Annali di Matematica pura ed applicata
(IV), Vol. CLXIX (1995), pp. 125-181

A Saint-Venant Type Principle for Dirichlet Forms on Discontinuous Media (*).

M. BIROLI - U. MOSCO

Abstract. – *We consider certain families of Dirichlet forms of diffusion type that describe the variational behaviour of possibly highly nonhomogeneous and nonisotropic bodies and we prove a structural Harnack inequality and Saint Venant type energy decays for their local solution. Estimates for the Green functions are also considered.*

Sunto. – *Si considerano certe famiglie di forme di Dirichlet di tipo diffusione che descrivono il comportamento di corpi fortemente non omogenei e non isotropi e si provano per le relative soluzione locali una diseguaglianza di Harnack strutturale e stime tipo Saint Venant della decrescita dell'energia. Si studiano inoltre stime per la funzione di Green.*

1. – Introduction and results.

We consider a body X with a very irregular internal structure, highly nonhomogeneous and possibly nonisotropic. We suppose that $u: X \to \boldsymbol{R}$ describes a physical state of X, whose equilibrium is subjected to a variational principle of a suitable nature.

In order to formulate such a principle with very few requirements about the internal structure of X, we shall assume that the energy functional to be minimized can be written as the quadratic functional associated with a Dirichlet form $a(u, v)$ in the Hilbert space

$$H = L^2(X, m),$$

for a suitable choice of a locally compact Hausdorff topology on X and of a positive

(*) Entrata in Redazione il 27 gennaio 1994.
Indirizzo degli AA.: M. BIROLI: Dipartimento di Matematica, Politecnico di Milano, P. Leonardo da Vinci 32, I-20133 Milano; U. MOSCO: Dipartimento di Matematica, Università di Roma «La Sapienza», P.le A. Moro 2, I-00185 Roma.

Radon measure m on X, with $\operatorname{supp} m = X$. The inner product of H will be denoted by $(\cdot, \cdot)$.

We recall that a *Dirichlet form* on H is a closed, non-negative definite, symmetric bilinear form $a(u, v)$ defined on a dense linear subspace $D[a]$ of H, which has in addition the following *Markovianity property*: if $u \in D[a]$, $v := 0 \vee u \wedge 1$, then $v \in D[a]$ and $a(v, v) \leq a(u, u)$.

We shall restrict our study to Dirichlet forms of diffusion type, that is to forms a that have the following *strong local property*: $a(u, v) = 0$ for every $u, v \in D[a]$ with v constant on $\operatorname{supp} u$.

Furthermore, we shall assume that the form a is *regular* in H, that is, there exists a subset C of $D[a] \cap C_0(X)$ which is both dense in $C_0(X)$ with the uniform norm and dense in $D[a]$ for the *intrinsic norm* $(a(u, u) + (u, u))^{1/2}$ ($C_0(X)$ denotes the space of continuous functions with compact support in X). Such a set C, that without restriction can be assumed to be a subalgebra of $D[a] \cap C_0(X)$, is called a *core* of a in H. The functions that belong to C play the role of test functions in our variational theory.

From the physical point of view the choice of the class of regular Dirichlet forms in order to state our variational principle for X is motivated by the fact that any such form can be given the following integral expression

$$a(u, v) = \int_X \mu(u, v)(dx) \tag{1.1}$$

for every $u, v \in D[a]$, where μ is a Radon-measure-valued nonnegative-definite bilinear form on $D[a]$, uniquely associated with a, called the *energy measure* of a [18], p. 152, [24].

One of the most important properties of μ is its local character, which is a consequence of the analogous property of the form a: the restriction of the measure $\mu(u, v)$ to any open subset of A of X depends only on the restriction of u and v to A. This property entitles us to interpret μ as a measure valued description of the physical characteristics of the body X. Moreover, it enables us to define in a natural way the space of functions u that belong locally to the domain of the form on a given open subset A of X (see Section 2). We shall denote this space by $D_{loc}[a, A]$ and simply by $D_{loc}[a]$ if $A = X$. We may suppress the explicit reference to the form a in our notation, if we are dealing with forms whose common domain has been previously specified.

By the local character of μ we can define a function u to be a *local minimizer* of the *energy functional*

$$E[u] = \frac{1}{2} a(u, u)$$

in a given arbitrary open subset X_0 of X, if u is a function on X that satisfies the minimality condition

(1.2) $$u \in D_{\rm loc}[a, X_0]: \frac{1}{2}\int_{X_0} \mu(u, u)(dx) \leq \frac{1}{2}\int_{X_0} \mu(u+\varphi, u+\varphi)(dx)$$

for every $\varphi \in C$ with $\operatorname{supp}\varphi \subset X_0$.

Clearly u is a solution of (1.2) if and only if u is a solution of the problem

(1.3) $$u \in D_{\rm loc}[a, X_0]: \int_{X_0} \mu(u, v)(dx) = 0 \qquad \text{for every } v \in D_0[a, X_0],$$

where by $D_0[a, X_0]$ we denote the closure of $D[a] \cap C_0(X_0)$ in $D[a]$ with its intrinsic norm. We will refer to any solution of (1.3) as to a local solution in X_0 of the equation formally written as

(1.4) $$Lu = 0,$$

where L is the self-adjoint operator in H associated with the form a according to the representation formula

$$a(u, v) = (\sqrt{L}u, \sqrt{L}v), \qquad u, v \in D[a].$$

Before going on, let us point out that our present interpretation of μ as measure valued characteristics of the body is further motivated by the special coordinate-invariant expression that is taken by the form a, whenever we are ready to introduce in X, or in some open portion of it, the additional structure of a (orientable) *differentiable manifold*. In this case, in fact, if there exist coordinate functions $x_1, \ldots, x_n$ that belong locally to $D[a]$ on there domain of definition, then any differentiable function u on X belongs locally to $D[a]$ and the energy measure $\mu(u, v)$ for every $u, v \in C^1(X)$ can be written in local coordinate as

$$\mu(u, v) = \sum_{i,j=1}^{n}\left(\frac{\partial u}{\partial x_i}(x_1, x_2, \ldots, x_n)\right)\left(\frac{\partial v}{\partial x_j}(x_1 x_2, \ldots, x_n)\right)\nu^{ij},$$

where

$$\nu^{ij} = \mu(x_i, x_j), \qquad i, j = 1, \ldots, n$$

defines a non-negative definite symmetric tensor ν on X (see the chain rule in Section 2). The form a, for every $u, v \in C_0^1(X)$, takes now the following more familiar invariant integral expression

$$a(u, v) = \int_X \sum_{i,j=1}^{n}\left(\frac{\partial u}{\partial x_i}(x_1, x_2, \ldots, x_n)\right)\left(\frac{\partial v}{\partial x_j}(x_1, x_2, \ldots, x_n)\right)\nu^{ij}(dx),$$

where the integral at the right hand side has to be intended as reduced to coordinate domains by means of a partition of unit in C and the following (degenerate) ellipticity condition is satisfied in the sense of measures on X:

$$0 \leq \sum_{i,j=1}^{n} \xi_i \xi_j \nu^{ij} \quad \text{for every } \xi \in \boldsymbol{R}^n,$$

see e.g. [26].

Let us now come back to our general setting. Our aim is to describe the behaviour of an arbitrary local solution of (1.4) in X_0 in a neighborhood of an arbitrary given point x_0 of X_0. Moreover, we want our theory and estimates have a *structural* character. By this we mean that the properties we will establish have to hold uniformly for a whole family of equivalent Dirichlet forms of diffusion type, in a sense that will be made precise below.

We will suppose that we are given a whole family of regular Dirichlet forms of diffusion type defined on a common domain $D \subset L^2(X, m)$, which are mutually *equivalent* in the following sense: however we choose a form b in the family, there exist two constants $0 < \lambda \leq \Lambda$, depending on b but whose ratio λ/Λ is independent of b, such that any other form a of the family is related to b by the condition

$$\lambda b(u, u) \leq a(u, u) \leq \Lambda b(u, u) \tag{1.5}$$

for every $u \in D = D[a] = D[b]$. We remark that by a well known *comparison principle* (see for instance [26]) condition (1.5) is equivalent to the condition

$$\lambda \mu_b(u, u) \leq \mu_a(u, u) \leq \Lambda \mu_b(u, u), \tag{1.6}$$

where μ_a and μ_b are the energy measures of a, b, respectively.

We will develop our theory under the assumption that the set of all test functions $\varphi \in C$, whose energy measures have a bounded density with respect to the measure m, is rich enough to separate the points of X. More precisely, we suppose that there is a form in the family, say b, that admits a *m-separating* core, that is, a core C that has the following separating property:

$$\text{for every } x, y \in X,\ x \neq y,\ \exists \varphi \in C \text{ with } \mu_b(\varphi, \varphi) \leq m \text{ on } X, \quad \text{such that } \varphi(x) \neq \varphi(y). \tag{1.7}$$

Clearly, if a set C is a core of b, then C is also a core for any other form a of the family; moreover, in view of (1.6), if in addition C has the separating property (1.7) with respect to a given form b, then C has the same property with respect to any other form $a$in the family.

We are now in a position to introduce the basic notion that is a the heart of our theory, that is, a family of (equivalent) metrics induced on the space X by the forms of the family. For related metric notions we refer to the fundamental paper [29] and to [13,33]. Given a form a we define the *distance function* $d = d_a \colon X \to [0, +\infty]$ by

$$d(x, y) = \sup \{\varphi(x) - \varphi(y) \colon \varphi \in C, \mu(\varphi, \varphi) \leq m \text{ on } X\}. \tag{1.8}$$

It is easy to verify that $d(x, y)$ satisfies the following properties: $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$, $d(x, y) \leq d(x, z) + d(z, y)$, with the usual convention on infinity. By $B = B_a$ we denote the metric balls given by the distance $d = d_a$ associated with the form a:

$$B(x, r) = \{y \in X, d(x, y) < r\}, \qquad r > 0. \tag{1.9}$$

It follows from (1.6) that the distance functions associated with two arbitrary forms of the family are mutually equivalent.

We point out that, while the energy measures in (1.1) are intrinsically defined in terms of the form itself, their densities occurring in the definition (1.8) of the distance are affected by the initial choice of the measure m. In any case, the metric balls $B_a(x, r)$ of a given form a in the family single out special regions of the space X, on which the form a, as well as its local solutions, should be expected to enjoy special properties that might not hold on other regions of X.

When taken up to the metric equivalence pointed out before, the intrinsic balls $B(x, r)$ play a basic role in two main regards. From one side, they allow us to formulate a *compatibility condition*, relating the whole family (1.5) to the initial topology on X and to the measure m initially chosen on X. This compatibility condition is expressed by *Assumption I* below. From the other side, the system of balls $B(x, r)$ allows us to formulate special scaling and embedding properties of the forms and their domains, that also have a structural character for the family (1.5). These properties, in the form of suitably scaled *Poincaré inequalities*, are expressed by *Assumption II* below.

ASSUMPTION I. – The forms in (1.5) admit a common m-separating core C in the space $L^2(X, m)$ and the following two properties hold:

(i) the metric topology induced by the distance (1.8) on X is equivalent to the initial topology of X;

(ii) the measure m is doubling with respect to the balls (1.9), that is, there exists a constant $c_0 > 0$ such that $0 < m(B(x, 2r)) \leq c_0 m(B(x, r)) < +\infty$ for every $x \in X$ and $0 < r \leq r_0$.

We remark that under this assumption the space X with the distance d acquires the structure of a *homogeneous space*, according to [8], Ch. III, Sect. 1. We observe that (ii) implies that $m(B(x, r)) \leq 2m(B(x, s))(r/s)^\nu$, $0 < s < r \leq \tau_0/2$. Where $\nu = \lg(c_0)/\lg 2$.

ASSUMPTION II. – Given a relatively compact open subset X_0 of X, there exist a constant $c_1 > 0$, and an integer $\kappa \geq 1$, such that for every $x \in X_0$ and every $r > 0$ with

$B(x, r) \subset X_0$ the following inequalities hold:

$$\text{(j)} \qquad \int_{B(x, r/\kappa)} |u - \bar{u}|^2 m(dx) \le c_1 r^2 \int_{B(x, r)} \mu(u, u)(dx)$$

for every $u \in D_{\text{loc}}[X_0]$, where $\bar{u} = 1/(m(B(x, \tau/\kappa))) \int_{B(x, \tau/\kappa)} um(dx)$.

REMARK. – We observe that if we suppose that X_0 is connected and $B(x, 2r) \subset X_0$, Assumptions I and II imply that the following inequality holds

$$\text{(k)} \qquad \left(\frac{1}{m(B(x, r))} \int_{B(x, r)} |u|^s m(dx) \right)^{1/s} \le c_2 r \left(\frac{1}{m(B(x, r))} \int_{B(x, r)} \mu(u, u)(dx) \right)^{1/2},$$

with constants $s = 2\nu/(\nu - 2)$ if $\nu > 2$, arbitrary $s > 2$ if $\nu \le 2$ and c_2 depending only on c_0, c_1, see [3] and [32]. Moreover from (k) it follows that

$$\int_{B(x, r)} |u|^2 m(dx) \le c_2 r^2 \int_{B(x, r)} \mu(u, u)(dx)$$

$\forall u \in D_{\text{loc}}[X_0]$ with $\operatorname{supp} u \subset B(x, r)$; this implies that our bilinear form a is coercive on $D_0[a, B(x, r)]$ for the intrinsic norm.

It is easily checked that if Assumption I and II are satisfied by a given form of the family (1.5) with some constants c_0, r_0, c_1, κ, then they are also satisfied by any other form of the family, with possibly new constants c_0', r_0', c_1', κ' depending on the initial c_0, r_0, c_1, κ and on the ratio Λ/λ. This observation allows us to check Assumption I and Assumption II for an *arbitrary* form in the family (1.5).

We can now state our main results. By X_0 we denote below a connected relatively compact open subset of X and by u an arbitrary solution of (1.3), where a is any form of a given family (1.5) for which Assumption I and Assumption II hold. We can also suppose, without loss of generality, that X_0 is contained in an intrinsic ball of radius less than R_0.

The *structural constants* c, possibly different ones, and the constants α, β in the estimates below only depend on Λ/λ and on the constants c_0, r_0, c_1, κ occurring in Assumptions I and II.

THEOREM 1.1 (Harnack inequality). – *If u is positive then*

$$\sup_{B(x, \tau)} u \le c \inf_{B(x, \tau)} u$$

for $B(x, r) \subset B(x, \bar{\kappa}r) \subset\subset X_0$, $\bar{\kappa} \ge 1$ depending on κ.

A standard consequence of Theorem 1.1 is the following

Corollary 1.2. – *There exists* $\alpha > 0$ *such that*

$$\operatorname*{osc}_{B(x,r)} u \leq c\left(\frac{r}{R}\right)^{\alpha} \operatorname*{osc}_{B(x,R)} u$$

for every $0 < r \leq R/4 \leq \bar{\kappa}^{-1} R_0$, $B(x_0, R_0) \subset\subset X_0$.

Therefore, since u is bounded, u is Hölder continuous with respect to the intrinsic distance of X, hence u is continuous with respect to the initial topology of X.

By taking into account the following L^{∞}-estimate:

$$\sup_{B(x,R)} |u| \leq cR^2 m(B(x,R))^{-1/p} \|f\|_{L^p(B(x,R),m)}, \tag{1.10}$$

that holds in $B(x,R) \subset R_0$ with $p > \max\{\nu/2, 2\}$ for every solution u of the equation

$$u \in D_0[a, B(x,R)]:\ a(u,v) = \int_{B(x,R)} fv\, m(dx), \qquad \forall v \in D_0[a, B(x,R)], \tag{1.11}$$

Corollary 1.2 (see also Theorem 5.13) enables us to define the *Green function* $G^{x_0}_{B(x,R)}$, for $x_0 \in B(x,R)$, as the unique function $G \in L^{p'}(B(x,R), m) \cap C(B(x,R) \setminus \{x_0\})$, $1/p + 1/p' = 1$, such that

$$u(x_0) = \int_{B(x,R)} Gf m(dx)$$

for every $f \in L^p(B(x,R), m)$ and $u = u_f$ solution of (1.11).

