$$a(\mathbf{u}, \lambda; \mathbf{v}, \mathbf{v}) = \langle \nabla \mathbf{v}, \nabla \mathbf{v} \rangle + \langle \mathbf{D}^2 F(u) \mathbf{v} \cdot \mathbf{v}, \lambda \rangle \ge \beta_1 \|\mathbf{v}\|_1^2, \quad \mathbf{v} \in \mathbf{Z}_{\mathbf{u}}. \tag{3.4}$$

Before we give the proof of the theorem we will establish an auxiliary result.

Lemma 3.1 *Let* $(\mathbf{u}, \lambda)$ *be as in Theorem 3.1 and define* $\mathbf{w} = (w_1, w_2)^t = \mathbf{D}F(\mathbf{u})$*. Then,*

$$\lambda \mathbf{D}^2 F(\mathbf{u}) \mathbf{w}^\perp \cdot \mathbf{w}^\perp = -\frac{w_1^2 |\nabla w_2|^2 + w_2^2 |\nabla w_1|^2 - 2 w_1 w_2 \nabla w_1 \cdot \nabla w_2}{|\mathbf{w}|^2}.$$

Proof. It follows from (2.4) and (2.8) that the multiplier λ can be expressed as $\lambda = -\nabla \mathbf{u} : \nabla \mathbf{w} / |\mathbf{w}|^2$. Hence,

$$\lambda \mathbf{D}^2 F(\mathbf{u}) \mathbf{w}^\perp \cdot \mathbf{w}^\perp = \frac{\nabla \mathbf{u} : \nabla \mathbf{w}}{|\mathbf{w}|^2} (F_{11} w_2^2 + F_{22} w_1^2 - 2 F_{12} w_1 w_2), \tag{3.5}$$

where $F_{ij} = \partial^2 F/\partial u_i \partial u_j$. Furthermore, since $\nabla F(\mathbf{u}) \equiv 0$ we have from (2.3) that

$$w_1 \nabla \mathbf{u}_1 + w_2 \nabla \mathbf{u}_2 = 0,$$

while (2.4) implies that

$$\nabla w_i = F_{i1} \nabla u_1 + F_{i2} \nabla u_2.$$

By combining these identities we obtain

$$\begin{aligned}
&(F_{11} w_2^2 + F_{22} w_1^2 - 2F_{12} w_1 w_2) \nabla u_1 \cdot \nabla w_1 \\
&\qquad = w_2^2 (F_{11} \nabla u_1 + F_{12} \nabla u_2) \cdot \nabla w_1 - w_1 w_2 (F_{22} \nabla u_2 + F_{12} \nabla u_1) \cdot \nabla w_1 \\
&\qquad = w_2^2 |\nabla w_1|^2 - w_1 w_2 \nabla w_1 \cdot \nabla w_2.
\end{aligned}$$

A similar argument shows that

$$(F_{11} w_2^2 + F_{22} w_1^2 - 2F_{12} w_1 w_2) \nabla u_2 \cdot \nabla w_2 = w_1^2 |\nabla w_2|^2 - w_1 w_2 \nabla w_1 \cdot \nabla w_2,$$

and hence the desired identity follows from (3.5). □

Proof of Theorem 3.1. As above we let $\mathbf{w} = \mathbf{D}F(\mathbf{u})$. For any $\mathbf{v} \in \mathbf{Z_u}$, there exists a $\alpha \in H_0^1(\Omega)$ such that $\mathbf{v} = \alpha \mathbf{w}^\perp$. The key identity we will use is the pointwise relation

$$|\nabla \mathbf{v}|^2 + \lambda \mathbf{D}^2 F(\mathbf{u}) \mathbf{v} \cdot \mathbf{v} = |\nabla(\alpha |\mathbf{w}|)|^2. \tag{3.6}$$

In order to verify this identity note that

$$\nabla(\alpha |\mathbf{w}|) = |\mathbf{w}| \nabla \alpha + \frac{\alpha}{|\mathbf{w}|} (w_1 \nabla w_1 + w_2 \nabla w_2).$$

Hence,

$$\begin{aligned}
|\nabla(\alpha |\mathbf{w}|)|^2 = |\mathbf{w}|^2 |\nabla \alpha|^2 + \frac{|\alpha|^2}{|\mathbf{w}|^2} |w_1 \nabla w_1 + w_2 \nabla w_2|^2 \\
+ 2\alpha (w_1 \nabla \alpha \cdot \nabla w_1 + w_2 \nabla \alpha \cdot \nabla w_2).
\end{aligned}$$

On the other hand,

$$|\nabla \mathbf{v}|^2 = |\mathbf{w}|^2 |\nabla \alpha|^2 + \alpha^2 |\nabla \mathbf{w}|^2 + 2\alpha (w_1 \nabla \alpha \cdot \nabla w_1 + w_2 \nabla \alpha \cdot \nabla w_2).$$

Therefore,

$$\begin{aligned}
|\nabla \mathbf{v}|^2 - |\nabla(\alpha |\mathbf{w}|)|^2 &= \alpha^2 \big(|\nabla \mathbf{w}|^2 - \frac{|w_1 \nabla w_1 + w_2 \nabla w_2|^2}{|\mathbf{w}|^2} \big) \\
&= \frac{\alpha^2}{|\mathbf{w}|^2} (w_1^2 |\nabla w_2|^2 + w_2^2 |\nabla w_1|^2 - 2 w_1 w_2 \nabla w_1 \nabla w_2) \\
&= -\lambda \mathbf{D}^2 F(\mathbf{u}) \mathbf{v} \cdot \mathbf{v},
\end{aligned}$$

where the last identity follows from Lemma 3.1. Hence, we have verified (3.6).

On the other hand, if $\mu = \alpha |\mathbf{w}|$ then $\mathbf{v} = \mu(\frac{w^\perp}{|\mathbf{w}|})$ and hence

$$\nabla \mathbf{v} = \frac{1}{|\mathbf{w}|} \mathbf{w}^\perp \cdot \nabla \mu + \mu \nabla (\frac{\mathbf{w}^\perp}{|\mathbf{w}|}).$$

Therefore, since $\mathbf{u}$ satisfies (3.1), Poincaré's inequality implies that

$$\|\nabla \mathbf{v}\|_0 \le c(\|\nabla \mu\|_0 + \|\mu\|_0) \le c \|\nabla(\alpha |\mathbf{w}|)\|_0,$$

where the constant c depends on $\mathbf{u}$. Together with (3.6) this implies the desired inequality of the theorem. □

Theorem 3.1 is one of the two required stability properties for a linear saddle point problem of the form (3.2). The second property is the so–called inf–sup condition established in the next theorem.

Theorem 3.2 *Let $(\mathbf{u}, \lambda)$ satisfy (3.1) and be related by (2.8). Then there is a positive constant β_2, depending on $\mathbf{u}$, such that*

$$\inf_{\mu \in H^{-1}(\Omega)} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{\langle \mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}, \mu \rangle}{\|\mathbf{v}\|_1 \|\mu\|_{-1}} \geq \beta_2. \tag{3.7}$$

Proof. For any $\mu \in H^{-1}(\Omega)$, there exists a $\varphi \in H_0^1(\Omega)$ such that

$$\frac{\langle \mu, \varphi \rangle}{\|\varphi\|_1} = \|\mu\|_{-1}. \tag{3.8}$$

Define $\mathbf{v} = \varphi \frac{\mathbf{w}}{|\mathbf{w}|^2}$, where as above $\mathbf{w} = \mathbf{D}F(\mathbf{u})$. Then, by Leibniz' rule there exists a $c > 0$, depending on $\mathbf{u}$, such that

$$\|\nabla \mathbf{v}\|_0 \leq c \|\varphi\|_1.$$

Furthermore,

$$\langle \mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}, \mu \rangle = \langle \varphi, \mu \rangle = \|\varphi\|_1 \|\mu\|_{-1}.$$

Hence, the desired inequality holds with $\beta_2 = 1/c$. □

4 A stable discretization

In the rest of the paper we assume that Ω is a polygonal domain. Given a shape regular and quasi–uniform family of triangulation $\{\mathcal{T}_h\}$ of Ω with a mesh size $h < 1$, let $\mathcal{N}_h$ denote the set of nodes associated with $\mathcal{T}_h$. We use V_h to denote the space of continuous piecewise linear functions and $V_{h,0} = V_h \cap H_0^1(\Omega)$. The notation $\mathbf{V}_h$ and $\mathbf{V}_{h,0}$ will be used for the vector version of the corresponding spaces. We will use π_h to denote the usual nodal interpolation operators onto the spaces V_h and $\mathbf{V}_h$. Standard approximation properties of spaces of piecewise linear functions will be used below. In particular, we will use the estimates

$$\|(I - \pi_h)v\|_1 \leq Ch|v|_2, \quad v \in H^2(\Omega), \tag{4.1}$$

and

$$\|(I - P_h)v\|_{-1} \leq Ch\|v\|_0, \quad v \in L^2(\Omega). \tag{4.2}$$

Here, $P_h : L^2(\Omega) \to V_{h,0}$ is the L^2 projection. Due to the quasi-uniformity of the mesh, the operator P_h can be extended to a uniformly bounded operator on H^{-1}. Moreover, the following inverse inequalities hold:

$$\|v\|_\infty \leq C \log(h^{-1})\|v\|_1, \quad \|v\|_1 \leq Ch^{-1}\|v\|_0, \quad v \in V_h. \tag{4.3}$$

Set $\mathbf{g}_h = \pi_h \mathbf{g}$ (on $\partial\Omega$). We define

$$\mathbf{V}_{h,\mathbf{g}} = \{\mathbf{v} \in \mathbf{V}_h : \ \mathbf{v}|_{\partial\Omega} = \mathbf{g}_h\}.$$

We will consider the following discretized minimization problem:

$$\min_{\mathbf{v} \in \mathbf{V}_{h,\mathbf{g}}} \mathcal{E}(\mathbf{v}) \text{ subject to } F(\mathbf{v}) = 0 \text{ on } \mathcal{N}_h. \tag{4.4}$$

The Lagrange functional $L : \mathbf{V}_{h,\mathbf{g}} \times V_{h,0} \mapsto R$ is

$$L(\mathbf{v}, \mu) = \mathcal{E}(\mathbf{v}) + \int_\Omega \mu \pi_h F(\mathbf{v}) d\mathbf{x} \quad (\mathbf{v}, \mu) \in \mathbf{V}_{h,\mathbf{g}} \times V_{h,0}. \tag{4.5}$$

The first order condition defining the critical points of L leads to the following discrete counter part of the nonlinear saddle point problem (2.7):

Find $(\mathbf{u}_h, \lambda_h) \in \mathbf{V}_{h,\mathbf{g}} \times V_{h,0}$ such that

$$\begin{aligned} \langle \nabla \mathbf{u}_h, \nabla \mathbf{v} \rangle + \langle \pi_h[\mathbf{D}F(\mathbf{u}_h) \cdot \mathbf{v}], \lambda_h \rangle = 0, \quad \mathbf{v} \in \mathbf{V}_{h,0}, \\ \langle \pi_h F(\mathbf{u}_h), \mu \rangle = 0, \quad \mu \in V_{h,0}. \end{aligned} \tag{4.6}$$

However, we shall first analyse the discrete counter part of the linearized system (3.2). For a given $(\hat{\mathbf{u}}, \hat{\lambda}) \in \mathbf{V}_{h,\mathbf{g}} \times V_{h,0}$, let us define the bilinear form $a_h(\hat{\mathbf{u}}, \hat{\lambda}; \cdot, \cdot)$ to be

$$a_h(\hat{\mathbf{u}}, \hat{\lambda}; \mathbf{v}, \hat{\mathbf{v}}) = \langle \nabla \mathbf{v}, \nabla \hat{\mathbf{v}} \rangle + \langle \pi_h[\mathbf{D}^2 F(\hat{\mathbf{u}})\mathbf{v} \cdot \hat{\mathbf{v}}], \hat{\lambda} \rangle.$$

Similarly as in (3.2) for the continuous problem, the linearized problem for (4.6) is to find $(\mathbf{v}, \mu) \in \mathbf{V}_{h,0} \times V_{h,0}$ such that

$$\begin{aligned} a_h(\hat{\mathbf{u}}, \hat{\lambda}; \mathbf{v}, \hat{\mathbf{v}}) + \langle \pi_h[\mathbf{D}F(\hat{\mathbf{u}}) \cdot \hat{\mathbf{v}}], \mu \rangle &= \langle \mathbf{f}, \hat{\mathbf{v}} \rangle, \quad \hat{\mathbf{v}} \in \mathbf{V}_{h,0} \\ \langle \pi_h[\mathbf{D}F(\hat{\mathbf{u}}) \cdot \mathbf{v}], \hat{\mu} \rangle &= \langle \sigma, \hat{\mu} \rangle, \quad \hat{\mu} \in V_{h,0}. \end{aligned} \tag{4.7}$$

For a given $\hat{\mathbf{u}} \in \mathbf{V}_{h,\mathbf{g}}$, define

$$Z_{h,\hat{\mathbf{u}}} = \{\mathbf{v} \in \mathbf{V}_{h,0} : \mathbf{D}F(\hat{\mathbf{u}}) \cdot \mathbf{v} = 0 \textit{ on } \mathcal{N}_h\}.$$

Lemma 4.1 *Let $\Phi : \mathbb{R}^2 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a smooth function. Then we have the following estimates for all $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in \mathbf{V}_h$:*

$$|\pi_h \Phi(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k)|_1 \le C \sum_{i=1}^{k} \|\mathbf{D}_{\mathbf{v}_i} \Phi\|_{0,\infty} |\mathbf{v}_i|_1; \tag{4.8}$$

$$\|(\pi_h - I)\Phi(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k)\|_0 \le Ch \sum_{i=1}^{k} \|\mathbf{D}_{\mathbf{v}_i} \Phi\|_{0,\infty} |\mathbf{v}_i|_1. \tag{4.9}$$

Above, the constant C is independent of h, Φ and $\mathbf{v}_i$. The norm $\|\mathbf{D}_{\mathbf{v}_i}\Phi\|_{0,\infty}$ stands for $\|\mathbf{D}_{\mathbf{v}_i}\Phi(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k)\|_{0,\infty}$.

Proof. For clarity, we shall only give the proof for $k = 2$. The extension of the proof for general cases is straight forward.

For an element $e \in \mathcal{T}_h$, let $p_i, i = 1, 2, 3$ be the vertexes of e. Under the condition that the finite element mesh $\mathcal{T}_h$ is regular and quasi-uniform, then we have the following equivalent H^1 norms for $\mathbf{v} \in \mathbf{V}_h$

$$|\mathbf{v}|_{1,e} \cong \sum_{i,j=1}^{3} |\mathbf{v}(p_i) - \mathbf{v}(p_j)|^2, \quad \mathbf{v} \in \mathbf{V}_h, e \in \mathcal{T}_h. \tag{4.10}$$

In particular,

$$|\pi_h \Phi(\mathbf{v}_1, \mathbf{v}_2)|_{1,e}^2 \le \sum_{i,j=1}^{3} |\Phi(\mathbf{v}_1(p_i), \mathbf{v}_2(p_i)) - \Phi(\mathbf{v}_1(p_j), \mathbf{v}_2(p_j))|^2.$$

Thus, we get (4.8) from the following estimate:

$$\begin{aligned} |\pi_h \Phi(\mathbf{v}_1, \mathbf{v}_2)|_{1,e}^2 \le 2 \sum_{i,j=1}^{3} \Big(&|\Phi(\mathbf{v}_1(p_i), \mathbf{v}_2(p_i)) - \Phi(\mathbf{v}_1(p_j), \mathbf{v}_2(p_i))|^2 \\ &+ |\Phi(\mathbf{v}_1(p_j), \mathbf{v}_2(p_i)) - \Phi(\mathbf{v}_1(p_j), \mathbf{v}_2(p_j))|^2 \Big) \\ \le 2 \sum_{i,j=1}^{3} \Big(&\|\mathbf{D}_{\mathbf{v}_1}\Phi\|_{0,\infty,e}^2 |\mathbf{v}_1(p_i) - \mathbf{v}_1(p_j)|^2 + \|\mathbf{D}_{\mathbf{v}_2}\Phi\|_{0,\infty,e}^2 |\mathbf{v}_2(p_i) - \mathbf{v}_2(p_j)|^2 \Big). \end{aligned}$$

Next, we estimate (4.9). By the definition of the interpolation operator π_h, we have:

$$(\pi_h - I)\Phi(\mathbf{v}_1, \mathbf{v}_2)(p) = \sum_{i=1}^{3} [\Phi(\mathbf{v}_1(p_i), \mathbf{v}_2(p_i)) - \Phi(\mathbf{v}_1(p), \mathbf{v}_2(p))]\chi_i(p) \quad p \in e,$$

where $\{\chi_i\}_{i=1}^3$ are the barycentric coordinates on e. From this, we see that

$$\begin{aligned}
&\|(\pi_h - I)\Phi(\mathbf{v}_1, \mathbf{v}_2)\|_{0,e}^2 \le C \sum_{i=1}^{3} \int_e |\big(\Phi(\mathbf{v}_1(p_i), \mathbf{v}_2(p_i)) - \Phi(\mathbf{v}_1, \mathbf{v}_2)\big)\chi_i|^2 \\
&\le C \sum_{i,j=1}^{3} \int_e \big(\|\mathbf{D}_{\mathbf{v}_1}\Phi\|_{0,\infty,e}^2 |\mathbf{v}_1(p_i) - \mathbf{v}_1|^2 + \|\mathbf{D}_{\mathbf{v}_2}\Phi\|_{0,\infty,e}^2 |\mathbf{v}_2(p_i) - \mathbf{v}_2|^2\big) \qquad (4.11) \\
&\le Ch^2 \sum_{i,j=1}^{3} \big(|\mathbf{D}_{\mathbf{v}_1}\Phi|_{0,\infty,e}^2 |\mathbf{v}_1|_{1,e}^2 + |\mathbf{D}_{\mathbf{v}_2}\Phi|_{0,\infty,e}^2 |\mathbf{v}_2|_{1,e}^2\big).
\end{aligned}$$

Thus, estimate (4.9) is verified. □

For the lemma above, it is essential that the functions $\mathbf{v}_i$ are finite element functions. If $\mathbf{v}_1 \in \mathbf{W}^{1,\infty}(\Omega)$ and $\mathbf{v}_2 \in \mathbf{V}_h$, then we obtain:

$$\|(\pi_h - I)\Phi(\mathbf{v}_1, \mathbf{v}_2)\|_0 \le Ch(\|\mathbf{D}_{\mathbf{v}_1}\Phi\|_{0,\infty}|\mathbf{v}_1|_{1,\infty} + \|\mathbf{D}_{\mathbf{v}_2}\Phi\|_{0,\infty}|\mathbf{v}_2|_1). \qquad (4.12)$$

The next results, which is essential for our analysis, is a discrete version of Theorem 3.1. As in the previous section $(\mathbf{u}, \lambda)$ is a solution of (2.7) satisfying (3.1).