Theorem 1.3 (Size of the Green function). – *For every* $B(x_0, R) \subset B(x_0 20R) \subset\subset X_0$ *and every* $0 < r \leq R/16$, *the following estimate holds for all* $x \in \partial B(x_0, r)$:

$$\frac{1}{c}\int_r^R \frac{s^2}{m(B(x_0,s))}\frac{ds}{s} \leq G^{x_0}_{B(x_0,R)}(x) \leq c\int_r^R \frac{s^2}{m(B(x_0,s))}\frac{ds}{s}.$$

The structural estimates in Theorem 1.4 and in Corollary 1.5 below are the analogue in our present setting of the *Saint-Venant principle* for the energy decay in linear elasticity.

Theorem 1.4 (Saint-Venant principle). – *Let* u *be as in Theorem* 1.1; *there exists* $\beta > 0$ *such that*

$$\frac{\displaystyle\int_{B(x_0,r)} G^{x_0}_{B(x_0,q^{-1}r)}\mu(u,u)(dx)}{\displaystyle\int_{B(x_0,R)} G^{x_0}_{B(x_0,q^{-1}R)}\mu(u,u)(dx)} \leq c\left(\frac{r}{R}\right)^{\beta}$$

for every $0 < r \leq q^2 R \leq q^3 R_0$, $B(x_0, 20R_0) \subset\subset X_0$ *and for every* $q \in (0, q_0]$, *for some* *fixed* $q_0 < 1$ *depending on* κ *and some structural constant* c *depending also on* q. *Furthermore*

$$\int_{B(x_0, R)} G^{x_0}_{B(x_0, q^{-1}R)} \mu(u, u)(dx) \leq c \frac{1}{m(B(x_0, R_0))} \|u\|^2_{L^2(B(x_0, R_0))}.$$

From Theorem 1.3 we obtain the more explicit special estimate:

COROLLARY 1.5. – *For* r, R, R_0 *as in Theorem* 1.4, *we have*

$$\frac{\dfrac{r^2}{m(B(x_0, r))} \displaystyle\int_{B(x_0, r)} \mu(u, u)(dx)}{\dfrac{R^2}{m(B(x_0, R))} \displaystyle\int_{B(x_0, R)} \mu(u, u)(dx)} \leq c\left(\frac{r}{R}\right)^{\beta}.$$

From the point of view of partial differential equations the theory discussed so far finds its main motivation in the study of the structural properties of second order degenerate elliptic operators. Existence, uniqueness and regularity in Sobolev spaces for such a type of PDE are studied in [14], [15], [22], [30], moreover in [5] regularity properties and maximum principles are studied in connection with the type of degeneration and the different axiomatic potential theoretic setting. We bound ourselves to illustrate our results by referring below to two important classes of operators on $\boldsymbol{R}^n$: (a) *Weighted* uniformly elliptic operators in divergence form with measurable coefficients, (b) *Uniformly subelliptic* (selfadjoint) second order operators with bounded measurable coefficients.

(a) We consider the form

$$a(u, v) = \int_{\boldsymbol{R}^n} \sum_{i, j=1}^{n} \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x)\, a^{ij}(x)\, dx$$

on $C^1_0(\boldsymbol{R}^n)$, where the (Lebesgue) measurable coefficients $a^{ij} = a^{ji}$, $i, j = 1, \ldots, n$, satisfy the condition

$$\lambda |\xi|^2 w(x) \leq \sum_{i, j=1}^{n} \xi_i \xi_j a^{ij}(x) \leq \Lambda |\xi|^2 w(x) \qquad \text{a.e. in } \boldsymbol{R}^n$$

for every $\xi \in \boldsymbol{R}^n$. Here w is a *weight* in the Muckenhoupt class A_2, or a weight $w(x) = |\det F'|^{1-2/n}$ associated with a quasi-conformal transformation F in $\boldsymbol{R}^n$. The domain $D[a]$ of this form is obtained by completion of $C^1_0(\boldsymbol{R}^n)$ with respect to the norm $(a(u, u) + (u, u))^{1/2}$, where $(\cdot, \cdot)$ is taken to be the inner product of the Hilbert space $H = L^2(\boldsymbol{R}^n, w(x)\, dx)$. This class of operators, that generalizes previous examples studied in [28], has been considered in [10], [11], where Theorem 1.1 and Theorem 1.2 were first obtained. Here $X = \boldsymbol{R}^n$, the ball $B(x, r)$ are the usual euclidean balls and $m(dx) = w(x)\, dx$. The doubling property of our Assumption I is a well known property of the weights considered above and the scaled Poincaré inequalities are given in [11].

(*b*) In the space $H = L^2(\boldsymbol{R}^n, \mathrm{d}x)$ we consider a family (1.5) of forms, where b is a given selfadjoint subelliptic form with smooth coefficients in $\boldsymbol{R}^n$, that is, b is given on $C_0^1(\boldsymbol{R}^n)$ by

$$b(u, v) = \int_{\boldsymbol{R}^n} \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x)\, b^{ij}(x)\,\mathrm{d}x$$

where $b^{ij} = b^{ji}$, $i, j = 1, \ldots, n$, are smooth functions that satisfy the degenerate ellipticity condition

$$0 \leq \sum_{i,j=1}^{n} \xi_i \xi_j b^{ij}(x) \quad \text{in } \boldsymbol{R}^n, \qquad \forall \xi \in \boldsymbol{R}^n$$

and b satisfies the following subellipticity estimate for some $\varepsilon \in (0, 1)$:

$$c\|u\|_{H^\varepsilon}^2 \leq b(u, u) + \|u\|_{L^2}^2 \quad \text{for every } u \in C_0^1(\boldsymbol{R}^n),$$

where H^ε denotes the usual fractional Sobolev space of order $\varepsilon \in (0, 1)$. The form a is any form of the following type

$$a(u, v) = \int_{\boldsymbol{R}^n} \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} a^{ij}(x)\,\mathrm{d}x, \qquad u, \ v \in C_0^1(\boldsymbol{R}^n)$$

with measurable coefficients $a^{ij} = a^{ji}$ that satisfy the *uniform subellipticity condition*

$$\lambda \sum_{i,j=1}^{n} \xi_i \xi_j b^{ij}(x) \leq \sum_{i,j=1}^{n} \xi_i \xi_j a^{ij}(x) \leq \Lambda \sum_{i,j=1}^{n} \xi_i \xi_j b^{ij}(x) \quad \text{a.e. in } \boldsymbol{R}^n \ \forall \xi \in \boldsymbol{R}^n$$

for some given constants $0 < \lambda \leq \Lambda$.

The distance d on $\boldsymbol{R}^n$ induced by the form b according to our definition (1.8) turns out to be equal to the distance d^* associated with the form b according to [12], [13] (see [21]). The distance d^* is known to satisfy the condition

$$\frac{1}{c}|x - y| \leq d^*(x, y) \leq c|x - y|^\varepsilon. \tag{1.12}$$

By noting that $m(\mathrm{d}x) = \mathrm{d}x$, it is shown in [29] that Assumption I (i) holds. The scaled Poincaré inequalities on the intrinsic balls for the form b have been proved in [20] and [21]. These authors also provide examples that show that Poincaré inequality may not hold on regions that do not coincide with an intrinsic ball, what may serve as illustration of a remark we made before.

The main example of uniform subelliptic operators to which the results above apply is that of an operator of uniform Hörmander type

$$L = X_k^* (\alpha^{hk}(x) X_h), \qquad x \in \mathbf{R}^n,$$

where X_h, $h = 1, \ldots, m$ are m smooth vector fields in $\mathbf{R}^n$ that satisfy the Hörmander condition and $\alpha = (\alpha^{hk})$ is any symmetlric $m \times m$ matrix of measurable functions on $\mathbf{R}^n$, such that $\lambda |\eta|^2 \leq \alpha(x) \eta \cdot \eta \leq \Lambda |\eta^2|$ for every $\eta \in \mathbf{R}^m$, a.e. on $\mathbf{R}^n$.

Theorems 1.1 and 1.3 above seem to be new in this uniform subelliptic setting even for Hörmander's square operators. Moreover the estimate on the energy decay of Theorem 1.4 is new also in the case of classical Hörmander's square operators with *smooth* coefficients, $\alpha^{hk} = \delta^{hk}$.

For related results see [23] and [16] and for a particular case of this setting, with dx replaced by $w(x)\,dx$, w as in (*a*), see [17].

(*c*) We now give an example of a form a in $\mathbf{R}^2$ whose local solutions are Hölder continuous with respect to the intrinsic distance of a, and only continuous with respect to the euclidean metric of $\mathbf{R}^2$. In the space $L^2(B, m(dx\,dy))$, where $B = \{(x, y) \in \mathbf{R}^2; 1/(\log|x|)^2 + |y|^2 < 1/4\}$ and $m(dx\,dy) = (1/(|x|(\log|x|)^2))\,dx\,dy$, we consider the form

$$a(u, v) = \int_B \left\{ x^2 (\log|x|)^4 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right\} \frac{1}{|x|(\log|x|)^2}\, dx\, dy,$$

where u, v are in the set

$$C = \left\{ w \in C_0(\mathbf{R}^2) : \frac{1}{|x|(\log|x|)^2}\frac{\partial w}{\partial x} \in C_0(\mathbf{R}^2), \quad \frac{\partial w}{\partial y} \in C_0(\mathbf{R}^2), \quad w \in W^{1,1}(B) \right\}.$$

We still denote by $a(u, v)$ the closure of this form in $L^2(B, m)$. Assumptions I and II can be easily checked, by suitably rescaling the form on a euclidean ball of $\mathbf{R}^2$. The intrinsic balls shrinks to $\{0\}$ as $\exp(-1/r)$ in the x-direction as $r \to 0$.

We finally mention that any family of forms satisfying a condition like (1.5) enjoys special *variational compactness properties*, that can be expressed in terms of Γ-convergence of the functionals $a(u, u)$, now defined on the whole H by extending them to $+\infty$ outside their domain. In case the domains $D[a]$ are uniformly compactly injected in H, as for instance in example (*a*) as well as in the subelliptic case (*b*), these convergence properties can be expressed in terms of the resolvent operators associated with the forms and are thus related to convergence of spectra and semigroups. These compactness properties play an important role in *homogenization theory* and, more generally, in the asymptotic variational approach to composite media, see [26]. The structural estimates presented before describe some properties of local solutions that are kept in the variational limit. This is of particular relevance in the asymptotic theory, due to the fact that the explicit expression of the limit energy form in terms of the *effective characteristics* of the composite body may not always be easily determined.

Finally we will remark that the results in this paper have been announced in [2].

We now give the plan of the paper. In Section 2 we recall some properties of Dirichlet forms which are used in the following. In Section 3 we construct a fundamental tool for our proofs, namely, cut-off functions between intrinsic balls or anuli. In Section 4 we obtain an L^{∞} global estimate for solutions of a Dirichlet problem with homogeneous boundary condition. In Section 5 we prove Theorem 1.1. The proof, which uses an L^{∞} local estimate, is obtained by De Giorgi-Stampacchia's truncation method [36], adapted to the metric structure of the homogeneous space X and by a simplified version of Moser's technique [27], which relies on an extension of John-Nirenberg's lemma to homogeneous spaces [6]. We notice that Corollary 1.2 can be obtained from Theorem 1.1 by standard methods, see for ex. [19]. In Section 6 we first estimate the relative size of Green functions, on homotetic balls, then we derive Theorem 1.3 as in [10]. Section 7 is devoted to the proof of Theorem 1.4. The Saint-Venant principle is proved by applying a modified version of the «hole filling argument», as in [4], which allows us to use the Poincaré inequality only on balls (and not on anuli).

Acknowledgments. Both authors are glad to acknowledge hospitality and partial support during the preparation of this work from the Institut for Mathematics and its Applications at the University of Minnesota in Minneapolis; the second author also from Sonderforschungsbereich 256 at the Institut für Angewandte Mathematik of the University of Bonn and from the Laboratoire d'Analyse Numerique of the Université Pierre et Marie Curie (Paris VI). This research is part of the project «Problemi variazionali irregolari. Strutture discontinue» of the Consiglio Nazionale delle Ricerche, Italy.

2. - Preliminaries on Dirichlet forms.

We shall denote by X a locally compact separable Hausdorff topological space and by m a positive Radon measure on X with $\operatorname{supp} m = X$.

Let us collect below a few relevant definitions and properties.

(a) *Dirichlet forms of diffusion type.* A *Dirichlet form* a, with domain $D[a]$ in the space $L^2(X, m)$, is a real valued nonnegative definite, symmetric, bilinear form $a(u, v)$, defined on a dense linear subspace $D[a]$ of $L^2(X, m)$, which has in addition the following properties: (i) a is closed in $L^2(X, m)$, i.e., $D[a]$ is complete under the inner product (*intrinsic metric*) $(u, v)_a = a(u, v) + (u, v)$, where (u, v) denotes the usual inner product of the space $L^2(X, m)$; (ii) a is Markovian, i.e., $T \circ u \in D[a]$ and $a(T \circ u, T \circ u) \leq a(u, u)$ whenever $u \in D[a]$ and T: $\boldsymbol{R} \to \boldsymbol{R}$, $T(0) = 0$, $|Tx - Ty| \leq$ $\leq |x - y|$ for every $x, y \in \boldsymbol{R}$.

If a is a Dirichlet form in $L^2(X, m)$ with domain $D[a]$, then $D[a] \cap L^{\infty}(X, m)$ is an algebra and we have:

$$a(uv, uv) \leq 2(\|u\|^2_{L^\infty(X,m)} a(v, v) + \|v\|_{L^\infty(X,m)} a(u, u))$$

for every $u, v \in D[a] \cap L^\infty(X, m)$ (see [18], Theorem 1.4.2).

A Dirichlet form a with domain $D[a] \subset L^2(X, m)$ is said of *diffusion type* of the following property holds

$$a(u, v) = 0, \quad \text{if } u, v \in D[a], \quad v = \text{constant } m\text{-a.e. on a neighborhood of supp } u.$$

By supp u we mean (compact) support of the measure $u \cdot m$ in X. We note that the diffusion property as stated above is a stronger property than the locality of the form a, where a is said to be *local* if $a(u, v) = 0$ whenever $u, v \in D[a]$ have disjoint supports in X.

A Dirichlet form is said *regular* in $L^2(X, m)$ if it possesses a core in $L^2(X, m)$, a *core* of a in $L^2(X, m)$ being any subalgebra C of $D[a] \cap C_0(X)$ which is dense both in $C_0(X)$ for the uniform norm and in $D[a]$ for the intrinsic norm $\|u\|_a = (a(u, u) + \|u\|^2_{L^2(X, m)})^{1/2}$. By $C_0(X)$ we denote the space of all continuous functions with compact support in X. Note that a form a is regular in $L^2(X, m)$ if and only if $C = D[a] \cap C_0(X)$ is a core in $L^2(X, m)$.

(*b*) *The intrinsic capacity.* Associated with a regular Dirichlet form of diffusion type in $L^2(X, m)$, a *capacity* set-function can be defined, in the Choquet sense, [18] Theorem 3.1.1. Related to this (intrinsic) capacity notion are the notions of null sets, i.e., subsets of X of capacity zero, quasi-continuous functions and sets, for which we refer to [18], Ch. 3. Every function $u \in D[a]$ admits a quasi-continuous modification $\tilde{u}$, i.e., there exists a quasi-continuous function $\tilde{u}$ on X, unique up to q.e. equality, such that $u = \tilde{u}$ m-a.e. on X [18], Theorem 3.1.4. Two quasi continuous function which are equal ($\leq$) m-q.e. on an open subset of X are equal ($\leq$) q.e. on that set, [18] Lemma 3.1.4.