Theorem 4.1 *There exists positive constants γ_0 and h_0 such that, for $(\hat{\mathbf{u}}, \hat{\lambda}) \in \mathbf{V}_{h,\mathbf{g}} \times V_{h,0}$ satisfying*

$$\|\hat{\mathbf{u}} - \pi_h \mathbf{u}\|_1 + \|\hat{\lambda} - P_h \lambda\|_{-1} \le \gamma / \log^2(h^{-1}) \qquad (4.13)$$

with $h \le h_0$ and $\gamma \le \gamma_0$, we have

$$a_h(\hat{\mathbf{u}}, \hat{\lambda}; \mathbf{v}, \mathbf{v}) \ge \beta_3 \|\mathbf{v}\|_1^2, \quad \mathbf{v} \in Z_{h,\hat{\mathbf{u}}}. \qquad (4.14)$$

Here the constants γ_0, h_0, β_3 depend on $\mathbf{u}$.

In order to prove the above theorem, we need to derive some auxiliary results. The main idea is to relate (4.14) to the continuous problem, and then use Theorem 3.1 and some approximate properties of the operators π_h and P_h. As before, we shall use $\mathbf{w} = \mathbf{D}F(\mathbf{u})$ with $\mathbf{u}$ being the true solution, see (3.1). Given a $(\hat{\mathbf{u}}, \hat{\lambda})$ satisfying (4.13), we define $\hat{\mathbf{w}} = \mathbf{D}F(\hat{\mathbf{u}})$. For any $\mathbf{v} \in \mathbf{Z}_{h,\hat{\mathbf{u}}}$, let us define

$$\alpha(p_i) = \frac{\mathbf{v}(p_i) \cdot \hat{\mathbf{w}}^\perp(p_i)}{|\hat{\mathbf{w}}(p_i)|^2}, \quad p_i \in \mathcal{N}_h. \qquad (4.15)$$

From the above definition, it is clear that

$$\alpha = \pi_h \left(\frac{\mathbf{v} \cdot \hat{\mathbf{w}}^\perp}{|\hat{\mathbf{w}}|^2} \right) \in V_{h,0} \ , \quad \mathbf{v} = \pi_h(\alpha \hat{\mathbf{w}}^\perp).$$

We have used the relation $\hat{\mathbf{w}} \cdot \mathbf{v} = 0$ on $\mathcal{N}_h$ in getting the last equality. Corresponding to the true solution $\mathbf{u}$ and a given $\hat{\mathbf{u}} \in \mathbf{Z}_{h,\hat{\mathbf{u}}}$, let $\varepsilon_h \in \mathbf{H}_0^1(\Omega)$ be the function given by $\varepsilon_h = \alpha \mathbf{w}^\perp - \mathbf{v}$. We see clearly that

$$\varepsilon_h + \mathbf{v} \in \mathbf{Z}_\mathbf{u}. \qquad (4.16)$$

For a given $\hat{\mathbf{u}}$ satisfying (4.13), one can verify by assumption (i), cf. (2.1), and the inverse estimate (4.3) that

$$|\mathbf{w}(p) - \hat{\mathbf{w}}(p)| = |\mathbf{D}F(\hat{\mathbf{u}}(p)) - \mathbf{D}F(\pi_h \mathbf{u}(p))| \le c_1 \gamma, \quad p \in \mathcal{N}_h.$$

Thus, by choosing γ small enough, one can guarantee that

$$0 < c|\mathbf{w}(p)| \le |\hat{\mathbf{w}}(p)| \le C|\mathbf{w}(p)|, \quad p \in \mathcal{N}_h. \qquad (4.17)$$

Lemma 4.2 *Let $(\hat{\mathbf{u}}, \hat{\lambda}) \in \mathbf{V}_{h,\mathbf{g}} \times V_{h,0}$ satisfy (4.13). Then we have the estimate*

$$\left|\pi_h\left(\varphi\frac{\hat{\mathbf{w}}}{|\hat{\mathbf{w}}|^2}\right)\right|_1 \le C|\varphi|_1, \quad \varphi \in V_{h,0},$$

where the constant C depends on $\mathbf{u}$.

Proof. Let $\psi = \pi_h\left(\varphi\frac{\hat{\mathbf{w}}}{|\hat{\mathbf{w}}|^2}\right)$. Using (4.10), we see that

$$\begin{aligned}|\psi|_{1,e}^2 &\le C\sum_{i,j}|\varphi(p_i)\tfrac{\hat{\mathbf{w}}(p_i)}{|\hat{\mathbf{w}}(p_i)|^2} - \varphi(p_j)\tfrac{\hat{\mathbf{w}}(p_j)}{|\hat{\mathbf{w}}(p_j)|^2}|^2 \\ &\le C\sum_{i,j}[\tfrac{|\varphi(p_i)-\varphi(p_j)|^2}{|\hat{\mathbf{w}}(p_i)|^2} + |\varphi(p_j)|^2\cdot|\tfrac{\hat{\mathbf{w}}(p_i)}{|\hat{\mathbf{w}}(p_i)|^2} - \tfrac{\hat{\mathbf{w}}(p_j)}{|\hat{\mathbf{w}}(p_j)|^2}|^2].\end{aligned} \tag{4.18}$$

It follows from (4.10) and (4.17) that

$$\sum_{i,j}\frac{|\varphi_h(p_i) - \varphi_h(p_j)|^2}{|\hat{\mathbf{w}}(p_i)|^2} \le C|\varphi|_{1,e}^2. \tag{4.19}$$

On the other hand, we have by (4.17) and assumption (iii), c.f. (2.2),

$$\begin{aligned}|\tfrac{\hat{\mathbf{w}}(p_i)}{|\hat{\mathbf{w}}(p_i)|^2} - \tfrac{\hat{\mathbf{w}}(p_j)}{|\hat{\mathbf{w}}(p_j)|^2}|^2 &\le C|\hat{\mathbf{w}}(p_i) - \hat{\mathbf{w}}(p_j)|^2 \le C|\hat{\mathbf{u}}(p_i) - \hat{\mathbf{u}}(p_j)|^2 \\ &\le C|(\hat{\mathbf{u}} - \pi_h\mathbf{u})(p_i) - (\hat{\mathbf{u}} - \pi_h\mathbf{u})(p_j)|^2 + |\pi_h\mathbf{u}(p_i) - \pi_h\mathbf{u}(p_j)|^2.\end{aligned}$$

Thus, we get by the inverse estimate (4.3) and (4.13) that

$$\begin{aligned}&\sum_{i,j}[|\varphi(p_j)|^2\cdot|\frac{\hat{\mathbf{w}}(p_i)}{|\hat{\mathbf{w}}(p_i)|^2} - \frac{\hat{\mathbf{w}}(p_j)}{|\hat{\mathbf{w}}(p_j)|^2}|^2] \\ &\le C\|\varphi\|_{0,\infty,e}^2\cdot|\hat{\mathbf{u}} - \pi_h\mathbf{u}|_{1,e}^2 + \|\varphi\|_{0,e}^2\cdot\|\pi_h\mathbf{u}\|_{1,\infty,e}^2 \\ &\le C(\gamma^2 + \|\mathbf{u}\|_{1,\infty,e}^2)\|\varphi\|_{1,e}^2.\end{aligned} \tag{4.20}$$

Substituting (4.19)-(4.20) into (4.18), we obtain the desired bound. □

Remark 4.1 *If we apply Lemma 4.1 on the function ψ defined by $\psi = \pi_h\left(\varphi\frac{\hat{\mathbf{w}}}{|\hat{\mathbf{w}}|^2}\right)$, we will get that*

$$|\psi|_1 \le C\log(h^{-1})|\varphi|_1.$$

The results we are getting here is better. We have removed the factor $\log(h^{-1})$.

Lemma 4.3 *Let $(\hat{\mathbf{u}}, \hat{\lambda}) \in \mathbf{V}_{h,\mathbf{g}} \times V_{h,0}$ satisfy (4.13). Then, there exist a h_0 and γ_0 depending on $\mathbf{u}$ such that the following estimate holds for $h \le h_0$ and $\gamma \le \gamma_0$*

$$a(\mathbf{u}, \lambda; \mathbf{v}, \mathbf{v}) \ge \frac{\beta_1}{2}|\mathbf{v}|_1^2, \quad \mathbf{v} \in \mathbf{Z}_{h,\hat{\mathbf{u}}}.$$

Proof. For any $\mathbf{v} \in \mathbf{Z}_{h,\hat{\mathbf{u}}}$, let α and ε_h be as defined in (4.15) and (4.16). From $\pi_h(\alpha\pi_h\mathbf{w}^\perp) = \pi_h(\alpha\mathbf{w}^\perp)$, we have

$$\varepsilon_h = (I - \pi_h)(\alpha\mathbf{w}^\perp) + \pi_h[\alpha\pi_h(\mathbf{w} - \hat{\mathbf{w}})^\perp]. \tag{4.21}$$

From (4.12) and also using the inverse inequality (4.3), we get that

$$\begin{aligned}|(I - \pi_h)(\alpha\mathbf{w}^\perp)|_1^2 &\le Ch^2\big(\|\mathbf{w}^\perp\|_{0,\infty}^2|\alpha|_1^2 + \|\alpha\|_{0,\infty}^2\|\mathbf{w}^\perp\|_{1,\infty}^2\big) \\ &\le Ch^2\log^2(h^{-1})\|\mathbf{u}\|_{1,\infty}^2|\alpha|_1^2.\end{aligned} \tag{4.22}$$

Note that there exists a ξ such that

$$\pi_h[\alpha\pi_h(\mathbf{w}-\hat{\mathbf{w}})^{\perp}] = \pi_h\Big[\alpha\pi_h\big(\pi_h\mathbf{D}^2F(\xi)(\pi_h\mathbf{u}-\hat{\mathbf{u}})\big)^{\perp}\Big]$$

A repeated application of (4.8) and (4.3) gives

$$|\pi_h[\alpha\pi_h(\mathbf{w}-\hat{\mathbf{w}})^{\perp}]|_1^2 \le C\log^4(h^{-1})|\alpha|_1^2|\pi_h\mathbf{u}-\hat{\mathbf{u}}|_1^2. \tag{4.23}$$

From Lemma 4.2, we see that

$$|\alpha|_1 \le C|\mathbf{v}|_1. \tag{4.24}$$

Combining (4.22)-(4.24) with (4.13), we see that

$$|\varepsilon_h|_1^2 \le C(h^2\log^2(h^{-1})\|\mathbf{u}\|_{1,\infty}^2+\gamma^2)|\alpha|_1^2 \le C(h^2\log^2(h^{-1})\|\mathbf{u}\|_{1,\infty}^2+\gamma^2)|\mathbf{v}|_1^2. \tag{4.25}$$

The following estimate follows from (3.3) and (3.4)

$$\begin{aligned} a(\mathbf{u},\lambda;\mathbf{v},\mathbf{v}) &= a(\mathbf{u},\lambda;\mathbf{v}+\varepsilon_h,\mathbf{v}+\varepsilon_h) - a(\mathbf{u},\lambda;\mathbf{v},\varepsilon_h) + a(\mathbf{u},\lambda;\varepsilon_h,\varepsilon_h) \\ &\ge C\beta_1|\mathbf{v}+\varepsilon_h|_1^2 - |\mathbf{v}|_1|\varepsilon_h|_1 - |\varepsilon_h|_1^2. \end{aligned} \tag{4.26}$$

Choosing h and γ small enough, we obtain the desired result from (4.25) and (4.26). $\square$

Proof of Theorem 4.1. In the proof, we always assume that h and γ are small. Note that

$$\begin{aligned} a_h(\hat{\mathbf{u}},\hat{\lambda};\mathbf{v},\mathbf{v}) - a(\mathbf{u},\lambda;\mathbf{v},\mathbf{v}) &= \langle\pi_h[\mathbf{D}^2F(\hat{\mathbf{u}})\mathbf{v}\cdot\mathbf{v}],\hat{\lambda}\rangle - \langle\mathbf{D}^2F(\mathbf{u})\mathbf{v}\cdot\mathbf{v},\lambda\rangle \\ &= \langle\pi_h[\mathbf{D}^2F(\hat{\mathbf{u}})\mathbf{v}\cdot\mathbf{v}],\hat{\lambda}-\lambda\rangle + \langle(\pi_h-I)[\mathbf{D}^2F(\hat{\mathbf{u}})\mathbf{v}\cdot\mathbf{v}],\lambda\rangle \\ &+ \langle(\mathbf{D}^2F(\hat{\mathbf{u}})-\mathbf{D}^2F(\mathbf{u}))\mathbf{v}\cdot\mathbf{v},\lambda\rangle = I_1+I_2+I_3. \end{aligned} \tag{4.27}$$

The meaning of I_i is self explainable. Since $\lambda\in L^2(\Omega)$, we get by (4.13)

$$\begin{aligned} \|\hat{\lambda}_h-\lambda\|_{-1} &\le \|\hat{\lambda}_h-P_h\lambda\|_{-1}+\|P_h\lambda-\lambda\|_{-1} \\ &\le \gamma/\log^2(h^{-1})+Ch\|\lambda\|_0. \end{aligned}$$

Using Lemma 4.1, we see that

$$|\pi_h[\mathbf{D}^2F(\hat{\mathbf{u}})\mathbf{v}\cdot\mathbf{v}]|_1 \le C|\mathbf{D}^2F(\hat{\mathbf{u}})\cdot\mathbf{v}|_{0,\infty}|\mathbf{v}|_1 + \|\mathbf{v}\|_{0,\infty}^2\|\mathbf{D}^3F(\hat{u})\|_{0,\infty}|\hat{u}|_1 \le C\log^2(h^{-1})|\mathbf{v}|_1^2.$$

For a small h, a combination of the above two inequalities leads to

$$|I_1| = |(\pi_h[\mathbf{D}^2F(\hat{\mathbf{u}})\mathbf{v}\cdot\mathbf{v}],\hat{\lambda}_h-\lambda)| \le C\log^2(h^{-1})\|\mathbf{v}\|_1^2(\gamma/\log^2(h^{-1})+Ch\|\lambda\|_0) \le C\gamma\|\mathbf{v}\|_1^2.$$

Again, we use Lemma 4.1 to prove that

$$\begin{aligned} |I_2| &= |((\pi_h-I)[\mathbf{D}^2F(\hat{\mathbf{u}})\mathbf{v}\cdot\mathbf{v}],\lambda)| \\ &\le \|(\pi_h-I)[\mathbf{D}^2F(\hat{\mathbf{u}})\mathbf{v}\cdot\mathbf{v}]\|_0\cdot\|\lambda\|_0 \le Ch\log^2(h^{-1})\|\mathbf{v}\|_1^2, \end{aligned}$$

and

$$\begin{aligned} |I_3| &= |((\mathbf{D}^2F(\hat{\mathbf{u}})-\mathbf{D}^2F(\mathbf{u}))\mathbf{v}\cdot\mathbf{v},\lambda)| \\ &\le \|(\mathbf{D}^2F(\hat{\mathbf{u}})-\mathbf{D}^2F(\mathbf{u}))\mathbf{v}\cdot\mathbf{v}\|_0\cdot\|\lambda\|_0 \le C\gamma\|\mathbf{v}\|_1^2. \end{aligned}$$

Choosing h and γ small enough, we obtain the desired result from Lemma 4.3 and the estimates above of the three terms appearing in (4.27). $\square$

Theorem 4.2 *Assume that $(\hat{\mathbf{u}},\hat{\lambda})\in\mathbf{V}_{h,\mathbf{g}}\times V_{h,0}$ satisfies the condition (4.13). There exists a constant β_4, which depends on $\mathbf{u}$, such that*

$$\inf_{\mu\in V_{h,0}}\sup_{\mathbf{v}\in\mathbf{V}_{h,0}}\frac{\langle\pi_h[\mathbf{D}F(\hat{\mathbf{u}})\cdot\mathbf{v}],\mu\rangle}{\|\mu\|_{-1}\|\mathbf{v}\|_1} \ge \beta_4. \tag{4.28}$$

Proof. For the φ given in (3.8), let $\varphi_h = P_h\varphi$. Then, we see that

$$\frac{\langle \mu_h, \varphi_h\rangle}{\|\varphi_h\|_1} \geq \beta_2 \|\mu_h\|_{-1}.$$

Define $\mathbf{v}_h = \pi_h\left[\varphi_h \frac{\mathbf{D}F(\hat{\mathbf{u}})}{|\mathbf{D}F(\hat{\mathbf{u}})|^2}\right]$. Then,

$$\langle \pi_h[\mathbf{D}F(\hat{\mathbf{u}})\cdot \mathbf{v}_h], \mu_h\rangle = \langle \mu_h, \varphi_h\rangle.$$

From Lemma 4.2, one gets that $|\mathbf{v}_h|_1 \leq C|\varphi_h|_1$. By collecting these estimates the theorem is established. □

Recall from the saddle point theory given [4] or [5], the two theorems, i.e. Theorems 4.1 and 4.2, assure existence, stability and uniqueness of the solution of the linearized saddle point system (4.7) when $(\hat{\mathbf{u}}, \hat{\lambda})$ satisfies (4.13). In the next section, we shall use these properties to prove some results for the corresponding nonlinear systems.

Remark 4.2 *If replacing $V_{h,0}$ by V_h in (4.28), the inf-sup condition (4.28) may not be satisfied. This is why we use the $V_{h,0}$, instead of V_h, as finite element space for the Lagrange multiplier.*

5 The discrete nonlinear problem

The main purpose of this section is to establish existence and uniqueness of solutions of the discretized nonlinear saddle point problem (4.6) in a neighborhood of a continuous solution $(\mathbf{u}, \lambda)$ of the system (2.7) satisfying the regularity assumption (3.1). Furthermore, we will show that the discrete solutions converge to the continuous solution with a linear rate with respect to the mesh parameter h. However, we start by summarizing some properties for the linearized saddle point systems.

For notational simplicity, we shall use X, X_h and $X_{h,\mathbf{g}}$ defined by $X = \mathbf{H}_0^1(\Omega) \times H^{-1}(\Omega)$, $X_h = \mathbf{V}_{h,0} \times V_{h,0}$, and $X_{h,\mathbf{g}} = \mathbf{V}_{h,\mathbf{g}} \times V_{h,0}$. Let $\|\cdot\|_X$ denote the norm on the product space $\mathbf{H}_0^1(\Omega) \times H^{-1}(\Omega)$, and let $\|\cdot\|_{X^*}$ denote the norm on the dual space $X^* = \mathbf{H}^{-1}(\Omega) \times H_0^1(\Omega)$. The norm $\|\cdot\|_{L(X,X^*)}$ will be used to denote the norm of a bounded linear operator from X to X^*. The spaces X_h and $X_{h,\mathbf{g}}$ are equipped with the norm of X, while X_h^* is equal to X_h as a set, but equipped with the dual norm of X with respect to the L^2 inner products. Similarly, the norm $\|\cdot\|_{L(X_h,X_h^*)}$ is the associated oprator norm.