(*c*) *The energy measures.* According to the fundamental theory of BEURLING-DENY [1], [9], and its extension due to SILVERSTEIN [34], [35], FUKUSHIMA [18], LE-JEAN [24], any regular Dirichlet form of diffusion type can be expressed on its domain $D[a]$ by an integral representation formula

$$a(u, v) = \int_X \mu(u, v)(dx), \qquad u, v \in D[a],$$

where $\mu(u, v)$ is a Radon-measure-valued symmetric bilinear form on $D[a]$, called the *energy measure* of a. For every $u, v \in D[a] \cap L^\infty(X, m)$ the signed Radon measure $\mu(u, v)$ is obtained by polarization from the positive Radon measure $\mu(u, u)$, uniquely defined for every $u \in D[a] \cap L^\infty(X, m)$ by the identity

$$\int_X \varphi(x)\mu(u, u)(dx) = a(u, \varphi u) - \frac{1}{2} a(u^2, \varphi) \quad \text{for every } \varphi \in C, \tag{2.1}$$

C being a core of a, ([34], [35], [18], p. 152). This identity is equivalent, in terms of the transition function $p_t(x, dy)$ associated with the form a, [18] p. 27, to

$$\int_X \varphi(x)\mu(u, u)(\mathrm{d}x) = \lim_{t \downarrow 0} \frac{1}{2t} \int_{X \times X - d} \varphi(x)(\tilde{u}(y) - \tilde{u}(x))^2 p_t(x, \mathrm{d}y)\, m(\mathrm{d}x), \tag{2.2}$$

where d is the diagonal in $X \times X$, $\tilde{u}$ the quasicontinuous modification of u and $\varphi \in C_0(X)$. The definition of $\mu(u, u)$ is extended to arbitrary $u \in D[a]$ as the increasing limit of the measures $\mu(u_n, u_n)$, where $u_n = \max\{-n, \min\{u, n\}\}$ as $n \to +\infty$, [24] Prop. 1.14.1. The measure $\mu(u, v)$ is then defined by polarization:

$$\mu(u, v) = \frac{1}{2}[\mu(u+v, u+v) - \mu(u, u) - \mu(v, v)], \qquad u, v \in D[a]. \tag{2.3}$$

The bilinear form $\mu(u, v)$ is nonnegative definite, that is

$$\mu(u, u) \geq 0 \qquad \text{for every } u \in D[a],$$

in the sense of measures on X. Moreover from (2.2) we have:

$$\int_X \varphi(x)\mu(u, v)(\mathrm{d}x) = \lim_{t \downarrow 0} \frac{1}{2t} \int_{X \times X - d} \varphi(x)(\tilde{u}(y) - \tilde{u}(x))(\tilde{v}(y) - \tilde{v}(x))\, p_t(x, \mathrm{d}y)\, m(\mathrm{d}x) \tag{2.4}$$

for every $u, v \in D[a] \cap L^\infty(X, m)$, $\varphi \in C_0(X)$.

From the characterization (2.4) and Lemma 5.4.3 in [18], we have the following *Schwarz rule*:

For every $u, v \in D[a] \cap L^\infty(X, m)$, if $f \in L^2(X, \mu(u, u))$ and $g \in L^2(X, \mu(v, v))$, then $f \cdot g$ is integrable with respect to the absolute variation of $\mu(u, v)$ and

$$\int_X |fg|\, |\mu(u, v)|(\mathrm{d}x) \leq \left(\int_X f^2 \mu(u, u)(\mathrm{d}x)\right)^{1/2} \left(\int_X g^2 \mu(v, v)(\mathrm{d}x)\right)^{1/2},$$

moreover

$$2|fg|\, |\mu(u, v)| \leq f^2 \mu(u, u) + g^2 \mu(v, v) \tag{2.5}$$

in the sense of measures on X. By approximation we can extend (2.5) to every $u, v \in D[a]$.

The measures $\mu(u, v)$, $u, v \in D[a]$, do not charge sets of capacity zero in X and have a local character in X, that is, the restriction of the measure $\mu(u, v)$ to any open subset A of X depends only on the restrictions of u and v to A. More precisely if $u_1, u_2 \in D[a]$ are such that $\tilde{u}_1 = \tilde{u}_2$ m-a.e. on A (then also q.e. on A), then

$$1_A \mu(u_1, u_1) = 1_A \mu(u_2, u_2)$$

as measures on X, where 1_A denote the characteristic function of A in X. Furthermore, we have

$$1_A \mu(u, v) = 0$$

for every $u \in D[a]$ which is constant m-a.e. on A and for every $v \in D[a]$. For these properties we refer to [24] Prop. 1.5.2 and [18] Lemma 5.4.6.

By $D_0[a, A]$ we denote the closure of $D[a] \cap C_0(A)$ in $D[a]$ for the intrinsic norm $(a(u, u) + \|u\|^2_{L^2(X, m)})^{1/2}$. By $D_{\text{loc}}[a]$ we denote the space of m-measurable functions on X, such that for every relatively compact open subset A of X there exists a function $w \in D[a]$ such that $u = w$ m-a.e. on A. The measure $\mu(u, u)$ for $u \in D_{\text{loc}}[a]$ is defined on X by putting on every A

$$1_A \mu(u, u) = 1_A \mu(w, w).$$

By $D_{\text{loc}}[a, A]$, or simply by $D_{\text{loc}}[A]$, we denote the space of the functions u on A, such that there exists $w \in D_{\text{loc}}[a]$ with $\varphi u = \varphi w$ for every $\varphi \in C$ with $\operatorname{supp} \varphi \subset A$.

(*d*) *The comparison principle.* Let a and b be two regular Dirichlet forms of diffusion type with same domain D, such that

$$\lambda b(u, u) \leq a(u, u) \leq \Lambda b(u, u), \qquad \forall u \in D$$

for some constants $0 < \lambda \leq \Lambda$. Then, an analogous inequality holds for the energy measures, that is,

$$\lambda \mu_b(u, u) \leq \mu_a(u, u) \geq \Lambda \mu_b(u, u), \qquad \forall u \in D$$

in the sense of measures on X, where μ_a and μ_b are the energy measures of the forms a and b, [24], [26].

(*e*) *The Leibnitz rule.* If a is any regular Dirichlet form of diffusion type and μ its energy measure, then

$$\mu(uv, w) = u\mu(v, w) + v\mu(u, w)$$

for every $u, v \in D[a] \cap L^\infty(X, m)$, $w \in D[a]$, see [24] 1.5.2, [18] Lemma 5.4.2.

(*f*) *The chain rule.* For any regular Dirichlet form of diffusion type a the following property holds:

For every $v \in D[a] \cap L^\infty(X, m)$ and every $u_1, \ldots, u_m \in D_{\text{loc}}[a] \cap L^\infty(X, m)$ and for energy $\eta \in C^1(\boldsymbol{R}^m)$, we have $\eta(u_1, \ldots, u_m) \in D_{\text{loc}}[a] \cap L^\infty(X, m)$ and

$$\mu(\eta(u_1, \ldots, u_m), v) = \sum_{i=1}^{m} \frac{\partial \eta}{\partial x_i}(u_1, \ldots, u_m) \mu(u_i, v).$$

The formula extends to arbitrary $u_1, \ldots, u_m \in D_{\text{loc}}[a]$, provided $\partial \eta / \partial x_i$ are in addition uniformly bounded in $\boldsymbol{R}^m$ and then $\eta(u_1, \ldots, u_m) \in D_{\text{loc}}[a]$, see [24] 2.1 (*a*), [18] Theorem 5.4.3. Note that, since the measure μ does not charge sets of capacity zero in X, the pointwise version of the functions occurring in the argument of $\partial\eta/\partial x_i$ can be taken equivalently in the m-a.e. sense or in the q.e. sense in X.

(*g*) *The truncation lemma.* Let a be a regular Dirichlet form of diffusion type and μ its energy measure. Then,

$$\mu(u^+, v) = 1_{\{\tilde{u} > 0\}}\mu(u, v) \tag{2.6}$$

for every $u \in D[a]$ and every $v \in D[a] \cap L^\infty(X, m)$, where $\tilde{u}$ is the quasi continuous version of u, [26]. From (2.6) we have easily

$$\mu(u^+, u^+) = 1_{\{\tilde{u} > 0\}}\mu(u, u), \qquad \forall u \in D[a]. \tag{2.7}$$

We recall that $k \in D_{\text{loc}}[a]$, $\forall k \in \boldsymbol{R}$. Therefore, for every $u \in D_{\text{loc}}[a]$ and every $v \in D_{\text{loc}}[a] \cap L^\infty(X, m)$, we have

$$\mu((u-k)^+, v) = 1_{\{\tilde{u} > k\}}\mu(u, v), \tag{2.8}$$

$$\mu((u-k)^+, (u-k)^+) = 1_{\{\tilde{u} > k\}}\mu(u, u). \tag{2.9}$$

Moreover, if $u_1, u_2 \in D[a]$ (or $\in D_{\text{loc}}[a]$) and $v \in D[a] \cap L^\infty(X, m)$, then

$$\mu(\max(u_1, u_2), v) = 1_{\{\tilde{u}_1 > \tilde{u}_2\}}\mu(u_1, v) + 1_{\{\tilde{u}_2 > \tilde{u}_1\}}\mu(u_2, v). \tag{2.10}$$

(*h*) *The space $D_0[\mathcal{O}]$. Let* a be a given regular form of diffusion type in X and let $\mathcal{O}$ be an open subset of X. By $D_0[\mathcal{O}]$ we denote the space $D_0[a, \mathcal{O}]$, as defined in (*c*) above. By Theorem 4.4.2 (i) of [18],

$$D_0[\mathcal{O}] = \{u \in D[a] \colon \tilde{u} = 0 \text{ q.e. on } X - \mathcal{O}\}.$$

We remark that, if $\mathcal{O}$ is relatively compact in X, if $u \in D_{\text{loc}}[a]$ and $\operatorname{supp} u \subset \mathcal{O}$, then $u \in D_0[\mathcal{O}]$. In particular $D_0[\mathcal{O}]$ coincides with the closure in $D[a]$ of the set if all $v \in D[a]$ with support in $\mathcal{O}$.

(*i*) *Maximum principles.* Let a be a form that satisfies Assumptions I and II. Let $\mathcal{O}$ be an open subset of some intrinsic ball $B \subset X_0$. Let $u \in D_{\text{loc}}[a] \cap C(\bar{\mathcal{O}})$ be such that

$$\int_{\mathcal{O}} \mu(u, v)(dx) \leq 0, \qquad \forall v \in D_0[\mathcal{O}], \qquad v \geq 0 \ m\text{-a.e. in } \mathcal{O}.$$

Then,

$$u \leq \max_{\partial\mathcal{O}} u \text{ in } \mathcal{O}.$$

In fact, let $M := \max u$ and $\varepsilon > 0$. Then, $(u - M - e)^+$ vanishes on a neighborhood of

$\partial\mathcal{O}$, hence, by (h), $(u - M - \varepsilon)^+ \in D_0[\mathcal{O}]$. Therefore,

$$\int_{\mathcal{O}} \mu(u, (u - M - \varepsilon)^+)(dx) \leqslant 0\,,$$

thus, by the truncation lemma

$$\int_{\mathcal{O}} \mu((u - M - \varepsilon)^+, (u - M - \varepsilon)^+)(dx) \leqslant 0\,.$$

By Poincaré's inequality, $(u - M - \varepsilon)^+ = 0$, m-a.e. in $\mathcal{O}$. Therefore $u \leqslant M$ m-a.e. in $\mathcal{O}$, hence the conclusion.

By a similar argument we prove that if $u \in D_0[B_R] \cap C(B_R - B_r)$, where $B_r \subsetneqq B_R$ are two concentric intrinsic balls in X_0, and u satisfies:

$$\int_{B_R} \mu(u, v)(dx) \leqslant 0\,, \qquad \forall v \in D_0[B_R - \overline{B}_r]\,, \qquad v \geqslant 0 \ m\text{-a.e. on } B_R\,,$$

then

$$u \leqslant \max_{\partial B_r} u \quad \text{ in } B_R - B_r\,.$$

We also need a deeper maximum principle, for local subsolutions which may not be continuous up to the boundary. Let $u \in D_{\mathrm{loc}}[X]$, such that

$$\int_{\mathcal{O}} \mu(u, v)(dx) \leqslant 0\,, \qquad \forall v \in D_0[\mathcal{O}]\,, \qquad v \geqslant 0 \ m\text{-a.e. in } \mathcal{O}\,.$$

Then,

$$\tilde{u}(x) \leqslant \sup_{X - \mathcal{O}} \tilde{u} \ \text{ q.e. } x \in \mathcal{O},$$

where $\tilde{u}$ is a q.c. version of u and the supremum on the (closed) subset $X - \mathcal{O}$ is taken in the essential capacity sense.

It suffices to prove that $u \leqslant M$ m-a.e. on $\mathcal{O}$ whenever $\tilde{u} \leqslant M$ q.e. on $X - \mathcal{O}$, for an arbitrary constant $M \in \boldsymbol{R}$. Now, the function $(\tilde{u} - M)^+$ vanishes q.e. on $X - \mathcal{O}$, hence, by (h), $(u - M)^+ \in D_0[\mathcal{O}]$. Therefore,

$$\int_{\mathcal{O}} \mu(u, (u - M)^+)(dx) \leqslant 0$$

thus, by the truncation lemma and Poincaré inequality in $D_0[\mathcal{O}]$, $u \leqslant M$ m-a.e. in $\mathcal{O}$, hence also $\tilde{u} \leqslant M$ q.e. in $\mathcal{O}$.

Similar minimum principles hold for local supersolutions in $\mathcal{O}$.

3. – Cut-off functions.

In this Section we consider a regular Dirichlet form of diffusion type a, with energy measure μ and separating core C, that satisfies Assumption I of Section 1. We observe that, due to the truncation lemma, if d is the distance defined by (1.8) we have

$$d(x, y) = \sup\{\varphi(x) - \varphi(y) : \varphi \in C^*, \mu(\varphi, \varphi) \leq m \text{ on } X\} = \\ = \sup\{\varphi(x) - \varphi(y) : \varphi + c \in C, c \in \boldsymbol{R}, \mu(\varphi, \varphi) \leq m \text{ on } X\},$$

where C^* denotes the set of all bounded continuous functions φ on X that in a neighborhood of their support coincide with the supremum or infimum of a finite number of functions belonging to C up to an additive constant. Clearly $C^* \subset D_{\text{loc}}[a]$ and $C^* \cap C_0(X)$ is a core. We recall that, due to Assumption I, X with the distance d is a space of homogeneous type, [8], in particular the following *covering lemma* holds (see [8] Ch. 3, Lemma 1.1). Below, we simply denote $D[a]$ by D.

LEMMA 3.1. – *For every $\varepsilon \in (0, 1)$, the ball $B(x, r)$, $x \in X$, $0 < r < r_0$, can be covered by the union of balls $B(y_i, \varepsilon r)$ with $y_i \in B(x, r)$, for $i = 1, 2, \ldots, l$ such that $l < c/\varepsilon^\alpha$ for suitable constants $c > 0$ and $\alpha > 0$. The constants c and α are independent of x, r and ε and depend only on c_0.*

Our construction of cut-off functions will be based on the following lemma:

LEMMA 3.2. – *Given $B(x, r) \subset\subset X$, $0 < r < r_0$ and $\delta > 1$, there exists a function $\varphi \in D[a] \cap C_0(X)$, $\varphi \in C^*$, such that $\varphi(x) = 1$, $\varphi \equiv 0$ on $X - B(x, r)$, $0 \leq \varphi \leq 1$ on X, moreover*

$$\mu(\varphi, \varphi) \leq \frac{\delta}{r^2} m \quad \text{on } X.$$

PROOF. – Let $\varepsilon > 0$ be such that $(1 - 3\varepsilon)^{-1} \leq \delta^{1/2}$. Let $B(y_i, \varepsilon r)$ with $y_i \in B(x, 2r)$, $i = 1, \ldots, l$ be the family of balls of a covering of $B(x, 2r)$ given by the Lemma 3.1. Let $B(y_j', \varepsilon r)$, $j = 1, \ldots, l'$, be the subfamily of all $B(y_i, \varepsilon r)$ such that $B(y_i, \varepsilon r) \cap B(x, 2r) \neq \emptyset$. Since $d(y_j', x) \geq r - \varepsilon r$, there exists φ_j, $\varphi_j + c \in C$ for some $c \in \boldsymbol{R}$, with $\mu(\varphi_j, \varphi_j) \leq m$, such that

$$\varphi_j(x) - \varphi_j(y_j') \geq (1 - \varepsilon) r - \varepsilon r .$$

As already remarked, any such φ_j can be modified by an arbitrary constant, so it is no restriction to assume in addition $\varphi_j(x) = (1 - \varepsilon) r$, hence $\varphi_j(y_j') \leq \varepsilon r$. Therefore, for every $y \in B(y_j', \varepsilon r)$ we have

$$\varphi_j(y) \leq d(y, y_j') + \varphi_j(y_j') < 2\varepsilon r .$$

We now define

$$\tilde{\varphi} = \min\{\varphi_j : j = 1, \ldots, l'\}.$$

We have $\tilde{\varphi} \in C^*$, $\tilde{\varphi}(x) = (1-\varepsilon)r$, $\tilde{\varphi}(y) < 2\varepsilon r$ for every point y in $\partial B(x, r)$. Since $\tilde{\varphi}$ is continuous on X and $B(x, r)$ is relatively compact in X, there exists a relatively compact neighborhood U of $\overline{B(x, r)}$, such that $\tilde{\varphi} < 2\varepsilon r$ on $U - B(x, r)$.