Let $x = (\mathbf{u}, \lambda)$ be a solution of (2.7). Corresponding to x, let $G(x) \in X^*$ to be given by

$$\langle G(x), y\rangle = \langle \nabla\mathbf{u}, \nabla\mathbf{v}\rangle + \langle \mathbf{D}F(\mathbf{u})\cdot\mathbf{v}, \mu\rangle + \langle F(\mathbf{u}), \mu\rangle, \quad y = (\mathbf{v}, \mu) \in X,$$

As usual, $\langle\cdot,\cdot\rangle$ is the duality pairing which extends the standard L^2 inner product. Associated with G, we define a mapping $G'(x) : X \to X^*$ by

$$\langle G'(x)\cdot y, \hat{y}\rangle = a(\mathbf{u}, \lambda; \mathbf{v}, \hat{\mathbf{v}}) + \langle \mathbf{D}F(\mathbf{u})\cdot\hat{\mathbf{v}}, \mu\rangle + \langle \mathbf{D}F(\mathbf{u})\cdot\mathbf{v}, \hat{\mu}\rangle, \tag{5.1}$$

for all $y = (\mathbf{v}, \mu), \hat{y} = (\hat{\mathbf{v}}, \hat{\mu}) \in X = \mathbf{H}_0^1(\Omega) \times H^{-1}(\Omega)$. The operator $G'(x)$ is formally the Fréchet differential of G at x.

Recall from the saddle point theory given in [4, 5] that Theorems 3.1-3.2 implies that the system (3.2) has a unique solution $(\mathbf{v}, \mu)$ which depends continuously on $(\mathbf{f}, \sigma) \in X^*$. Thus we have the following result.

Theorem 5.1 *If $(\mathbf{u}, \lambda)$ satisfies the regularity assumption (3.1) then the map $G'(x)$ defined by (5.1) is an isomorphism from $X = \mathbf{H}_0^1(\Omega) \times H^{-1}(\Omega)$ to $X^* = \mathbf{H}^{-1}(\Omega) \times H_0^1(\Omega)$.*

For the discretized saddle point problem, we define $G_h : X_{h,\mathbf{g}} \to X_h^*$ to be the map defined by (4.6). For any $\hat{x} = (\hat{u}, \hat{\lambda}) \in X_{h,\mathbf{g}}$, $G_h(\hat{x})$ is the operator that satisfies

$$\langle G_h(\hat{x}), \hat{y}\rangle = \langle \nabla\hat{\mathbf{u}}, \nabla\hat{\mathbf{v}}\rangle + \langle \pi_h[\mathbf{D}F(\hat{\mathbf{u}})\cdot\hat{\mathbf{v}}], \hat{\lambda})\rangle + \langle \pi_h F(\hat{\mathbf{u}}), \hat{\mu}\rangle, \quad \hat{y} = (\hat{\mathbf{v}}, \hat{\mu}) \in X_h.$$

Thus, problem (4.6) is in fact to find $x_h = (\mathbf{u}_h, \lambda_h) \in X_{h,\mathbf{g}}$ such that

$$\langle G_h(x_h), y\rangle = 0, \quad y = (\hat{\mathbf{v}}, \hat{\mu}) \in X_h. \tag{5.2}$$

Let $G_h'(\hat{x})$ be the Fréchet derivative of G_h at $\hat{x} = (\hat{\mathbf{u}}, \hat{\lambda}) \in X_{h,\mathbf{g}}$. Then, $G_h'(\hat{x}) : X_h \to X_h^*$ is the linear operator given by

$$\langle G_h'(\hat{x})y, \hat{y}\rangle = a_h(\hat{\mathbf{u}}, \hat{\lambda}; \mathbf{v}, \hat{\mathbf{v}}) + \langle \pi_h[\mathbf{D}F(\hat{u}) \cdot \hat{\mathbf{v}}], \mu\rangle + \langle \pi_h[\mathbf{D}F(\hat{u}) \cdot \mathbf{v}], \hat{\mu}\rangle,$$
$$y = (\mathbf{v}, \mu) \in X_h, \quad \hat{y} = (\hat{\mathbf{v}}, \hat{\mu}) \in X_h. \tag{5.3}$$

By Theorem 4.1-4.2, the following result is a consequence of the theory of [4, 5]:

Theorem 5.2 *Assume that $\hat{x} = (\hat{\mathbf{u}}, \hat{\lambda}) \in X_{h,\mathbf{g}}$ satisfies the condition (4.13). For sufficiently small h and γ, the map $G_h'(\hat{x})$ is an isomorphism from X_h to X_h^*. Moreover,*

$$\|G_h'(\hat{x})^{-1}\|_{L(X_h^*, X_h)} \le M, \tag{5.4}$$

where M is a constant independent of h and $\hat{x} = (\hat{\mathbf{u}}, \hat{\lambda})$.

Define $x_* = (\pi_h \mathbf{u}, P_h\lambda)$, and set $y_* = G_h(x_*)$. We can use similar techniques as for Theorems 4.1 to prove the following lemma.

Lemma 5.1 *For any $\hat{x} = (\hat{\mathbf{u}}, \hat{\lambda}) \in X_{h,\mathbf{g}}$ satisfying*(4.13)*, we have*

$$\|G_h'(\hat{x}) - G_h'(x_*)\|_{L(X_h, X_h^*)} \le C \log^2(h^{-1})\|\hat{x} - x_*\|_X.$$

Proof. By the definition of G_h', we have for any $y = (\mathbf{v}, \mu) \in X_h$ and $\hat{y} = (\hat{\mathbf{v}}, \hat{\mu}) \in X_h$

$$\begin{aligned}\langle (G_h'(\hat{x}) - G_h'(x_*))y, \hat{y}\rangle &= \langle \pi_h[\mathbf{D}^2F(\hat{\mathbf{u}})\mathbf{v} \cdot \hat{\mathbf{v}}], \hat{\lambda} - P_h\lambda\rangle \\ &+\langle \pi_h[(\mathbf{D}^2F(\hat{\mathbf{u}}) - \mathbf{D}^2F(\pi_h\mathbf{u}))\mathbf{v} \cdot \hat{\mathbf{v}}], P_h\lambda\rangle \\ &+\langle \pi_h[(\mathbf{D}F(\hat{\mathbf{u}}) - \mathbf{D}F(\pi_h\mathbf{u})) \cdot \hat{\mathbf{v}}], \mu\rangle \\ &+\langle \pi_h[(\mathbf{D}F(\hat{\mathbf{u}}) - \mathbf{D}F(\pi_h\mathbf{u})) \cdot \mathbf{v}], \hat{\mu}\rangle.\end{aligned} \tag{5.5}$$

From Lemma 4.1, (4.13) and (4.3), we see that

$$\begin{aligned}\langle \pi_h[\mathbf{D}^2F(\hat{\mathbf{u}})\mathbf{v} \cdot \hat{\mathbf{v}}], \hat{\lambda} - P_h\lambda\rangle &\le C\|\pi_h[\mathbf{D}^2F(\hat{\mathbf{u}})\mathbf{v} \cdot \hat{\mathbf{v}}]\|_1 \ \|\hat{\lambda} - P_h\lambda\|_{-1} \\ &\le C\log^2(h^{-1})\|\hat{\mathbf{u}}\|_1\|\mathbf{v}\|_1\|\hat{\mathbf{v}}\|_1\|\hat{\lambda} - P_h\lambda\|_{-1} \le C\gamma\|\mathbf{v}\|_1\|\hat{\mathbf{v}}\|_1.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\langle \pi_h[(\mathbf{D}^2F(\hat{\mathbf{u}}) - \mathbf{D}^2F(\pi_h\mathbf{u}))\mathbf{v} \cdot \hat{\mathbf{v}}], P_h\lambda\rangle \\ &\quad \le C\|\pi_h[(\mathbf{D}^2F(\hat{\mathbf{u}}) - \mathbf{D}^2F(\pi_h\mathbf{u}))\mathbf{v} \cdot \hat{\mathbf{v}}]\|_1\|P_h\lambda\|_{-1} \\ &\quad \le C\|\pi_h[(\mathbf{D}^3F(\xi)(\hat{\mathbf{u}} - \pi_h\mathbf{u}))\mathbf{v} \cdot \hat{\mathbf{v}}]\|_1\|\lambda\|_{-1} \\ &\quad \le C\log^4(h^{-1})\|\xi\|_1\|\hat{\mathbf{u}} - \pi_h\mathbf{u}\|_1\|\mathbf{v}\|_1\|\hat{\mathbf{v}}\|_1\|\lambda\|_{-1} \\ &\quad \le C\gamma\log^2(h^{-1})\|\mathbf{v}\|_1\|\hat{\mathbf{v}}\|_1.\end{aligned}$$

Estimating the last two terms in (5.5) similarly using Lemma 4.1, (4.3) and (4.13), we get the result. The constants C in the estimates depend on $(\mathbf{u}, \lambda)$. $\square$