Let $\alpha \in D \cap C_0(X)$, $\alpha \geq 0$, be such that $\alpha \equiv 1$ on $\overline{B(x, r)}$, $\alpha \equiv 0$ on $X - U$ ([17] Lemma 1.4.2), and let

$$\tilde{\tilde{\varphi}} = (\tilde{\varphi} - 2\varepsilon r)\alpha.$$

We have $\tilde{\tilde{\varphi}} \in D \cap C_0(X)$, $\tilde{\tilde{\varphi}}(x) = (1-3\varepsilon)r$, $\tilde{\tilde{\varphi}} \leq 0$ in $U - B(x, r)$, $\tilde{\tilde{\varphi}} = 0$ in $X - U$, $\tilde{\tilde{\varphi}}(y) < 0$ for every $y \in \partial B(x, r)$. We now define

$$\varphi = \left(\frac{1}{(1-3\varepsilon)r}\,\tilde{\tilde{\varphi}} \vee 0\right) \wedge 1.$$

We have $\varphi \in D \cap C_0(X)$, $\varphi(x) = 1$, $\varphi(y) = 0$ for every $y \in X - B(x, r)$. By the truncation lemma,

$$\mu(\varphi, \varphi) \leq \mathbf{1}_{\{\tilde{\tilde{\varphi}} > 0\}} \frac{1}{(1-3\varepsilon)^2 r^2}\mu(\tilde{\tilde{\varphi}}, \tilde{\tilde{\varphi}}) \leq \mathbf{1}_{B(x,r)} \frac{1}{(1-3\varepsilon)^2 r^2}\mu(\tilde{\varphi}, \tilde{\varphi}) \leq \frac{1}{(1-3\varepsilon)^2 r^2} m = \frac{\delta}{r^2}$$

on X. Moreover, the support of φ is contained in $B(x, r)$ and φ coincides on $B(x, r)$ with $\{[(1-3\varepsilon)r]^{-1}(\tilde{\varphi} - 2\varepsilon r) \vee 0\} \wedge 1$, therefore $\varphi \in C^*$.

PROPOSITION 3.3. - *Given $B(x, r) \subset B(x, 2r) \subset\subset X$, $0 < r < r_0$, $q \in (0, 1)$, there exists $\varphi \in C^*$ with the properties: $\varphi \equiv 1$ on $B(x, qr)$, $\varphi \equiv 0$ on $X - B(x, r)$, $0 \leq \varphi \leq 1$ and $\mu(\varphi, \varphi) \leq (10/(1-q)^2 r^2)\, m$ on X.*

PROOF. - Let ϕ be the function associated, according to Lemma 3.2, with $B(x, r)$ and let $\delta > 1$ to be chosen later. Let

$$\bar{\varphi} = \frac{2}{1-q}\phi.$$

Then

$$\mu(\bar{\varphi}, \bar{\varphi}) \leq \frac{4\delta}{(1-q)^2 r^2} m \qquad \text{on } X,$$

hence

$$\mu\left(\frac{(1-q)r}{2\delta^{1/2}}\bar{\varphi}, \frac{(1-q)r}{2\delta^{1/2}}\bar{\varphi}\right) \leq m \qquad \text{on } X.$$

Moreover $(((1-q)r)/2\delta^{1/2})\bar{\varphi} \in C^*$. Therefore, for every $y \in B(x, qr)$ we have

$$\bar{\varphi}(y) \geqslant \bar{\varphi}(x) - \frac{2\delta^{1/2}}{(1-q)r} qr .$$

Since $\bar{\varphi}(x) = 2/(1-q)$, this implies

$$\bar{\varphi}(y) \geqslant \frac{2 - 2q\delta^{1/2}}{1-q} \geqslant 1$$

provided we choose $\delta^{1/2} \in (1, (1+q)/2q)$, what is allowed because $q \in (0, 1)$. Since $(1+q)/2q$ decreases from $+\infty$ to 1 as q increase from 0 to 1, we can additionally chose $\delta = \delta(q)$ in the above range such that, for example, $4\delta \leqslant 10$ for every $q \in (0, 1)$. Thus

$$\mu(\bar{\varphi}, \bar{\varphi}) \leqslant \frac{10}{(1-q)^2 r^2} m \quad \text{on } X .$$

We now take $\alpha \in D \cap C_0(X)$ such that $\alpha \equiv 1$ on a neighborhood of $\overline{B(x, r)}$, $\alpha \geqslant 0$, and we define

$$\varphi = \bar{\varphi} \wedge \alpha .$$

Since $\operatorname{supp} \varphi \subset \overline{B(x, r)}$, we have $\varphi = \bar{\varphi} \wedge 1$ on a neighborhood of $\operatorname{supp} \varphi$, hence $\varphi \in C^*$. Since $\bar{\varphi} \geqslant 1$ on $B(x, qr)$, we have $\varphi \equiv 1$ on $B(x, qr)$. Moreover, $\varphi \equiv 0$ on $X - B(x, r)$ and $0 \leqslant \varphi \leqslant 1$ on $B(x, r)$ hence on X. Finally, by the truncation lemma

$$\mu(\varphi, \varphi) = \mathbf{1}_{\{\varphi > 0\}} \mu(\varphi, \varphi) \leqslant \mu(\bar{\varphi}, \varphi) \leqslant \frac{10}{(1-q)^2 r^2} m \quad \text{on } X .$$

We will refer to the function φ of Proposition 3.3 as to the cut-off function of the ball $B(x, qr)$ in the ball $B(x, r)$.

COROLLARY 3.4. – *Given $B(x, sr) \subset B(x, 2tr) \subset\subset X$, $0 < 2tr < r_0$, $0 < s < t$, there exists a function $\varphi \in C^*$, such that $\varphi \equiv 1$ on $B(x, sr)$, $\varphi \equiv 0$ on $X - B(x, tr)$, $0 \leqslant \varphi \leqslant 1$ on X, and moreover*

$$\mu(\varphi, \varphi) \leqslant \frac{10}{(t-s)^2 r^2} m \quad \text{on } X .$$

To prove the result it is enough to apply Proposition 3.3, with r replaced by tr and $q = s/t$.

COROLLARY 3.5. – *Given $B(x, 2(2t-s)r) \subset\subset X$ and the anuli $B(x, tr) - B(x, sr) \subset B(x, (2t-s)r) - B(x, (s^2/t)r)$, $0 < 4tr < r_0$. $0 < s < t$, there exists a function $\varphi \in C^* - C^*$ such that $\varphi \equiv 1$ on $B(x, tr)$, $\varphi \equiv 0$ on $B(x, (s^2/t)r)$ and on $X -$*

$-B(x,(2t-s)r)$, $0 \leqslant \varphi \leqslant 1$ *on* X, *and*

$$\mu(\varphi,\varphi) \leqslant \frac{20}{(s/t)^2(t-s)^2 r^2}m \quad \text{on } X.$$

PROOF. – Let $\tilde{\varphi}$ be the cut-off function of $B(x, tr)$ with respect to $B(x,(2t-s)r)$ and let $\underset{\sim}{\varphi}$ be the cut-off function of $B(x,(s^2/t)r)$ with respect to $B(x, sr)$. We have

$$\mu(\tilde{\varphi},\tilde{\varphi}) \leqslant \frac{10}{(t-s)^2 r^2}m \quad \text{on } X,$$

as well as

$$\mu(\underset{\sim}{\varphi},\underset{\sim}{\varphi}) \leqslant \frac{10}{(s/t)^2(t-s)^2 r^2}m \quad \text{on } X.$$

We define $\varphi = \tilde{\varphi} - \underset{\sim}{\varphi}$. We have $\varphi = 0$ on $B(x,(s^2/t)r)$ and on $X - B(x,(2t-s)r)$, moreover

$$\mu(\varphi,\varphi) = \mu(\tilde{\varphi},\tilde{\varphi}) + \mu(\underset{\sim}{\varphi},\underset{\sim}{\varphi})$$

because $\tilde{\varphi} = 1$ on a neighborhood of supp $\underset{\sim}{\varphi}$. Therefore,

$$\mu(\varphi,\varphi) \leqslant \frac{20}{(s/t)^2(t-s)^2 r^2}m \quad \text{on } X.$$

As an application of a cut-off function argument we give the following generalization of the Sobolev-Poincaré inequality (k) of Section 1:

PROPOSITION 3.6. – *Let* u *be in* $D_{\text{loc}}[X_0]$ *and* $B(x, 2tr) \subset\subset X_0$. *Then,*

$$\left(\fint_{B(x,sr)} |u|^s m(dz)\right)^{1/s} \leqslant c\left[r\left(\fint_{B(x,tr)} \mu(u,u)(dz)\right)^{1/2} + \frac{1}{(t-s)}\left(\fint_{B(x,tr)} u^2 m(dz)\right)^{1/2}\right],$$

$0 < s_0 < s < t < 1$, *where* c *is a constant depending only on* s_0, c_0 *and* c_2.

Here and in the following of the paper we use the notation

$$\fint_B = \frac{1}{m(B)}\int_B$$

for every open subset B of X.

PROOF. – It is not restrictive to assume $u \in L^\infty(X, m)$. Moreover, we approximate u in the intrinsic norm with a sequence $u \in C$, $n = 1, 2, \ldots$. Let φ be the cut-off function of $B(x, sr)$ with respect to $B(x, tr)$. From the Sobolev-Poincaré inequality (k) of Section 1, we obtain by the Leibnitz rule and by (2.5):

$$\left(\fint_{B(x,tr)} |u_n|^s \varphi^s m(\mathrm{d}x)\right)^{1/s} \leq c_2 r \left(\fint_{B(x,tr)} \mu(u_n\varphi, u_n\varphi)\, m(\mathrm{d}x)\right)^{1/2} \leq$$

$$\leq c_2 r \left[\fint_{B(x,tr)} (\varphi^2 \mu(u_n, u_n)(\mathrm{d}x) + 2u_n \varphi\mu(u_n, \varphi)(\mathrm{d}z) + u_n^2 \mu(\varphi, \varphi)(\mathrm{d}x))\right]^{1/2} \leq$$

$$\leq 2c_2 r \left[\fint_{B(x,tr)} \varphi^2 \mu(u_n, u_n)(\mathrm{d}x) + \fint_{B(x,tr)} u_n^2 \mu(\varphi, \varphi)(\mathrm{d}x)\right]^{1/2} \leq$$

$$\leq 2c_2 r \left[\fint_{B(x,tr)} \varphi^2 \mu(u_n, u_n)(\mathrm{d}x) + \frac{10}{(t-s)^2 r^2} \fint_{B(x,tr)} u_n^2 m(\mathrm{d}x)\right]^{1/2}.$$

By letting $n \to \infty$, we obtain:

$$\left(\fint_{B(x,tr)} |u|^s \varphi^s m(\mathrm{d}x)\right)^{1/s} \leq 2c_2 \left[r^2 \fint_{B(x,tr)} \varphi^2 \mu(u, u)(\mathrm{d}x) + \frac{10}{(t-s)^2 r^2} \fint_{B(x,tr)} u^2 m(\mathrm{d}x)\right]^{1/2},$$

from which the result follows.

4. - Global L^∞-estimates.

In this section we will prove a global L^∞-estimate for the solution of the problem

$$a(u, v) = \int_{B_R} fv\, m(\mathrm{d}x) \tag{4.1}$$

$\forall v \in D_0[B_R]$, $u \in D_0[B_R]$ with $D_0[B_R] = D_0[a, B_R]$, where B_R is an intrinsic ball of radius R and the closure of the concentric ball B_{2R} is contained in X_0.

THEOREM 4.1. - *Let u be a solution to* (4.1) *with* $f \in L^p(B_R, m)$, $p > \max(\nu/2, 2)$, *where ν is the constant occurring in* (k). *Then,*

$$\sup_{B_R} |u| \leq cR^2 m(B_R)^{-1/p} \|f\|_{L^p(B_R, m)}, \tag{4.2}$$

where c depends only on the constants c_0 and c_1 of Section 1.

PROOF. – The proof uses Stampacchia's method for the usual elliptic case [36]. Let

$$\beta(\tau) = (\operatorname{sign}\tau)\max(|\tau| - k, 0) = \begin{cases} \tau - k & \text{if } \tau > k, \\ 0 & \text{if } |\tau| \leqslant k, \\ \tau + k & \text{if } \tau < -k, \end{cases}$$

where $k > 0$. We observe that

$$\beta(\tau) = (\tau - k)^+ - (\tau + k)^- .$$

For every $n \in N$, let $\zeta_n := \beta(u_n)$, where

$$u_n = \begin{cases} u & \text{if } |u| \leqslant n, \\ n & \text{if } u > n, \\ -n & \text{if } u < -n . \end{cases}$$

We have $u_n,\ \zeta_n \in D_0[B_R]$ and by repeatedly applying the truncation lemma we find

$$a(u_n, \zeta_n) = \int_{B_R} \mu(u_n, \zeta_n)(\mathrm{d}x) = \int_{B_R} \mu(\zeta_n, \zeta_n)(\mathrm{d}z) = a(\zeta_n, \zeta_n).$$

For k fixed, as $n \to +\infty$, u_n converges strongly to u is the intrinsic metric and ζ_n converges weakly in the same metric to $\zeta := \beta(u)$, [18] theorem 1.4.2 (iii) and (v), and $\zeta \in D_0[B_R]$. Therefore, we find

$$a(\zeta, \zeta) \leqslant \liminf_{n \to +\infty} a(\zeta_n, \zeta_n) = a(u, \zeta) = \int_{B_R} f\zeta m(\mathrm{d}x).$$

For every $k > 0$, let $A(k) := \{x \in B_R,\ |u| > k\}$. Again by the truncation lemma and by Sobolev-Poincaré inequality, we have

$$\begin{aligned} a(\zeta, \zeta) = \int_{B_R} \mu(\zeta, \zeta)(\mathrm{d}x) &\leqslant \left(\int_{A(k)} |f|^{s'} m(\mathrm{d}x)\right)^{1/s'} \left(\int_{B_R} |\zeta|^s m(\mathrm{d}x)\right)^{1/s} \leqslant \\ &\leqslant c_2 R m(B_R)^{1/s - 1/2} \left(\int_{A(k)} |f|^{s'} m(\mathrm{d}x)\right)^{1/s'} \left(\int_{B_R} \mu(\zeta, \zeta)(\mathrm{d}x)\right)^{1/2} \leqslant \\ &\leqslant c_2^2 \frac{R^2}{2} m(B_R)^{2/s - 1} \left(\int_{A(k)} |f|^{s'} m(\mathrm{d}x)\right)^{2/s'} + \frac{1}{2}\int_{B_R} \mu(\zeta, \zeta)(\mathrm{d}x) \leqslant \\ &\leqslant c_2^2 \frac{R^2}{2} m(B_R)^{2/s - 1} \|f\|^2_{L^p(B_R, m)} m(A(k))^{(2/s' - 2/p)} + \frac{1}{2}\int_{B_R} \mu(\zeta, \zeta)(\mathrm{d}x), \end{aligned}$$

where $s' = s/(s-1)$ and $p > s'$, therefore

$$a(\zeta, \zeta) = \int_{B_R} \mu(\zeta, \zeta)(dx) \leq c_2^2 R^2 m(B_R)^{2/s-1} \|f\|^2_{L^p(B_R, m)} m(A(k))^{(2/s' - 2/p)}. \tag{4.3}$$