At this point, we need to recall the implicit function theorem as for example given in Lemma 1 of [6]. From the implicit function theorem, we can conclude that if there is a $\delta > 0$ such that

$$\hat{x} \in X_h, \ \|\hat{x} - x_*\|_X \le \delta \text{ implies } \|G_h'(\hat{x}) - G_h'(x_*)\|_{L(X_h, X_h^*)} \le \frac{1}{2M}, \tag{5.6}$$

then the equation

$$G_h(\hat{x}) = \hat{y} \tag{5.7}$$

has a unique solution for all $\hat{y}$ satisfying

$$\|\hat{y} - y_*\|_{X^*} \leq \frac{\delta}{2M}.$$

Here $M > 0$ is the positive constant appearing in Theorem 5.2. From Lemma 5.1, we see that the implication (5.6) is fulfilled if we choose $\delta = 1/(2MC\log^2(h^{-1}))$. Hence, we have that the equation (5.7) has a unique solution $\hat{x}$ satisfying

$$\|\hat{x} - x_*\|_X \leq \frac{1}{2MC\log^2(h^{-1})}$$

for all $\hat{y}$ such that

$$\|\hat{y} - y_*\|_{X^*} \leq \frac{1}{4M^2C\log^2(h^{-1})}.$$

Furthermore, we can conclude from Lemma 1 of [6] that

$$\|\hat{x} - x_*\|_X \leq 2M\|\hat{y} - y_*\|_{X^*}. \tag{5.8}$$

Note that our desired equation is $G_h(x) = 0$. Thus, if we can verify that

$$\|G_h(x_*)\|_{X^*} = \|y_*\|_{X^*} \leq \frac{1}{4M^2C\log^2(h^{-1})}, \tag{5.9}$$

we can conclude existence and uniqueness of solution of this equation. If we assume more smoothness on $\mathbf{u}$, this is a consequence of the following lemma.

Lemma 5.2 *Assume that* $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{W}^{1,\infty}(\Omega)$. *Then we have*

$$\|G_h(x_*)\|_{X^*} \leq Ch \text{ with } x_* = (\pi_h\mathbf{u}, P_h\lambda).$$

Proof. It suffices to prove that

$$|\langle G_h(x_*), \hat{x}\rangle| \leq Ch\|\hat{x}\|_X, \quad \hat{x} = (\mathbf{v}, \mu) \in X_h. \tag{5.10}$$

We have by (2.7) and the definition of G_h

$$\begin{aligned} \langle G_h(x_*), \hat{x}\rangle &= \langle\nabla(\pi_h\mathbf{u} - \mathbf{u}), \nabla\mathbf{v}\rangle + \langle\pi_h F(\pi_h\mathbf{u}), \mu\rangle - \langle F(\mathbf{u}), \mu\rangle \\ &\quad + \langle\pi_h[\mathbf{D}F(\pi_h\mathbf{u}) \cdot \mathbf{v}], P_h\lambda\rangle - \langle\mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}, \lambda\rangle. \end{aligned} \tag{5.11}$$

It is clear that

$$|\langle\nabla(\pi_h\mathbf{u} - \mathbf{u}), \nabla\mathbf{v}\rangle| \leq |\pi_h\mathbf{u} - \mathbf{u}|_1 \cdot |\mathbf{v}|_1 \leq Ch\|\mathbf{u}\|_2 \cdot |\mathbf{v}|_1. \tag{5.12}$$

Note that since $\pi_h F(\pi_h\mathbf{u}) = \pi_h F(\mathbf{u})$ we obtain from (4.1) that

$$\begin{aligned} |\langle\pi_h F(\pi_h\mathbf{u}), \mu\rangle - \langle F(\mathbf{u}), \mu\rangle| &= |\langle(\pi_h - I)F(\mathbf{u}), \mu\rangle| \\ &\leq \|(\pi_h - I)F(\mathbf{u})\|_1 \cdot \|\mu\|_{-1} \leq Ch\|F(\mathbf{u})\|_2 \cdot \|\mu\|_{-1}. \end{aligned} \tag{5.13}$$

Furthermore, by the assumptions on F and the estimates (4.1), (4.2) and (4.12) we get

$$\begin{aligned} &|\langle\pi_h[\mathbf{D}F(\pi_h\mathbf{u}) \cdot \mathbf{v}], P_h\lambda\rangle - \langle\mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}, \lambda\rangle| \\ &\quad \leq |\langle(\pi_h - I)[\mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}], P_h\lambda\rangle| + |\langle\mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}, P_h\lambda - \lambda\rangle| \\ &\quad \leq \|(\pi_h - I)[\mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}]\|_0 \cdot \|P_h\lambda\|_0 + \|\mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}\|_1 \cdot \|P_h\lambda - \lambda\|_{-1} \\ &\quad \leq Ch\|\mathbf{D}F(\mathbf{u}) \cdot \mathbf{v}_h\|_1 \cdot \|\lambda\|_0 \leq Ch\|\mathbf{D}F(\mathbf{u})\|_{1,\infty} \cdot \|\lambda\|_0 \cdot \|\mathbf{v}\|_1. \end{aligned} \tag{5.14}$$

Substituting (5.12)-(5.14) into (5.11), gives (5.10). □

From this lemma, we see that y_* satisfies (5.9) for small h. Thus, there exists a unique solution for equation (4.6). Moreover, the solution satisfies the estimate (5.8). We state this conclusion more clearly in the following theorem.

Theorem 5.3 *Assume that $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{W}^{1,\infty}(\Omega)$. Then, for sufficiently small h, there exists a unique saddle point $(\mathbf{u}_h, \lambda_h) \in X_h$ for (4.6) in a small neighborhood of $(\pi_h \mathbf{u}, P_h \lambda)$. Moreover, the following error estimate holds:*

$$\|\mathbf{u}_h - \mathbf{u}\|_1 + \|\lambda_h - \lambda\|_{-1} \le Ch.$$

6 Preconditioned iterative methods

We shall propose two iterative methods to solve the nonlinear saddle point problem (4.6). The first one is the classical Newton's method, cf. for example [15, chapter 7]. Let $x_0 = (\mathbf{u}_0, \lambda_0) \in X_h$ be a suitable initial guess. The Newton iteration is given by

$$x_{n+1} = x_n - G_h'(x_n)^{-1} G_h(x_n), \quad n = 0, 1, \cdots. \tag{6.1}$$

Assume that the initial guess $(\mathbf{u}_0, \lambda_0)$ satisfies (4.13) with a small γ. Using Theorem 5.2, combined with Lemma 5.1, and the standard properties of Newton's method, it follows that all $(\mathbf{u}_n, \lambda_n)$ satisfy (4.13) with the same γ, and all $G_h'(\mathbf{u}_n, \lambda_n)$ are invertible. Moreover, the sequence $\{(\mathbf{u}_n, \lambda_n)\}$ converges with almost order 2, i.e.

$$\|\mathbf{u}_{n+1} - \mathbf{u}_h\|_1 + \|\lambda_{n+1} - \lambda_h\|_{-1} \le C \log^2(h^{-1})(\|\mathbf{u}_h - \mathbf{u}_n\|_1 + \|\lambda_h - \lambda_n\|_{-1})^2.$$

For the iteration (6.1), we need to invert $G_h'(x_n)$, i.e. we need to solve the system

$$G_h'(x_n)(x_{n+1} - x_n) = -G(x_n). \tag{6.2}$$

From Theorem 5.2, we obtain that $G_h'(x_n)$ is an isomorphism from X_h to X_h^*. Moreover, $\|G_h'(x_n)\|_{L(X_h, X_h^*)}$ is bounded and the bound is independent of h and n if the initial value is chosen close enough to the true solution. This property can be utilized to construct good preconditioners for system (6.2). Let $\mathbf{\Delta}_h$ and Δ_h be the finite element discretizations for the vector and scalar Laplacian operators $\mathbf{\Delta}$ and Δ on $\mathbf{V}_{h,0}$ and $V_{h,0}$ respectively. To be precise, $\mathbf{\Delta_h} : \mathbf{V}_{h,0} \mapsto \mathbf{V}_{h,0}$ is the mapping defined by

$$(\mathbf{\Delta}_h \mathbf{u}_h, \mathbf{v}) = -(\nabla \mathbf{u}_h, \nabla \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_{h,0}.$$

Then the operator

$$T_h = \begin{pmatrix} -\mathbf{\Delta}_h^{-1} & 0 \\ 0 & -\Delta_h \end{pmatrix},$$

is an isomorphism from X_h^* to X_h with associated operator norm bounded independently of h. Thus, T_h can be used as a preconditioner to solve system (6.2). However, to simplify the computation we replace $\mathbf{\Delta}_h^{-1}$ by another spectral equivalent operator, i.e. by a preconditioner for the discrete Laplacian. The system (6.2) is then solved by the preconditioned minimum residual method, with the modified T_h operator as the preconditioner, cf. [16].

A disadvantage with Newton's method is that the linear system (6.2) has to be solved for each iteration. As an alternative approach for the nonlinear saddle point problem (4.6), we will consider a preconditioned fixed–point iteration, which can be seen as another nonlinear version of the minimum residual method. This method is described in a general setting in the Appendix. For the problem (4.6) we will apply this method to the preconditioned equation

$$N(x) = T_h G_h(x) = 0,$$

but where the operator $\mathbf{\Delta}_h^{-1}$ in T_h is replaced by a spectral equivalent operator. For the algorithm given in (8.2) let us take $H = X$, $N(x) = T_h G_h(x)$ and use the following inner product for H:

$$\langle x, y\rangle_H = \langle T_h^{-1} x, y\rangle. \tag{6.3}$$

Then $N'(x) = T_h G_h'(x)$ and $N'(x)^* = T_h G_h'(x)^*$. Here $N'(x)^*$ is the adjoint of $N'(x)$ with respect to the inner product (6.3), while $G_h'(x)^*$ is the adjoint operator of $G_h'(x)$ with respect to the L^2 inner product. The iteration (8.2) will then take the form

$$x_{n+1} = x_n - \theta_n T_h G_h'(x_n)^* T_h G_h(x_n), \quad n = 0, 1, \cdots, \tag{6.4}$$

where the relaxation factor θ_n is given by

$$\theta_n = \frac{\|T_h G_h'(x_n)^* T_h G_h(x_n)\|_H^2}{\|T_h G_h'(x_n) T_h G_h'(x_n)^* T_h G_h(x_n)\|_H^2}.$$

From Lemma 8.1 given in the Appendix, together with Theorem 5.2 and Lemma 5.1, we conclude that there exists a positive number $\tilde{\gamma}_0$ such that, when the initial guess x_0 satisfies

$$\|T_h G_h(x_0)\|_H \le \tilde{\gamma}_0 / \log^2(h^{-1}), \tag{6.5}$$

the iteration converges linearly with a rate independent of h for the error $\|T_h G_h(x_n)\|_H$.