From the Sobolev-Poincaré inequality, if we put $M = 2s/(s-2)$ (then $s = 2M/(M-2)$), we have

$$\left(⨍_{B_R} |\zeta|^s m(dx) \right)^{2/s} = \left(⨍_{B_R} [(|u| - k)^+]^{2M/(M-2)} m(dx) \right)^{(M-2)/M} \leq \tag{4.4}$$

$$\leq c_2^2 R^2 ⨍_{B_R} \mu(\zeta, \zeta) \leq c_2^4 R^4 m(B_R)^{2/(s-2)} \|f\|^2_{L^p(B_R, m)} m(A(k))^{(2/s' - 2/p)},$$

therefore

$$\left(\int_{B_R} [(|u| - k)^+]^{2M/(M-2)} \right)^{(M-2)/M} \leq c_2^4 R^4 m(B_R)^{-4/M} \|f\|^2_{L^p(B_R, m)} m(A(k))^{(2/s' - 2/p)}. \tag{4.5}$$

If $h > k > 0$ we have

$$(h-k)^2 m(A(h))^{(M-2)/M} \leq \left(\int_{A(k)} [(|u| - k)^+]^{2M/(M-2)} \right)^{(M-2)/M} \leq$$

$$\leq c_2^4 R^4 m(B_R)^{-4/M} \|f\|^2_{L^p(B_R, m)} m(A(k))^{(2/s' - 2/p)},$$

then

$$m(A(h)) \leq (c_2^4 R^4 m(B_R)^{-4/M} \|f\|^2_{L^p(B_R, m)})^{M/(M-2)} \cdot \frac{1}{(h-k)^{2M/(M-2)}} m(A(k))^\beta \tag{4.6}$$

with

$$\beta = \left(\frac{2}{s'} - \frac{2}{p} \right) \frac{M}{M-2} = \left(1 + \frac{2}{M} - \frac{2}{p} \right) \frac{M}{M-2},$$

which is greater than 1 since $p > M/2 = s/(s-2)$. From Lemma 4.1, p. 93 of [36], this implies

$$m(A(d)) = 0 \tag{4.7}$$

for

$$d = (c_2^4 R^4 m(B_R)^{-4/M} \|f\|^2_{L^p(B_R, m)})^{1/2} m(B_R)^{(2/M - 1/p)} = c_2^2 R^2 \|f\|_{L^p(B_R, m)} m(B_R)^{-1/p}.$$

From (4.7) we have the result.

We now define the *regularized Green function* for the form a, relative to a ball

$B(x, r) \subset X_0$ and to ball $B(y, \rho) \subset\subset B(x, R)$, both being intrinsic balls for a, as the solution of the problem

$$(4.8)\quad a(G^y_{\rho, B(x,R)}, v) = \fint_{B(y,\rho)} v m(dx) \quad \forall v \in D_0[a, B(x, R)], \quad G^y_{\rho, B(x,R)} \in D_0[a, B(x, R)].$$

We recall that, due to the Poincaré inequality, problem (4.8) has a unique solution. Moreover, $G^y_{\rho, B(x,R)}$ is *nonnegative*, as it can be seen by a simple truncation argument.

LEMMA 4.2. – *We have* $G^y_{\rho, B(x,R)} \in L^\infty(B(x, R), m)$ *and*

$$(4.9)\quad \left(\frac{1}{m(B(x,R))} \int_{B(x,R)} (G^y_{\rho, B(x,R)})^{p'} m(dx) \right)^{1/p'} \leq c \frac{R^2}{m(B(x,R))}$$

$\forall y \in B(x, R - \rho)$, *where* c *is a structural constant. Then*

$$(4.10)\quad \int_{B(x,R)} G^y_{\rho, B(x,R)} m(dx) \leq cR^2.$$

$p' = p/(p-1)$, p *as in Theorem* 4.1.

PROOF. – The first part of the result in an easy consequence of the L^∞ global estimate of Theorem 4.1.

Again from Theorem 4.1 we have

$$\int_{B(x,R)} G^y_{\rho, B(x,R)} f m(dx) \leq cR^2 \|f\|_{L^p(B(x,R), m)} m(B(x,R))^{-1/p}$$

uniformly in ρ. Then

$$(4.11)\quad \left(\int_{B(x,R)} (G^y_{\rho, B(x,R)})^{p'} m(dx) \right)^{1/p'} \leq cR^2 m(B(x,R))^{-1/p}$$

and (4.9) easily follows from (4.11).

Finally (4.10) is an easy consequence of (4.9).

5. – Local estimates and Harnack inequality.

We first prove a *Caccioppoli type* inequality:

PROPOSITION 5.1. – *Let u be a local solution (positive subsolution) in a open bounded set $\mathcal{O} \subset X_0$ of the problem:*

$$a(u, v) = 0\,, \qquad u \in D_{\text{loc}}[\mathcal{O}]\,, \qquad \forall v \in D_0[\mathcal{O}]$$

$$(a(u, v) \leqslant 0\,,\ \ u \in D_{\text{loc}}[\mathcal{O}]\,,\ \ u \geqslant 0\,,\ \ \forall v \in D_0[\mathcal{O}]\,,\ \ v \geqslant 0)\,.$$

Let $B_s \subset B_t \subset B_{2t} \subset\subset \mathcal{O}$ be concentric balls, $0 < s < t \leqslant 1$. Then

$$\int_{B_s} \mu(u, u)(\mathrm{d}x) \leqslant \frac{40}{(t-s)^2} \int_{B_t - B_s} u^2 m(\mathrm{d}x)\,.$$

PROOF. – We choose as test function $u_k \varphi^2$, where φ is the cut-off function of B_s relative to B_t and

$$u_k = \begin{cases} u & \text{if } |u| \leqslant k\,, \\ k & \text{if } u > k\,, \\ -k & \text{if } u < -k \end{cases}$$

for every $k \in N$. We have, by the Leibnitz rule and the chain rule and by (2.5):

$$0 \geqslant a(u, u_k \varphi^2) = \int_{\mathcal{O}} \mu(u, u_k \varphi^2)(\mathrm{d}x) =$$

$$= \int_{\mathcal{O}} \varphi^2 \mu(u, u_k)(\mathrm{d}x) + \int_{\mathcal{O}} u_k \mu(u, \varphi^2)(\mathrm{d}x) =$$

$$= \int_{\mathcal{O}} \varphi^2 \mu(u, u_k)(\mathrm{d}x) + 2\int_{\mathcal{O}} u_k \varphi \mu(u, \varphi)(\mathrm{d}x) \geqslant$$

$$\geqslant \int_{\mathcal{O}} \varphi^2 \mu(u, u_k)(\mathrm{d}x) - \frac{1}{2}\int_{\mathcal{O}} \varphi^2 \mu(u, u)(\mathrm{d}x) - 2\int_{\mathcal{O}} \varphi_k^2 \mu(\varphi, \varphi)(\mathrm{d}x)\,.$$

Then

$$\int_{\mathcal{O}} \varphi^2 \mu(u, u_k)(\mathrm{d}x) - \frac{1}{2}\int_{\mathcal{O}} \varphi^2 \mu(u, u)(\mathrm{d}x) \leqslant 2\int_{\mathcal{O}} u_k^2 \mu(\varphi, \varphi)(\mathrm{d}x) \leqslant$$

$$\leqslant \frac{20}{(t-s)^2} \int_{B_t - B_s} u_k^2\, m(\mathrm{d}x) \leqslant \frac{20}{(t-s)^2} \int_{B_t - B_s} u^2 m(\mathrm{d}x)\,.$$

As $k \to +\infty$, by the truncation lemma

$$\int_{\mathcal{O}} \varphi^2 \mu(u, u_k)(\mathrm{d}x) \to \int_{\mathcal{O}} \varphi^2 \mu(u, u)(\mathrm{d}x).$$

Therefore, from the preceding inequality

$$\int_{B_s} \mu(u, u)(\mathrm{d}x) \leq \int_{\mathcal{O}} \varphi^2 \mu(u, u)(\mathrm{d}x) \leq \frac{40}{(t-s)^2} \int_{B_t - B_s} u^2 m(\mathrm{d}x).$$

We recall now a Real Analysis lemma, whose proof we give for sake of completeness.

LEMMA 5.2. – *Let $u \in L^\infty(B(x, r), m)$ and assume there exists positive constants C, L such that for every s, t with $1/2 \leq s < t \leq 1$ we have*

$$\sup_{B(x, sr)} |u| \leq \frac{C}{(t-s)^L} \left(\fint_{B(x, tr)} |u|^{2d} m(\mathrm{d}x) \right)^{1/2d}, \quad d > 0. \tag{5.1}$$

Then for every $p > 0$ there exists a constant c_p, which depends on p, C, L, d and on the constant c_0 in Assumption I (ii) *buty does not depend on u, such that*

$$\sup_{B(x, \frac{1}{2}r)} |u| \leq c_p \left(\fint_{B(x, r)} |u|^p m(\mathrm{d}x) \right)^{1/p}.$$

PROOF. – If $p \geq 2d$ the result follows from (5.1). Assume $p < 2d$ and denote $I(s) = \left(\fint_{B(x, s\tau)} |u|^{2d} m(\mathrm{d}x) \right)^{1/2d}$. It is not restrictive to assume $\fint_{B(x, \tau)} |u|^p m(\mathrm{d}x) = 1$. If $1/2 \leq s < 1$ we have, by taking the duplication property of Assumption I into account:

$$I(s) = \left(\fint_{B(x, sr)} |u|^p |u|^{2d-p} m(\mathrm{d}x) \right)^{1/d} \leq (\sup_{B(x, sr)} |u|)^{1-p/2d} \left(\fint_{B(x, sr)} |u|^p m(\mathrm{d}x) \right)^{1/2d} \leq$$

$$\leq c(\sup_{B(x, s)} |u|)^{1-p/2d} \left(\fint_{B(x, r)} |u|^p m(\mathrm{d}x) \right)^{1/2d} = c(\sup_{B(x, sr)} |u|)^{1-p/2d}.$$

From (5.1) we have for $1/2 \leq s < t < 1$:

$$I(s) \leq \frac{cC^{\theta}}{(t-s)^{L\theta}} I(t)^{\theta}, \qquad \theta = 1 - \frac{p}{2d}, \tag{5.2}$$

therefore,

$$\log I(s) \leq \log c + \theta \log C - L\theta \log(t-s) + \theta \log I(t). \tag{5.3}$$

We choose $s = t^b$, $b > 1$, and we obtain

$$\log I(t^b) \leq \log c + \theta \log C - L\theta \log(t - t^b) + \theta \log I(t). \tag{5.4}$$

We divide (5.4) by t and we integrate from $(2/3)^{1/b}$ to 1:

$$\int_{(2/3)^{1/b}}^{1} \frac{\log I(t^b)}{t} dt \leq C(\theta, L) + \theta \int_{(2/3)^{1/b}}^{1} \frac{\log I(t)}{t} dt \leq C(\theta, L) + \theta \int_{2/3}^{1} \frac{\log I(t)}{t} dt. \tag{5.5}$$

If there exists $\bar{t} \in (2/3, 1)$ with $I(\bar{t}) \leq 1$, the result follows from (5.1). Thus, we suppose $I(t) > 1$ in $(2/3, 1)$.

Choosing $\tau = t^b$, we have

$$\int_{(2/3)^{1/b}}^{1} \frac{\log I(t^b)}{t} dt = \frac{1}{b} \int_{2/3}^{1} \frac{\log I(\tau)}{\tau} d\tau.$$

Then, from (5.5),

$$\frac{1}{b} \int_{2/3}^{1} \frac{\log I(t)}{t} dt \leq c(\theta, L) + \theta \int_{2/3}^{1} \frac{\log I(t)}{t} dt.$$

Choosing b such that $1/b - \theta > 0$,

$$\int_{2/3}^{1} \frac{\log I(t)}{t} dt \leq c(\theta, L) \left(\frac{1}{b} - \theta \right)^{-1}.$$

Then there exists $\bar{t} \in (2/3, 1)$ with

$$I(\bar{t}) \leq c$$

and the result follows.

By the same methods of Lemma 5.2 we obtain the following result

PROPOSITION 5.3. - *Let u be a local solution or a positive local subsolution, as in*

Proposition 5.1. *Then, for every* $r > 0$ *with* $B(x, 2r) \subset\subset \mathcal{O}$, *we have*

$$\left(\fint_{B(x, r/2)} |u|^2 m(dx) \right)^{1/2} \leq c \int_{B(x, r)} |u| m(dx),$$

where c *is a structural constant.*

PROOF. – Without loss of generality we can suppose $\fint_{B(x, r)} |u| m(dx) = 1$. Denote

$$I(s) = \left(\fint_{B(x, sr)} |u|^2 m(dx) \right)^{1/2}, \qquad \frac{1}{2} \leq s < 1 .$$

By Hölder's inequality, we have for $0 < \theta < 1$:

$$I(s) \leq \left(\fint_{B(x, sr)} |u|^{(2-\theta)/(1-\theta)} m(dx) \right)^{(1-\theta)/2} \left(\fint_{B(x, sr)} |u| m(dx) \right)^{\theta/2} \leq$$

$$\leq c \left(\fint_{B(x, sr)} |u|^{(2-\theta)/(1-\theta)} m(dx) \right)^{(1-\theta)/2} .$$

We now choose $\theta \in (0, 1)$ such that $(2-\theta)/(1-\theta) = s$, where here $s > 2$ is the exponent in the Sobolev-Poincaré inequality. We have, by Proposition 3.6,

$$\left(\fint_{B(x, sr)} |u|^{(2-\theta)/(1-\theta)} m(dx) \right)^{(1-\theta)/(2-\theta)} \leq$$

$$\leq c \left[\left(\frac{1}{(t-s)^2} \fint_{B(x, (t-(t-s)/2)r)} |u|^2 m(dx) \right)^{1/2} + \left(r^2 \fint_{B(x, (t-(t-s)/2)r)} \mu(u, u)(dx) \right)^{1/2} \right]$$

and by Caccioppoli's inequality

$$\left(\fint_{B(x, sr)} |u|^{(2-\theta)/(1-\theta)} m(dx) \right)^{(1-\theta)/(2-\theta)} \leq \frac{c}{(t-s)} \left(\fint_{B(x, tr)} |u|^2 m(dx) \right)^{1/2} .$$

Then

$$I(s) \leq c\left\{\frac{I(t)}{t-s}\right\}^{(1-\theta)/2}, \qquad \frac{1}{2} \leq s < t < 1 .$$

The proof can be completed by proceeding as in the proof of Lemma 5.2.

Now we prove a boundedness result for local solutions or positive subsolutions relative to $a(u, v)$, which may be interesting in itself.

THEOREM 5.4. – *Let u be a local solution (positive subsolution) in $B(x, 4r) \subset\subset X_0$, i.e.,*

$$a(u, v) = 0 (\leq 0) \quad \forall v \in D_0[B(x, 4r)](v \geq 0), \qquad u \in D_{\mathrm{loc}}[B(x, 4r)](u \geq 0).$$

Then, for every $p > 0$ we have

$$\sup_{B(x, r/2)} |u| \leq c_p \left(\fint_{B(x, r)} |u|^p m(\mathrm{d}z) \right)^{1/p}$$

where c_p is a structural constant depending on p.

PROOF. – By Lemma 5.2 it is enough to prove that for a suitable $\beta > 0$ and for every s, t with $1/2 \leq s < t \leq 1$, we have

$$\sup_{B(x, sr)} |u| \leq \frac{c}{(t-s)^\beta} \left(\fint_{B(x, tr)} |u|^2 m(\mathrm{d}z) \right)^{1/2} . \tag{5.6}$$

Let φ be the cut-off function of $B(x, (s+\varepsilon) r)$ with respect to $B(x, (t-\varepsilon) r)$ where $\varepsilon = (t-s)/4$. By Corollary 3.4, we have

$$\mu(\varphi, \varphi) \leq \frac{10}{(t-s)^2 r^2} m .$$

We consider the regularized Green function $G_\rho^y = G_{\rho, B(x, 4r)}^y$, $y \in B(x, sr)$.

We have

$$\fint_{B(y, \rho)} u_k \varphi m(\mathrm{d}x) = a(G_\rho^y, u_k \varphi) = \int_{B(x, 4r)} \mu(u_k \varphi, G_\rho^y)(\mathrm{d}x),$$

where

$$u_k = \begin{cases} k & u > k, \\ u & |u| \leq k, \\ -k & u < -k. \end{cases}$$

We recall that $G_\rho^y \in D_0[B(x, 4r)] \cap L^\infty(B(x, 4r), m)$. By the Leibnitz rule and by Proposition 5.4.3 in [18], we obtain

$$\fint_{B(y,\rho)} u_k \varphi m(\mathrm{d}x) = \int_{B(x,4r)} [\varphi \mu(u_k, G_\rho^y)(\mathrm{d}x) + u_k \mu(\varphi, G_\rho^y)(\mathrm{d}x)] =$$

$$= \int_{B(x,4r)} [\mu(u_k, \varphi G_\rho^y)(\mathrm{d}x) - G_\rho^y \mu(u_k, \varphi)(\mathrm{d}x) + u_k \mu(\varphi, G_\rho^y)(\mathrm{d}x)] \leqslant \int_{B(x,4r)} \mu(u_k, \varphi G_\rho^y)(\mathrm{d}x) +$$

$$+ \frac{c}{(t-s)r} \left(\int_{B(x,(t-\varepsilon)r) - B(x,(s+\varepsilon)r)} (G_\rho^y)^2 m(\mathrm{d}x) \right)^{1/2} \left(\int_{B(x,(t-\varepsilon)r) - B(x,(s+\varepsilon)r)} \mu(u_k, u_k)(\mathrm{d}x) \right)^{1/}$$

$$+ \frac{c}{(t-s)r} \left(\int_{B(x,tr)} |u_k|^2 m(\mathrm{d}x) \right)^{1/2} \left(\int_{B(x,(t-\varepsilon)r) - B(x,(s+\varepsilon)r)} \mu(G_\rho^y, G_\rho^y)(\mathrm{d}x) \right)^{1.}$$

Passing to the limit as $k \to +\infty$, we find

$$\fint_{B(y,\rho)} u \varphi m(\mathrm{d}x) \leqslant \frac{c}{(t-s)r} \left(\int_{B(x,(t-\varepsilon)r) - B(x,(s+\varepsilon)r)} (G_\rho^y)^2 m(\mathrm{d}x) \right)^{1/2} \cdot$$

$$\cdot \left(\int_{B(x,(t-\varepsilon)r) - B(x,(s+\varepsilon)r)} \mu(u, u)(\mathrm{d}x) \right)^{1/2} +$$

$$+ \frac{c}{(t-s)r} \left(\int_{B(x,tr)} |u|^2 m(\mathrm{d}x) \right)^{1/2} \left(\int_{B(x,(t-\varepsilon)r) - B(x,(s+\varepsilon)r)} \mu(G_\rho^y, G_\rho^y)(\mathrm{d}x) \right)^{1/2}.$$

If u is a local solution the same relation holds also for $-u$, then in general we can write

$$\left| \fint_{B(y,\rho)} u \varphi m(\mathrm{d}x) \right| \leqslant \frac{c}{(t-s)r} \left(\int_{B(x,(t-\varepsilon)r) - B(x,(s+\varepsilon)r)} (G_\rho^y)^2 m(\mathrm{d}x) \right)^{1/2} \cdot$$

$$\cdot \left(\int_{B(x,(t-\varepsilon)r) - B(x,(s+\varepsilon)r)} \mu(u, u)(\mathrm{d}x) \right)^{1/2} +$$

$$+ \frac{c}{(t-s)r} \left(\int_{B(x,tr)} |u|^2 m(\mathrm{d}x) \right)^{1/2} \left(\int_{B(x,(t-\varepsilon)r) - B(x,(s+\varepsilon)r)} \mu(G_\rho^y, G_\rho^y)(\mathrm{d}x) \right)^{1/2}.$$

From Caccioppoli's inequality we have

$$\left(\int_{B(x,(t-\varepsilon)r)} \mu(u,u)(dx)\right)^{1/2} \leqslant \frac{c}{(t-s)r}\left(\int_{B(x,tr)} |u|^2 m(dx)\right)^{1/2}.$$

By the same methods used to obtain the Caccioppoli inequality and using the cut-off function of $B(x,(t-\varepsilon)r) - B(x,(s+\varepsilon)r)$ with respect to $B(x,(t-\varepsilon/2)r) - B(x,(s+\varepsilon/2)r)$. we find

$$\left(\int_{B(x,(t-\varepsilon)r)-B(x,(s+\varepsilon)r)} \mu(G_\rho^y, G_\rho^y)(dx)\right)^{1/2} \leqslant$$

$$\leqslant \frac{c}{(t-s)r}\left(\int_{B(x,(t-\varepsilon/2)r)-B(x,(s+\varepsilon/2)r)} (G_\rho^y)^2 m(dx)\right)^{1/2}$$

Therefore, we have

$$\left|\int_{B(y,\rho)} u\varphi m(dx)\right| \leqslant \tag{5.7}$$

$$\leqslant \frac{c}{(t-s)^2 r^2}\left(\int_{B(x,(t-\varepsilon/2)r)-B(x,(s+\varepsilon/2)r)} (G_\rho^y)^2 m(dx)\right)^{1/2}\left(\int_{B(x,tr)} |u|^2 m(dx)\right)^{1/2}$$

Taking into account the structure of homogeneous space of X, Lemma 3.1, we can cover $B(x,(t-\varepsilon/2)r)$ by balls $B(x_i, \varepsilon/32r)$, $i = 1, \ldots, l$, where $l \leqslant \tilde{c}/\varepsilon^\alpha \leqslant c/(t-s)^\alpha$, for some $c > 0$ and $\alpha > 0$ depending only on c_0, $x_i \in B(x,(t-\varepsilon/2)r)$. Moreover we observe that, due to the duplication property,

$$c\varepsilon^{-\nu} m\left(B\left(x_i, \frac{\varepsilon}{32}r\right)\right) \geqslant m(B(x,tr)). \tag{5.8}$$

[Suppose $2^{-N} < \varepsilon < 2^{-(N-1)}$, then

$$c_0^{N+2} m(B(x_i, \varepsilon r)) \geqslant m(B(x_i, 4r)) \geqslant m(B(x,tr)).$$

Now $(N-1)\log 2 \leqslant -\log\varepsilon \Rightarrow N \leqslant 1 - (\log\varepsilon)/(\log 2)$. Taking $c = c_0^8$, $\nu = (\log c_0)/(\log 2)$ we obtain (5.8).] We consider the balls $B(x_i,(\varepsilon/32)r)$, $i = i_1, \ldots, i_n$, whose union covers $B(x,(t-\varepsilon/2)r) - B(x,(s+\varepsilon/2)r)$, with $B(x,(t-\varepsilon/2)r) - B(x,(s+\varepsilon/2)r) \cap B(x_i,(\varepsilon/32)r) \neq \emptyset$ for every $i = i_1, \ldots i_n$. We have $B(x_i, \varepsilon/4r) \subset B(x,tr) - B(x,sr)$ for every $i = i_1, \ldots, i_n$. From Proposition 5.3 and

(5.8) we obtain

$$\left(\int_{B(x_i,(\varepsilon/32)r)} (G_\rho^y)^2 m(\mathrm{d}x)\right)^{1/2} \leqslant c\frac{m(B(x_i,(\varepsilon/32)r))^{1/2}}{m(B(x_i,(\varepsilon/16)r))}\int_{B(x_i,(\varepsilon/16)r)} G_\rho^y m(\mathrm{d}x) \leqslant$$

$$\leqslant \frac{c}{m(B(x_i,(\varepsilon/16)r))^{1/2}}\int_{B(x_i,(\varepsilon/16)r)} G_\rho^y m(\mathrm{d}x) \leqslant \frac{c}{(t-s)^{\nu/2} m(B(x,tr))^{1/2}}\int_{B(x,tr)} G_\rho^y m(\mathrm{d}x).$$

Summing up with respect to $i = i_1, \ldots, i_n$ we find

$$\left(\int_{B(x,(t-\varepsilon/2)r)-B(x,(s+\varepsilon/2)r)} (G_\rho^y)^2 m(\mathrm{d}x)\right)^{1/2} \leqslant \frac{c}{(t-s)^\gamma} m(B(x,tr))^{1/2} \fint_{B(x,tr)} G_\rho^y m(\mathrm{d}x) \tag{5.9}$$

where $\gamma = \alpha + \nu/2$.

From (5.7), (5.9), we have

$$\left|\fint_{B(y,\rho)} u\varphi m(\mathrm{d}x)\right| \leqslant \frac{c}{(t-s)^{\gamma+2} r^2}\int_{B(x,tr)} G_\rho^y m(\mathrm{d}x)\left(\fint_{B(x,tr)} |u|^2 m(\mathrm{d}x)\right)^{1/2},$$

hence, by Lemma 4.2

$$\left|\fint_{B(y,\rho)} u\varphi m(\mathrm{d}x)\right| \leqslant \frac{c}{(t-s)^{\gamma+2}}\left(\fint_{B(x,tr)} |u|^2 m(\mathrm{d}x)\right)^{1/2}, \tag{5.10}$$

that holds uniformly with respect to $y \in B(x, sr)$ and $\rho > 0$. In the limit as $\rho \to 0$, by Lebesgue theorem, [7], we find

$$\sup_{B(x,sr)} |u| \leqslant \frac{c}{(t-s)^{\gamma+2}}\left(\fint_{B(x,tr)} |u|^2 m(\mathrm{d}x)\right)^{1/2},$$

what proves (5.6).

We recall that a function u on X is said to be of *bounded mean oscillation* in an open subset $\mathcal{O}$ of X, if $u \in L^1(\mathcal{O}, m)$ and the following seminorm is finite

$$\|u\|_{\mathrm{BMO},\mathcal{O}} := \sup m(B)^{-1}\int_B |u(x) - \bar{u}|^1 m(\mathrm{d}x),$$

where $\bar{u}$ denotes the average

$$\bar{u}_B = m(B)^{-1}\int_B u(x)\, m(\mathrm{d}x)$$

and the supremum is taken over the family of all intrinsic balls $B = B(x, r)$ contained in $\mathcal{O}$.

In [1] the classic John-Nirenberg's lemma in euclidean spaces has been extended to BMO functions on homogeneous spaces, such as X. In the following we need, however, a *local* version of this result, Proposition 5.5 below. We omit the proof, that can be given by relying on Theorem 1.2 of [8] and adapting the arguments of [6] to the present local setting.

PROPOSITION 5.5. – *Let $B(x_0, 12r) \subset X$, $0 < 12r < r_0$ and $u \in L^1(B(x_0, 12r), m)$. Then for every $x \in B(x_0, r)$ and every $\sigma > 0$ we have*

$$m(\{y \in B(x, r) \colon |u(y) - \bar{u}_{B(x,r)}| > \sigma\} \leq A \exp\left(-\frac{\alpha}{\|u\|_*}\sigma\right) m(B(x, r)),$$

where $\|u\|_* = \|u\|_{\mathrm{BMO}, B(x_0, 12r)}$ and $A \geq 1$, $\alpha > 0$ are suitable constants, depending only on the constant c_0 in Assumption I.

COROLLARY 5.6. – *Let u be a function of bounded mean oscillation in $B(x_0, 12r) \subset X$, $0 < 12r < r_0$. Then,*

$$\fint_B \exp\left(\frac{\alpha}{2M} u\right) m(dx) \fint_B \exp\left(-\frac{\alpha}{2M} u\right) m(dx) \leq A \tag{5.11}$$

for every $B = B(x, r)$, $x \in B(x_0, r)$ and arbitrary $M > \|u\|_{\mathrm{BMO}, B(x_0, 12r)}$, where $\alpha > 0$ and $A \geq 1$ are the constants occurring in Proposition 5.5.

PROOF. – For every $a > 0$ we have

$$\int_B \exp(a|u(x) - \bar{u}_B|) m(dx) = \int_0^{+\infty} m(L_\sigma) a \exp(a\sigma) d\sigma,$$

where $L_\sigma := \{x \in B \colon |u(x) - \bar{u}_B| > \sigma\}$, $\sigma > 0$. Therefore, by Proposition 5.5,

$$\int_B \exp(a|u - \bar{u}_B|) m(dx) \leq \int_0^{+\infty} A \exp\left(-\frac{\alpha}{\|u\|_*}\sigma\right) m(B) a \exp(a\sigma) d\sigma \leq$$

$$\leq Aam(B) \int_0^{+\infty} \exp\left(a\sigma - \frac{\alpha}{M}\sigma\right) d\sigma,$$

for arbitrary $M > \|u\|_*$.

By choosing $a := (1/2)(\alpha/M)$, we obtain

$$\fint_B \exp\left(\frac{\alpha}{2M}|u-\bar{u}_B|\right) m(\mathrm{d}x) \leq A\frac{\alpha}{2M}\frac{2M}{\alpha} = A. \tag{5.12}$$

From (5.12), both inequalities below follow:

$$\int_B \exp\left(\frac{\alpha}{2M}(u(x)-\bar{u}_B)\right) m(\mathrm{d}x) \leq Am(B),$$

$$\int_B \exp\left(-\frac{\alpha}{2M}(u(x)-\bar{u}_B)\right) m(\mathrm{d}x) \leq Am(B).$$

Therefore

$$\fint_B \exp\left(\frac{\alpha}{2M}u\right) m(\mathrm{d}x) \leq A\exp\left(\frac{\alpha}{2M}\bar{u}_B\right),$$

$$\fint_B \exp\left(-\frac{\alpha}{2M}u\right) m(\mathrm{d}x) \leq A\exp\left(-\frac{\alpha}{2M}\bar{u}_B\right),$$

hence the conclusion (5.11).

The proposition below affirms that a suitable power of any non-negative supersolution is locally a weight in the class A_2 of Mukenhoupt (see [11] for the definition).

We consider an open subset $\mathcal{O}$ of X_0 and a function v that satisfies:

$$\begin{aligned} &v \in D_{\mathrm{loc}}[\mathcal{O}] \cap L^\infty(\mathcal{O}, m), \qquad v \geq 0 \ \ m\text{-a.e. in } \mathcal{O}, \\ &a(v, w) \geq 0 \quad \forall w \in D_0[\mathcal{O}], \quad w \geq 0 \ \ m\text{-a.e. in } \mathcal{O}. \end{aligned} \tag{5.13}$$

PROPOSITION 5.7. – *Let v satisfy* (5.13) *and let* $B(x_0, (4\kappa + 12)R) \subset \mathcal{O}$, $x \in B(x_0, R)$, $0 < r \leq R$, $v \not\equiv 0$ *m-a.e. in* $B(x, r)$. *Then,*

$$\fint_{B(x,r)} v^\gamma m(\mathrm{d}x) \fint_{B(x,r)} v^{-\gamma} m(\mathrm{d}x) \leq A,$$

where $\gamma \in (0, 1)$ *is a suitable constant depending only on* c_0, c_1, κ, *and* $A \geq 1$ *is a constant depending only on* c_0.

As a consequence, as stated in Corollary 5.8 below, the $v^\gamma \cdot m$ inherits the duplication property of the measure m, with a duplication constant c_0' that depends only on the duplication constant c_0 of m.

COROLLARY 5.8. – *Let v satisfy* (5.13) *and let $B(x_0, (4\kappa + 12)R) \subset \mathcal{O}$, $x \in B(x_0, R)$, $0 < r \leq R$. Then,*

$$\int_{B(x,r)} v^\gamma m(dx) \leq c_0' \int_{B(x,r/2)} v^\gamma m(dx), \tag{5.14}$$

where $c_0' = Ac_0^2$ and γ and A are the constant occurring in Proposition 5.7.

In order to prove Proposition 5.7 we need some preliminary lemmas.