7 Numerical experiments

Numerical experiments for the harmonic map problem with $\mathcal{M} = \mathbf{S}^1$, i.e. the unit circle, will be done. The domain Ω is always a square. The sequence of grids is made as a refinements of a 2×2 partition of Ω , which is further divided into triangles by the diagonal with a negative slope. When refining the mesh, each triangle is divided into four equal smaller triangles. The finite element problem (4.6) is to find $(\mathbf{u}_h, \lambda_h) \in \mathbf{V}_{h,\mathbf{g}} \times V_{h,0}$ such that

$$\begin{aligned} \langle \nabla \mathbf{u}_h, \nabla \hat{\mathbf{v}}_h\rangle + \langle \pi_h(\mathbf{u}_h \cdot \hat{\mathbf{v}}_h), \lambda_h\rangle &= 0, \quad \hat{\mathbf{v}}_h \in \mathbf{V}_{h,0}, \\ \langle \pi_h(|\mathbf{u}_h|^2 - 1), \hat{\mu}_h\rangle &= 0, \quad \hat{\mu}_h \in V_{h,0}. \end{aligned} \tag{7.1}$$

For the finite element method, we need to integrate over each element $e \in \mathcal{T}_h$. If we use the three vertices of e as the integration points, then the mass matrix reduces to a diagonal matrix. Correspondingly, the system (7.1) is reduced to:

$$\begin{aligned} -\mathbf{L_h} \mathbf{u}_h + \lambda_h \mathbf{u}_h = \mathbf{0} \quad &\text{on } \mathcal{N}_h, \\ |\mathbf{u}_h|^2 - 1 = 0 \quad &\text{on } \mathcal{N}_h. \end{aligned}$$

Above $\mathbf{L}_h$ is the standard five-point finite difference discrete Laplacian approximation. For the Newton iteration (6.1), we need to solve:

$$\begin{pmatrix} -\mathbf{L_h} + \mathbf{\Lambda_n} & \mathbf{diag}(\mathbf{u}_n) \\ \mathbf{diag}(\mathbf{u}_n)^t & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{n+1} - \mathbf{u}_n \\ \lambda_{n+1} - \lambda_n \end{pmatrix} = \begin{pmatrix} \mathbf{L_h} \mathbf{u}_n - \lambda_n \mathbf{u}_n \\ (1 - |\mathbf{u}_n|^2)/2 \end{pmatrix} \tag{7.2}$$

on $\mathcal{N}_h$. Here, $\mathbf{\Lambda_n}$ and $\mathbf{diag}(\mathbf{u}_n)$ are the matrix representations of the operators $\mathbf{v} \mapsto \pi_h(\lambda_n \mathbf{v})$ and $\mu \mapsto \pi_h(\mu \mathbf{u}_n)$ respectively. From Theorem 5.2, it is interesting to observe that the block-diagonal matrix $T_h = \mathrm{diag}(\mathbf{L}_h^{-1}, L_h)$ is a uniform preconditioner for the matrix of system (7.2).

For the preconditioned iterative solver (6.4) and the Newton iteration (6.2), the matrix $\mathbf{L}_h^{-1}$ in T_h replaced by an spectrally equivalent operator again. In our simulations, $\mathbf{L}_h^{-1}$ is replaced by the domain decomposition or multigrid preconditioners for $\mathbf{L}_h$ based on the theory of [18, 20]. For the multigrid case, the preconditioner for $\mathbf{L}_h$ is composed of one multigrid sweep with one pre- and post smoothing sweeps with symmetric Gauss-Seidel. The matrix L_h is simply a discrete Laplacian with homogeneous Dirichlet boundary conditions. Thus, the iteration (6.4) is rather inexpensive to compute. By doing so, no matrix needs to be inverted during the iterations. The cost per iteration is $O(N)$, where N is the degree of freedom for the discretization.

In the following, we compare the behavior of three different nonlinear iterative solvers:

- The exact Newton solver: this refers to the scheme where we solve the linear system (6.2) with a preconditioned Minimum Residual method which is terminated when the residual is reduced by a factor of 10^{10}.
- The inexact Newton solver: this refers to the scheme where the Newton iterations (6.2) are terminated when the residual is reduced by a factor of 10^2.
- The iterative solver (6.4).

In the tables, we show the numerical errors e_n versus the iteration number n, where e_n is defined as

$$e_n = \|\mathbf{u}_h^n - \mathbf{u}_h\|_{\mathbf{H}_h^1} + \|\lambda_h^n - \lambda_h\|_{H_h^{-1}}, \tag{7.3}$$

where $\|x_h\|^2_{\mathbf{H}_h^1} = (\pi_h x_h)^t(I - \mathbf{L}_h)\pi_h x_h$ and $\|y_h\|^2_{\mathbf{H}_h^{-1}} = (\pi_h y_h)^t(I - L_h)^{-1}\pi_h y_h$.

7.1 A smooth harmonic map

In the first example we consider a smooth harmonic map

$$\mathbf{u} = (\sin(\theta(x,y)), \cos(\theta(x,y)))$$

with $\theta = k\log(\sqrt{(x-a)^2 + (y-b)^2})$ and $\lambda = -|\nabla \mathbf{u}|^2$ on $\Omega = [0,1] \times [0,1]$. We have used $a = b = -0.1$ and $k = 3$. The initial guess was $\mathbf{u}_0 = 2(\pi_h \mathbf{u} + \epsilon)$, where ϵ is a random noise vector field with values between -0.3 and 0.3, and $\lambda_0 = 0$.

When using the inexact Newton solver the stop criteria is obtained in less than 20 iterations, with a few exceptions in the first nonlinear iterations where the maximum was 80. For the exact Newton solver the stop criteria is obtained in less than 50 iterations with a few exceptions in the first nonlinear iterations where as much as 300 iterations were required on the finest mesh. Hence, except for the first iterations the required number of iterations seems to be bounded independent of the mesh size. This is due to the property of the preconditioner.

In Table 1 we estimate the L^2 and H^1 error of $\mathbf{u} - \mathbf{u}_h$ in terms of h. We have linear convergence in $\mathbf{H}^1$ and quadratic convergence in L_2, respectively. This is in accordance with the error estimate of Theorem 5.3. Also $\lambda - \lambda_h$ seems to converge more than linearly in L_2.

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\|\mathbf{u} - \mathbf{u}_h\|_0$	6.7e-1	3.6e-2	9.4e-3	2.4e-3	6.0e-4
$\|\mathbf{u} - \mathbf{u}_h\|_1$	4.6	1.1	5.7e-1	2.9e-1	1.4e-1
$\|\lambda - \lambda_h\|_0$	4.2e-1	2.2e-2	1.6e-3	1.5e-4	1.2e-5

Table 1: The L_2 and H_1 error of $\mathbf{u}$ and the L_2 of λ with respect to h.

A comparison of the exact Newton and inexact Newton solvers is shown in Table 2 for mesh size $h = 2^{-4}$. The convergence for other mesh sizes is similar. These tests indicate that the inexact Newton solver is nearly as efficient as the exact Newton solver. In Table 3, the convergence of the inexact Newton solver with different mesh sizes are shown. It shows the mesh independence property of the iterative solver and the preconditioner.

The iteration (6.4) seems to be more unstable (i.e. the domain of attraction is smaller) than for both Newton variants and the convergence is slower, see Table 4. The results with the other mesh sizes shows that the convergence rate is bounded independent of the mesh size.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
Exact	3.2e+1	9.3	1.7	2.3e-1	4.0e-3	3.4e-6	2.6e-9	-
Inexact	3.2e+1	9.5	1.7	2.4e-1	3.5e-3	1.1e-5	1.0e-7	2.7e-9

Table 2: Convergence for the exact Newton solver with $h = 2^{-4}$.

$h \backslash it.$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
2^{-2}	9.2	2.6	4.7e-1	2.8e-2	1.9e-4	9.9e-7	7.7e-9	7.6e-10
2^{-3}	1.6e+1	4.7	9.1e-1	7.6e-2	8.8e-4	4.0e-6	7.9e-8	1.4e-9
2^{-4}	3.2e+1	9.5	1.7	2.4e-1	3.5e-3	1.1e-5	1.0e-7	2.7e-9
2^{-5}	6.4e+1	2.4e+1	3.6	9.6e-1	1.5e-2	4.7e-5	1.5e-6	6.6e-9

Table 3: Convergence for the the Inexact Newton solver

7.2 A harmonic map with singularity

Here, we test a non-smooth problem with a solution that has a singularity, i.e. $\mathbf{u} = (x/r, y/r)$ with $r = k\sqrt{x^2 + y^2}$ and $\lambda = -|\nabla \mathbf{u}|^2$ on $\Omega = [-0.5, 0.5] \times [0.5, 0.5]$. For this example, we have $\|\mathbf{u}\|_1 = \infty$. The Dirichlet boundary conditions are obtained from the analytical solution, while the start value for λ is $\lambda_0 = 0$ everywhere except in $(0, 0)$ where $\lambda = 1$. The initial value for $\mathbf{u}$ is shown in Figure 1.a. The numerical errors are shown in Table 5. The errors indicate that both $\mathbf{u}_h$ and λ_h converge linearly to the solution when measured in L_2. The H^1 norm of the $\mathbf{u} - \mathbf{u}_h$ is fixed independent of h, but this is reasonable since $\|\mathbf{u}\|_1 = \infty$. The computed solution is shown in Figure 1.b.