LEMMA 5.9. – *Let v satisfy* (5.13) *and let $B(x, 4r) \subset \mathcal{O}$. Then,*

$$\int_{B(x,r)} \mu(\log(v+\varepsilon), \log(v+\varepsilon))(dx) \leq 20c_0 \frac{m(B(x,r))}{r^2},$$

for every $\varepsilon > 0$.

PROOF. – For every $\varepsilon > 0$, $v_\varepsilon := v + \varepsilon \in D_{\text{loc}}[\mathcal{O}] \cap L^\infty(\mathcal{O}, m)$ and, by (5.13),

$$\int_{B(x,2r)} \mu(v_\varepsilon, \psi)(dx) \geq 0 \tag{5.15}$$

for every $\psi \in D_0[B(x, 2r)]$, $\psi \geq 0$ m-a.e. Moreover, by the chain rule,

$$\log v_\varepsilon \in D_{\text{loc}}[\mathcal{O}] \cap L^\infty(\mathcal{O}, m), \qquad v_\varepsilon^{-1} \in D_{\text{loc}}[\mathcal{O}] \cap L^\infty(\mathcal{O}, m)$$

and

$$\mu(\log v_\varepsilon, \log v_\varepsilon) = v_\varepsilon^{-2}\mu(v_\varepsilon, v_\varepsilon) = -\mu(v_\varepsilon, v_\varepsilon^{-1})$$

in $\mathcal{O}$.

Now, let φ be the cut-off function of $B(x, r)$ in $B(x, 2r)$. Then,

$$\int_{B(x,2r)} \varphi^2\mu(\log v_\varepsilon, \log v_\varepsilon)(dx) = -\int_{B(x,2r)} \varphi^2\mu(v_\varepsilon, v_\varepsilon^{-1})(dx) =$$

$$= -\int_{B(x,2r)} \mu(v_\varepsilon, \varphi^2 v_\varepsilon^{-1})(dx) + \int_{B(x,2r)} v_\varepsilon^{-1}\mu(v_\varepsilon, \varphi^2)(dx) \leq 2\int_{B(x,2r)} \varphi v_\varepsilon^{-1}\mu(v_\varepsilon, \varphi)(dx),$$

where we have applied (5.15) with $\psi := \varphi^2 v_\varepsilon^{-1}$.

By the Schwarz rule, we get

$$\int_{B(x,2r)} \varphi^2 \mu(\log v_\varepsilon, \log v_\varepsilon)(dx) \leqslant 2\left(\int_{B(x,2r)} \varphi^2 v_\varepsilon^{-2} \mu(v_\varepsilon, v_\varepsilon)(dx)\right)^{1/2}\left(\int_{B(x,2r)} \mu(\varphi, \varphi)(dx)\right)^{1/2} \leqslant$$

$$\leqslant \frac{1}{2}\int_{B(x,2r)} \varphi^2 \mu(\log v_\varepsilon, \log v_\varepsilon)(dx) + 2\int_{B(x,2r)} \mu(\varphi, \varphi)(dx).$$

Therefore, by Proposition 3.3 and the duplication property,

$$\int_{B(x,2r)} \varphi^2 \mu(\log v_\varepsilon, \log v_\varepsilon)(dx) \leqslant 20 c_0 \frac{m(B(x,r))}{r^2}$$

and since $\varphi \equiv 1$ on $B(x, r)$, the conclusion follows.

The following lemma establishes a uniform bound for the local BMO seminorms of the functions $\log(v+\varepsilon)$, $\varepsilon > 0$.

LEMMA 5.10. – *Let v satisfy* (5.13) *and let $B(x, 4\kappa r) \subset \mathcal{O}$. Then,*

$$\fint_{B(x,r)} |\log(v+\varepsilon) - \overline{\log(v+\varepsilon)}|^2 m(dx) \leqslant c$$

for every $\varepsilon > 0$, where c is a constant depending only on c_0, c_1 and κ.

By $\overline{\log(v+\varepsilon)}$ we are denoting the average of $\log(v+\varepsilon)$ in $B(x, r)$.

PROOF OF LEMMA 5.10. – By Poincaré's inequality (j) of Assumption II,

$$\int_{B(x,r)} |\log(v+\varepsilon) - \overline{\log(v+\varepsilon)}|^2 m(dx) \leqslant c_1 r^2 \int_{B(x,kr)} \mu(\log(v+\varepsilon), \log(v+\varepsilon))(dx),$$

hence the conclusion follows from Lemma 5.9 and the duplication property of m.

We are now ready for the

PROOF OF PROPOSITION 5.7. – Let $B \equiv B(x, r)$ be an arbitrary ball in $B(x_0, 12r)$. Then, $B(x, 4kr) \subset B(x, (4k+12)r) \subset \mathcal{O}$. By Lemma 5.10,

$$\|\log(v+\varepsilon)\|_{\mathrm{BMO}, B(x_0, 12r)} \leqslant c_{\mathrm{BMO}}$$

uniformly in $\varepsilon > 0$, c_{BMO} depending only on c_0, c_1, k.

By Corollary 5.6, for every $x \in B(x_0, r)$ we have

$$\fint_{B(x,r)} \exp\left(\frac{\alpha}{2M}\log(v+\varepsilon)\right) m(dx) \fint_{B(x,r)} \exp\left(-\frac{\alpha}{2M}\log(v+\varepsilon)\right) m(dx) \leq A$$

uniformly in $\varepsilon > 0$, where we choose $M = (c_{\mathrm{BMO}} + 1) \vee (\alpha/2)$, with $A \geq 1$ the constants, depending only on C, occurring in Lemma 5.10. Thus,

$$\fint_{B(x,r)} (v+\varepsilon)^{\gamma} m(dx) \fint_{B(x,r)} (v+\varepsilon)^{-\gamma} m(dx) \leq A$$

uniformly in $\varepsilon > 0$, where $\gamma = \alpha/(2M) \in (0, 1)$. Since $v \not\equiv 0$ in $B(x, r)$,

$$\fint_{B(x,r)} (v+\varepsilon)^{-\gamma} m(dx) \leq A \left[\fint_{B(x,r)} v^{\gamma} m(dx)\right]^{-1}$$

and letting $\varepsilon \downarrow 0$, by the monotone convergence theorem we find that $v^{-\gamma} \in L^1(B(x, r), m)$ and

$$\fint_{B(x,r)} v^{-\gamma} m(dx) \leq A \left[\fint_{B(x,r)} v^{\gamma} m(dx)\right]^{-1}$$

and this concludes the proof.

For completeness, we give also the proof of Corollary 5.8, which is based on the same argument as in Lemma 4 of [7].

PROOF OF COROLLARY 5.8. – We can obviously assume that $v \not\equiv 0$ in $B(x, r)$, hence Proposition 5.7 applies. In particular, $v^{\gamma/2}$ and $v^{-\gamma/2}$ belong both to $L^2(B(x, r), m)$, moreover

$$m\left(B\left(x, \frac{r}{2}\right)\right) = \int_{B(x,r/2)} v^{\gamma/2} v^{-\gamma/2} m(dx) \leq$$

$$\leq \left(\int_{B(x,r/2)} v^{\gamma} m(dx)\right)^{1/2} \left(\int_{B(x,r/2)} v^{-\gamma} m(dx)\right)^{1/2} \leq$$

$$\leq \left(\int_{B(x,r/2)} v^{\gamma} m(dx)\right)^{1/2} m(B(x,r)) A^{1/2} \left(\int_{B(x,r)} v^{\gamma} m(dx)\right)^{-1/2}$$

therefore

$$\int_{B(x,r)} v^\gamma m(\mathrm{d}x) \leq A\left[\frac{m(B(x,r))}{m(B(x,r/2))}\right]^2 \int_{B(x,r/2)} v^\gamma m(\mathrm{d}x)$$

hence the conclusion, by the duplication property of m.

We are now in a position to prove Harnack's inequality, namely Theorem 1.1 of the Introduction.

PROOF OF THEOREM 1.1. – We introduce the function $z := (u+\varepsilon)^q$, where $\varepsilon > 0$ and $q < 0$. We first observe that, if $B(c, 64r) \subset\subset \mathcal{O}$, then $z \in D_{\mathrm{loc}}[B(x, 8r)] \cap \cap L^\infty(B(x, 8r), m)$ and

$$a(z, w) \leq 0, \quad \forall w \in D_0[B(x, 8r)], \quad w \geq 0 \ m\text{-a.e.},$$

i.e., z is a positive subsolution in $B(x, 8r)$. In fact $z \in D_{\mathrm{loc}}[B(x, 8r)] \cap \cap L^\infty(B(x, 8r), m)$. By the chain rule, if $w \in D_0[B(x, 8r)] \cap L^\infty(B(x, 8r), m)$

$$a(z, w) = \int_{B(x,8r)} \mu((u+\varepsilon)^q, w)(\mathrm{d}x) = q \int_{B(x,8r)} (u+\varepsilon)^{q-1} \mu(u+\varepsilon, w)(\mathrm{d}x) =$$

$$= q \int_{B(x,8r)} \mu(u, (u+\varepsilon)^{q-1} w)(\mathrm{d}x) - q \int_{B(x,8r)} w\mu(u+\varepsilon, (u+\varepsilon)^{q-1})(\mathrm{d}x) \leq$$

$$\leq -q(q-1) \int_{B(x,8r)} w(u+\varepsilon)^{q-2} \mu(u+\varepsilon, u+\varepsilon)(\mathrm{d}x) \leq 0\,.$$

We can thus apply Theorem 5.4 again, this time to z, and we find

$$\sup_{B(x,r)} z \leq c_p \left(\fint_{B(x,2r)} z^p m(\mathrm{d}x) \right)^{1/p} \tag{5.16}$$

for arbitrary $p > 0$, where c_p is a constant depending only on c_0, c_2, s and p. From (5.16), by taking $q = -1$, we obtain

$$\inf_{B(x,r)} (u+\varepsilon) \geq c_p^{-1} \left(\fint_{B(x,2r)} (u+\varepsilon)^{-p} m(\mathrm{d}x) \right)^{-1/p},$$

and letting $\varepsilon \downarrow 0$, by the monotone convergence theorem, $u^{-p} \in L^1(B(x, 2r), m)$ and

$$\inf_{B(x,r)} u \geq c_p^{-1} \left(\fint_{B(x,2r)} u^{-p} m(\mathrm{d}x) \right)^{-1/p}.$$

We now choose $p=\gamma$, where $\gamma>0$ is the constant in Proposition 5.7, and suppose $B(x,(8k+24)2r)\subset \mathcal{O}$. Then,

$$\left(\fint_{B(x,2r)} u^{-\gamma} m(dx)\right)^{-1/\gamma} \geq A^{-1/\gamma}\left(\fint_{B(x,2r)} u^{\gamma} m(dx)\right)^{1/\gamma},$$

which together with the previous inequality yields

$$\inf_{B(x,r)} u \geq c_\gamma^{-1} A^{-1/\gamma}\left(\fint_{B(x,2r)} u^{\gamma} m(dx)\right)^{-1/\gamma}.$$

We apply Theorem 5.4 to u, with $p=\gamma$, and we finally obtain

$$\inf_{B(x,r)} u \geq c_\gamma^{-2} A^{-1/\gamma} \sup_{B(s,r)} u,$$

that is,

$$\sup_{B(x,r)} u \leq c \inf_{B(x,r)} u,$$

where $c=c_\gamma^2 A^{1/\gamma}$ depends only on c_0, c_1, k, c_2, s.

If u is a non-negative local solution,

$$(5.17)\qquad \begin{array}{ll} u\in D_{\mathrm{loc}}[\mathcal{O}], & u\geq 0 \ m\text{-a.e. in } \mathcal{O} \\ a(u,w)=0, & \forall w\in D_0[\mathcal{O}], \end{array}$$

$\mathcal{O}$ an open subset of X_0, then a refinement of Theorem 5.4 holds, that follows from Theorem 5.4 by taking Corollary 5.8 into account.

THEOREM 5.12. – *Let u satisfy* (5.17) *and let $B(x,(8k+24)2r)\subset\subset \mathcal{O}$. Then, for every $p>0$*

$$\sup_{B(x,r)} u \leq c_p'\left(\fint_{B(x,r)} u^{p} m(dx)\right)^{1/p},$$

where c_p' is a constant depending only on p and on c_0, c_1, k, c_2, s.

PROOF. – By Theorem 5.4

$$\sup_{B(x,r)} u \leq c_p\left(\fint_{B(x,2r)} u^{p} m(dx)\right)^{1/p},$$

for every $p>0$. We choose $p=\gamma$, where γ is the constant occurring in Corollary 5.8.

Then, by applying Corollary 5.8 twice, we get

$$\left(\fint_{B(x,2r)} u^\gamma m(\mathrm{d}x) \right)^{1/\gamma} \leqslant (c_0')^{1/\gamma} \left[\frac{m(B(x,r))}{m(B(x,2r))} \right]^{1/\gamma} \left(\fint_{B(x,r)} u^\gamma m(\mathrm{d}x) \right)^{1/\gamma} \leqslant$$

$$\leqslant (c_0')^{2/\gamma} \left(\fint_{B(x,r/2)} u^\gamma m(\mathrm{d}x) \right)^{1/\gamma} \leqslant (c_0')^{2/\gamma} \sup_{B(x,r/2)} u\,,$$

hence, by applying Theorem 5.4 again with arbitrary $p > 0$:

$$\sup_{B(x,r)} u \leqslant c_\gamma (c_0')^{2/\gamma} \sup_{B(x,r/2)} u \leqslant c_\gamma (c_0')^{2/\gamma} c_p \left(\fint_{B(x,r)} u^p m(\mathrm{d}x) \right)^{1/p}.$$

THEOREM 5.13. – *Let u be a solution of the problem*

$$a(u,v) = \int_{\mathcal{O}} fv m(\mathrm{d}x)\,, \qquad \forall v \in D_0[\mathcal{O}]\,, \quad u \in D_{\mathrm{loc}}[\mathcal{O}]\,, \tag{5.18}$$

where $\mathcal{O} \subset X_0$ is an open set and $f \in L^p(\mathcal{O}, m)$, $p \geqslant p_0$, $p_0 > \max\{\nu/2, 2\}$. Then u is locally Hölder continuous in $\mathcal{O}$ with respect to the intrinsic distance, with structural Hölder exponent and constant.

PROOF. – By standard methods [19], from Theorem 1.1 and Theorem 4.1 we obtain that if $B(x, 8k + 24)8r \subset\subset \mathcal{O}$

$$\operatorname{osc}(u, B(x,r)) \leqslant \sigma \operatorname{osc}(u, B(x,4r)) + cr^2 \|f\|_{L^p(Bx,4r),m)} \operatorname{mis}(B(x,r))^{-1/p}\,,$$

where $\sigma \in (0,1)$ and $c > 0$ are structural constants and $p > \nu/2 \vee 2$. We remark that $r^2 \operatorname{mis}(B(x,r))^{-1/p} \leqslant C$ where C in some constant depending on $\inf_{x \in X_0} m(B(x,1)) > 0$ (see (5.8)), we obtain

$$\operatorname{osc}(u, B(x,r)) \leqslant \sigma \operatorname{osc}(u, B(x,4r)) + cC \|f\|_{L^p(B(x,4r),m)} r^\delta\,,$$

where $\delta \in (0,1)$ depends on p; the Hölder continuity with respect to the intrinsic distance follows by standard iteration methods [19].

We now prove a Reverse-Hölder inequality for the Green function, which is not used in the following, but may be of some interest in itself.