For this example, the Newton solvers are unstable and do not always converge. Thus, we have used the following iteration to produce the initial value for the Newton solvers:

$$\begin{pmatrix} -\mathbf{L_h} & \mathbf{diag}(\mathbf{u}_n) \\ \mathbf{diag}(\mathbf{u}_n)^t & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{n+1} - \mathbf{u}_n \\ \lambda_{n+1} - \lambda_n \end{pmatrix} = \begin{pmatrix} \mathbf{L_h}\mathbf{u}_n - \lambda_n \mathbf{u}_n \\ (1 - |\mathbf{u}_n|^2)/2 \end{pmatrix}, \tag{7.4}$$

Compared with (7.2), the matrix $\mathbf{\Lambda_n}$ has been dropped. This iterative scheme is globally convergent and is normally slower than the Newton solvers. Its convergence will be analyzed and discussed elsewhere. We do ten iterations of (7.4) and the inexact Newton solver is then turned on. The results are shown in Table 6 for $h = 2^{-4}$, where it is clear that we have quadratic convergence in the last iterations.

For the smooth problem tested in Section 7.1, it seems that the iterative solution always converges to the same solution no matter what kind of initial solution we use. For the problem here, we have noticed that the saddle point problem may have multiple solutions. With another initial solution as shown in Figure 1.c, we get another solution which is shown in Figure 1.d.

8 Appendix

Consider a general nonlinear equation of the form

$$N(x) = 0, \tag{8.1}$$

where N is a (locally) Fréchet differentiable map of a real Hilbert space H into itself. We let $\|\cdot\|_H$ and $\langle \cdot, \cdot \rangle_H$ be the corresponding norm and inner product, and $\mathcal{L}(H)$ the set of bounded linear operators mapping H into itself.

e_0	e_{10}	e_{20}	e_{30}	e_{40}	e_{50}	e_{60}	e_{70}	e_{80}
5.3	2.2e-1	1.9e-2	5.3e-3	2.2e-3	1.1e-3	5.3e-4	2.7e-4	1.4e-4

Table 4: Convergence for iterative solver (6.4) with $h = 2^{-4}$.

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\Vert\mathbf{u} - \mathbf{u}_h\Vert_0$	2.2e-1	1.3e-1	7.4e-2	4.0e-2
$\Vert\mathbf{u} - \mathbf{u}_h\Vert_1$	3.8	3.8	3.8	3.9
$\Vert\lambda - \lambda_h\Vert_0$	8.3e-1	4.1e-1	2.1e-1	1.0e-1

Table 5: The L_2 and H_1 error of $\mathbf{u}$ and the L_2 of λ with respect to h.

A disadvantage with Newton methods for such equations is that for each iteration the Fréchet derivative, $N'(x) \in \mathcal{L}(H)$, has to be recomputed, and a linear system has to be solved. Therefore, sometimes a simple fixed–point iteration is more effective, even if the converenge is slower. Throughout this appendix we will consider an iteration of the form

$$x_{n+1} = x_n - \theta_n N'(x_n)^* N(x_n), \tag{8.2}$$

where $N'(x_n)^*$ is the adjoint operator of $N'(x_n)$ for the inner product $\langle \cdot, \cdot \rangle_H$ and the real parameter θ_n will be chosen as

$$\theta_n = \frac{\Vert N'(x_n)^* N(x_n)\Vert_H^2}{\Vert N'(x_n) N'(x_n)^* N(x_n)\Vert_H^2}. \tag{8.3}$$

This method can be seen as a variant of the steepest descent method, where the functional to be minimized is the norm of the residual. We will make the following assumptions on the map N:

The equation (8.1) has a solution $x \in H$ and there is a ball B around x, and positive constants L and κ such that

$$\Vert N'(y) - N'(z)\Vert_{\mathcal{L}(H)} \leq L\Vert y - z\Vert_H \quad y, z \in B \tag{8.4}$$

$$\mathrm{Cond}(N'(y)^* N'(y)) \leq \kappa \quad y \in B. \tag{8.5}$$

Here $\mathrm{Cond}(N'(y)^* N'(y))$ denotes the spectral condition number of the operator $N'(y)^* N'(y)$.

Note that it follows from part (8.4) that if y and $y + z$ both are in B then

$$\begin{aligned} N(y+z) - N(y) - N'(y)z &= \int_0^1 \frac{d}{dt} N(y + zt)\, dt - N'(y)z \\ &= \int_0^1 (N'(y + zt) - N'(y))z\, dt, \end{aligned}$$

and as a consequence

$$\Vert N(y+z) - N(y) - N'(y)z\Vert_H \leq \frac{L}{2}\Vert z\Vert_H^2. \tag{8.6}$$

The main convergence result for iteration (8.2)–(8.3) can now be derived from the following lemma.

Lemma 8.1 *If x_{n+1} and x_n both are in B then we have*

$$\Vert N(x_{n+1})\Vert_H \leq \left(\frac{\kappa - 1}{\kappa + 1} + \frac{L}{2}\Vert N(x_n)\Vert_H\right)\Vert N(x_n)\Vert_H.$$

e_1	e_5	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}
1.1e+1	6.4e-1	1.1e-1	8.1e-2	9.7e-4	2.4e-7	1.2e-8

Table 6: Convergence for the Inexact Newton solver for the singular problem.

Proof. By using (8.2) we have

$$N(x_{n+1}) = N(x_n) - \theta_n N'(x_n)N'(x_n)^* N(x_n) + R_n$$

where the error term R_n is given by

$$R_n = N(x_n - \theta_n N'(x_n)^* N(x_n)) - N(x_n) + \theta_n N'(x_n)N'(x_n)^* N(x_n).$$

Hence, it follows from (8.3) and (8.6) that

$$\begin{aligned} \|R_n\|_H &\le \frac{L}{2}\theta_n^2 \|N'(x_n)N'(x_n)^* N(x_n)\|_H^2 \\ &\le \frac{L}{2} \frac{\|N'(x_n)^* N(x_n)\|_H^4}{\|N'(x_n)N'(x_n)^* N(x_n)\|_H^2}, \end{aligned}$$

which gives

$$\|R_n\|_H \le \frac{L}{2}\|N(x_n)\|_H^2. \tag{8.7}$$

It remains to bound $\|N(x_n) - \theta_n N'(x_n)N'(x_n)^* N(x_n)\|_H$. A direct computation, using (8.3), shows that

$$\begin{aligned} \|(I - \theta_n N'(x_n)N'(x_n)^*)N(x_n)\|_H^2 &= \|N(x_n)\|_H^2 \\ &- 2\theta_n \|N'(x_n)^* N(x_n)\|_H^2 + \theta_n^2 \|N'(x_n)N'(x_n)^* N(x_n)\|_H^2 \\ &= \left(1 - \frac{\|N'(x_n)^* N(x_n)\|_H^4}{\|N'(x_n)N'(x_n)^* N(x_n)\|_H^2 \|N(x_n)\|_H^2}\right)\|N(x_n)\|_H^2. \end{aligned}$$

However, for any positive definite self-adjoint operator $A \in \mathcal{L}(H)$ we have

$$\frac{4\lambda_{\min}(A)\lambda_{\max}(A)}{(\lambda_{\min}(A) + \lambda_{\max}(A))^2} \le \frac{\|x\|_H^4}{\langle Ax, x\rangle_H \langle A^{-1}x, x\rangle_H} \le 1, \quad x \in H.$$

The left inequality here is usually referred to as the Kantorovich inequality, cf. [21], while the right inequality is just Cauchy–Schwarz inequality. Hence, we obtain that

$$\|(I - \theta_n N'(x_n)N'(x_n)^*)N(x_n)\|_H \le \frac{\kappa - 1}{\kappa + 1}\|N(x_n)\|_H.$$

However, together with (8.7) this implies the desired bound. □

Note that if the initial value x_0 is chosen such that

$$\frac{\kappa - 1}{\kappa + 1} + \frac{L}{2}\|N(x_0)\|_H < 1$$

then the sequence $\{\|N(x_n)\|_H\}$ will converge at least linearly to 0.

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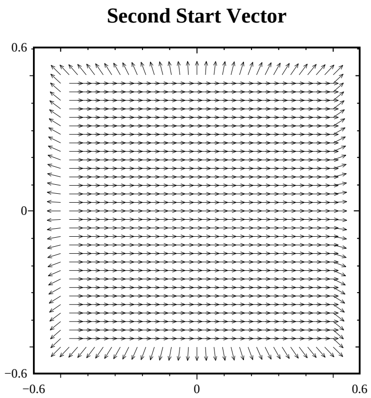
a) b)



c) d)

Figure 1: Plot of the initial solutions and the computed solutions. a) The first initial solution. b) The solution for a). c) The second initial solution. d) The solution for c).

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