LEMMA 5.14. – *Let $G^y_{\varrho, B(x,5r)}$ be the regularized Green function at y with respect to*

$B(x, 5r) \subset\subset X_0$. *There exists a structural constant* $c > 0$ *such that for every* $\rho < r$

$$\inf_{y \in B(x, 3r)} \left(\fint_{B(x, r)} (G^y_{\rho, B(x, 5r)})^\gamma m(dx) \right)^{1/\gamma} \geq cr^2 .$$

PROOF. – Let φ be the cut-off function of $B(x, 7/2\,r)$ with respect to $B(x, 2r)$. By denoting $G^y_\rho = G^y_{\rho, B(x, 5r)}$, we have

$$1 = \int_{B(x, 4r)} \mu(G^y_\rho, \varphi)(dx) \leq \frac{c}{r} m(B(x, r))^{1/2} \left(\int_{B(x, 4r) - B(x, 7/2r)} \mu(G^y_\rho, G^y_\rho)(dx) \right)^{1/2} .$$

By the same methods used in the proof of the Caccioppoli's inequality we obtain for $y \in B(x, 3r)$,

$$1 \leq c \frac{m(B(x, r))^{1/2}}{r^2} \left(\int_{B(r, 9/2r) - B(x, 13/4r), m)} (G^y_\rho)^2 m(dx) \right)^{1/2} \leq$$

$$\leq c \frac{m(B(x, r))}{r^2} \|G^y_\rho\|_{L^\infty(B(x, 9/27r) - B(x, 13/4r), m)} \leq c \frac{m(B(x, r))}{r^2} \left(\fint_{B(x, 5r)} (G^y_\rho)^\gamma m(dx) \right)^{1/\gamma} .$$

Here we apply Theorem 5.12 with $p = \gamma$, γ the exponent of Proposition 5.7 and Corollary 5.8. Moreover, we use the fact that the anulus $B(x, 9/2r) - B(x, 13/4r)$ can be covered by a finite fixed number of balls of radius $1/((8k + 24)4r)$ by Lemma 3.1, together with the duplication property of m.

By the duplication property of $(G^y_\rho)^\gamma$, Corollary 5.8, and by Theorem 5.12 with $p = 1$ we have from Lemma 5.14

$$(5.19) \qquad 1 \leq c \frac{m(B(x, r))}{r^2} \left(\fint_{B(x, r)} (G^y_\rho)^\gamma m(dx) \right)^{1/\gamma} \leq$$

$$\leq c \frac{m(B(x, r))}{r^2} \fint_{B(x, r)} G^y_\rho m(dx) \leq \frac{c}{r^2} \int_{B(x, r)} G^y_\rho m(dz) .$$

By taking Lemma 4.2 into account we then find

PROPOSITION 5.15. – *Let* $G^y_{\rho, B(x, 5r)}$ *be the regularized Green function at* y *with re-*

spect to the ball $B(x, 5r) \subset\subset X_0$. *Then for* $y \in B(x, 3r)$ *we have*

$$\left(\fint_{B(x,r)} (G^y_{\rho, B(x,5r)})^{p'} m(dz) \right)^{1/p'} \leq c \fint_{B(x,r)} G^y_{\rho, B(x,5r)} m(dz),$$

where c *is a structural constant and* $p' = p/(p-1)$, p *as in Theorem* 4.1.

6. – Estimates of the Green functions and capacities of balls.

We define the *Green function* $G^x_{\mathcal{O}}$ for the problem

$$\text{(6.1)} \qquad \begin{cases} a(u, v) = \int_{\mathcal{O}} fv\, m(dx), \\ u \in D_0[\mathcal{O}], \qquad \forall v \in D_0[\mathcal{O}], \end{cases}$$

where $\mathcal{O}$ is a given ball $B(x_0, R_0)$ with $B(x_0, 2R_0) \subset\subset X_0$ and $x \in \mathcal{O}$.

We recall that, in Section 4, we have defined the regularized Green functions $G^x_{\rho, \mathcal{O}}$, $\rho > 0$, $B(x, \rho) \subset \mathcal{O}$, as the solution of the problem

$$\text{(6.2)} \qquad \begin{cases} a(G^x_{\rho, \mathcal{O}}, v) = \fint_{B(x,\rho)} v\, m(dx), \\ F^x_{\rho, \mathcal{O}} \in D_0[\mathcal{O}], \qquad \forall v \in D_0[\mathcal{O}], \end{cases}$$

which exists and is unique for each $\rho > 0$, due to the Poincaré inequality (jj) of Assumption II.

By Lemma 4.2, the functions $G^x_\rho = G^x_{\rho, \mathcal{O}}$ are bounded in $L^{p'}(\mathcal{O}, m)$ uniformly in ρ, where $p' = p/(p-1)$, $p > \max\{\nu/2, 2\}$, ν being as in the inequality in Assumption II.

Therefore, possibly after extraction of a subsequence, we have that $G^x_{\rho, \mathcal{O}}$ converges weakly in $L^{p'}(\mathcal{O}, m)$ to some function $G^x = G^x_{\mathcal{O}} \in L^{p'}(\mathcal{O}, m)$, as $\rho \to 0$. In particular, we have

$$\text{(6.3)} \qquad \int_{\mathcal{O}} G^x_\rho f m(dy) \to \int_{\mathcal{O}} G^x f m(dy) \quad \text{as } \rho \to 0,$$

for every $f \in L^p(\mathcal{O}, m)$, $p > \max\{\nu/2, 2\}$.

By (6.1) and the definition of G^x_ρ, for every $f \in L^p(\mathcal{O}, m)$ we have

$$\text{(6.4)} \qquad \int_{\mathcal{O}} G^x_\rho f m(dy) = \fint_{B(x,\rho)} u m(dy),$$

where u is the unique solution of (6.1). By Theorem 5.13, for every $f \in L^p(\mathcal{O}, m)$ u is

Hölder continuous in $\mathcal{O}$, therefore by letting $\rho \to 0$ in (6.4) we find

$$\int_{\mathcal{O}} G^x fm(dy) = u(x), \tag{6.5}$$

for every $x \in \mathcal{O}$.

This implies, in particular, that (6.2) uniquely defines the function $G_{\mathcal{O}}^x \in L^{p'}(\mathcal{O}, m)$, which is then the weak limit in $L^{p'}(\mathcal{O}, m)$ of the whole family $G_{\rho,\mathcal{O}}^x$, as $\rho \to 0$. We say that $G_{\mathcal{O}}^x$ is the *Green function* in $\mathcal{O}$, with singularity at x, for the problem (6.1).

Clearly, $G_{\mathcal{O}}^x$ is the unique function $G^x \in L^{p'}(\mathcal{O}, m)$ for which the identity (6.5) holds, for every $f \in L^p(\mathcal{O}, m)$ and $u = u_f$ solution of (6.1).

We now remark that $G_{\rho,\mathcal{O}}^x$ for each $\rho > 0$ is a solution of the problem

$$\begin{cases} a(G_{\rho,\mathcal{O}}^x, v) = 0\,, \\ G_{\rho,\mathcal{O}}^x \in D_0[\mathcal{O}]\,, \qquad \forall v \in D_0[\mathcal{O} \setminus \overline{B(x,\rho)}]\,, \end{cases} \tag{6.6}$$

Therefore, by Theorem 5.13, for every $\rho_0 > 0$ the function $G_{\rho,\mathcal{O}}^x$ is Hölder continuous in $\mathcal{O} - \overline{B(x, \rho_0)}$, uniformly in $\rho \in (0, \rho_0)$. Thus

$$\lim_{\rho \to 0} G_{\rho,\mathcal{O}}^x = G_{\mathcal{O}}^x \quad \text{in } C_{\text{loc}}^{\alpha}(\mathcal{O} \setminus \{x\}) \tag{6.7}$$

for some $\alpha \in (0, 1)$. Moreover, if $G_{\rho,\mathcal{O}}^x$ is extended by 0 on $X_0 - \mathcal{O}$, $\phi G_{\rho,\mathcal{O}}^x \in D_0[a, \mathcal{O}]$ for every $\phi \in D_0[a, \mathcal{O}] \cap C_{\text{loc}}^{\alpha}(\mathcal{O} \setminus \{x\})$ and $\phi G_{\rho,\mathcal{O}}^x \to \phi G_{\mathcal{O}}^x$ in $D_0[a]$.

We now proceed to estimate the size of $G_{B(x,dr)}^x$ on $\partial B(x, r)$, by estimating first the size of the approximate $G_{\rho, B(x,dr)}^x$ and then passing to the limit as $\rho \to 0$.

We define the *capacity* of the ball $B(x, r)$ with respect to the ball $B(x, dr)$, $d > 1$, relative to the form a, by setting

$$\begin{aligned} &\operatorname{cap}(B(x, r), B(x, dr)) = \\ &\qquad = \min\{a(v, v) \colon v \in D_0[B(x, dr)],\ \ v \geq 1\ \ m\text{-a.e. on } B(x, r)\}\,, \end{aligned} \tag{6.8}$$

We observe that, by Sobolev-Poincaré's inequality (jj), the minimum is achieved and the unique minimizer $u \equiv u_{B(x,r)}$ is called the *equilibrium potential* of $B(x, r)$ with respect to $B(x, dr)$, relative to the form a. We also notice that, again by Sobolev-Poincaré inequality, the capacity (6.8) is equivalent to the analogue capacity defined according to Section 2(*b*) and to [18], where the form $a(v, v)$ is replaced by $a(v, v) + (v, v)$.

Since the cut-off function φ of $B(x, r)$ in $B(x, (1 + (d-1)/2)r)$ is an admissible test function in (6.8), by Proposition 3.3 we have

$$\operatorname{cap}(B(x, r), B(x, dr)) \leq \int_{B(x,dr)} \mu(\varphi, \varphi)(dy) \leq \frac{10}{(d-1)^2} \frac{m(B(x, r))}{r^2}\,. \tag{6.9}$$

We recall from [18] that there exists a positive Radon measure $\nu \equiv \nu_{B(x,r)}$, calaled the *equilibrium measure* of $B(x, r)$ in $B(x, \mathrm{d}r)$ relative to a, such that

$$a(u_{B(x,r)}, v) = \int_{B(x,\mathrm{d}r)} \tilde{v}(y)\,\nu_{B(x,r)}(\mathrm{d}y) \tag{6.10}$$

for every $v \in D_0[B(x, \mathrm{d}r)]$, where $\tilde{v}$ in the q.c. version of v, see [18] Lemma 3.3.1 and (3.2.2).

Moreover,

$$\operatorname{supp} \nu_{B(x,r)} \subset \partial B(x, r) \tag{6.11}$$

and

$$\operatorname{cap}(B(x, r), B(x, \mathrm{d}r)) = a(u_{B(x,r)}, u_{B(x,r)}) = \nu_{B(x,r)}(\partial B(x, r)), \tag{6.12}$$

see [18] Lemma 3.1.1, (iv).

Since $u_{B(x,r)} \equiv 1$ m-a.e. on $B(x, r)$, [17] Lemma 3.1.1 (ii), and for $\rho < r/2$ we have $G^x_{\rho, B(x,\mathrm{d}r)} \in C(B(x, \mathrm{d}r) - B(x, r/2)) \cap D_0[B(x, \mathrm{d}r)]$, then by (6.10), (6.11)

$$1 = a(u_{B(x,r)}, G^x_{\rho, B(x,\mathrm{d}r)}) = \int_{\partial B(x,r)} G^x_{\rho, B(x,\mathrm{d}r)}(y)\,\nu_{B(x,r)}(\mathrm{d}y),$$

therefore, by (6.12)

$$\inf_{\partial B(x,r)} G^x_{\rho, B(x,\mathrm{d}r)} \le \frac{1}{\operatorname{cap}(B(x, r), B(x, \mathrm{d}r))} \le \sup_{\partial B(x,r)} G^x_{\rho, B(x,\mathrm{d}r)}. \tag{6.13}$$

We now observe that, by Sobolev-Poincaré inequality, for $d \ge 2$ we have

$$\operatorname{cap}(B(x, r), B(x, \mathrm{d}r)) = a(u_{B(x,r)}, u_{B(x,r)}) \ge \frac{c_2}{d^2 r^2} \int_{B(x,r)} u^2_{B(x,r)} m(\mathrm{d}y) = \frac{c_2}{d_2} \frac{m(B(x, r))}{r^2},$$

which together with (6.9) shows that

$$\frac{(d-1)^2}{10} \frac{r^2}{m(B(x, r))} \le \frac{1}{\operatorname{cap}(B(x, r), B(x, \mathrm{d}r))} \le \frac{d^2}{c_2} \frac{r^2}{m(B(x, r))}. \tag{6.14}$$

Suppose $B(x, 4r) \subset\subset X_0$. Since $\partial B(x, r)$ can be covered by a finite number l of balls of radius $r/2\bar{\kappa}$, with l depending only on c_0, the following Harnack inequality on $\partial B(x, r)$

$$\sup_{\partial B(x,r)} G^x_{\rho, B(x,\mathrm{d}r)} \le c \frac{r^2}{m(B(x, r))} \left(\int_{B(x,r)} (G^y_{\rho, B(x,5r)})^\gamma m(\mathrm{d}x) \right)^{1/\gamma} \le \inf_{\partial B(x,r)} G^x_{\rho, B(x,\mathrm{d}r)} \tag{6.15}$$

is a consequence of Harnack's inequality on balls of Theorem 1.1, c being a structural constant, depending on d. In fact the second inequality follows from Lemma 5.14 and from the first part of the proof of Theorem 1.1; the first inequality follows by applying (6.13) and Harnack inequality to each ball of the covering of $\partial B(x, r)$.

From (6.13), (6.14) and (6.15), by letting $\rho \to 0$ and taking (6.7) into account, we obtain the

THEOREM 6.1. – *Let* $G^x_{B(x,\,dr)}$ *be the Green function of problem* (6.1), $\mathcal{O} = B(x, dr)$, *with singularity at* x, $d \geq 2$, $B(x, 4r) \subset\subset X_0$. *Then, there exist structural constants* $c > 0$ *such that the following estimates hold:*

$$\frac{1}{c}[\operatorname{cap}(B(x, r), B(x, dr))]^{-1} \leq G^x_{B(x,\,dr)}(y) \leq c[\operatorname{cap}(B(x, r), B(x, dr))]^{-1} \tag{6.16}$$

for every $y \in \partial B(x, r)$ *and*

$$\frac{(d-1)^2}{c}\,\frac{r^2}{m(B(x, r))} \leq [\operatorname{cap}(B(x, r), B(x, dr))]^{-1} \leq cd^2\,\frac{r^2}{m(B(x, r))}\,. \tag{6.17}$$

We now prove the global estimate of the size of the Green function as stated in Theorem 1.3 of the Introduction.

PROOF OF THEOREM 1.3. – We adapt the argument of Theorem 3.3 of [10]. Let $n \in N$ be such that $2^n r < R < 2^{n+1} r$. For every $j = 0, 1, \ldots, n$, let G^x_j be the Green function in $B(x, 2^j r)$, with singularity at x. By Theorem 6.1 we have

$$F^x_j(y) \simeq \frac{(2^{j-1} r)^2}{m(B(x, 2^{j-1} r))}\,, \qquad y \in \partial B(x, 2^{j-1} r)\,, \tag{6.18}$$

where by $\simeq$ we mean that the two quantities can be estimated one each other by some structural constant. We now introduce the function

$$u_j := G^x_j - G^x_{j-1} \quad \text{ in } B(x, 2^{j-1} r)\,,$$

which is a solution of

$$a(u_j, v) = 0\,,$$

$$u_j \in D_{\mathrm{loc}}[B(x, 2^{j-1} r)]\,, \qquad \forall v \in D_0[B(x, 2^{j-1} r)]\,.$$

Therefore, by the corollary to Theorem 1.1, u_j is Hölder continuous in $B(x, 2^{j-1} r)$. Moreover, $\phi(u_j - G^x_j) = -\phi G^x_{j-1} \in D_0[B(x, 2^{j-1} r)]$, for every ϕ in $D[a] \cap C_0(\mathcal{O} \setminus \{x\})$. Therefore by Section 2 (h),

$$\tilde{u}_j(y) - G^x_j(y) = -G^x_{j-1}(y) = 0 \quad \text{ q.e. } \; y \in \partial B(x, 2^{j-1} r)\,,$$

where $\tilde{u}$, denotes the quasi continuous version of u_j, hence by (6.18)

$$\tilde{u}_j \simeq \frac{(2^{j-1} r)^2}{m(B(x, 2^{j-1} r))} \quad \text{ q.e. on } \partial B(x, 2^{j-1} r)\,, \tag{6.19}$$

for every $j = 1, \ldots, n$.

By the maximum principles of Section 2(*i*), by (6.19) we have

$$u_j(y) \simeq \frac{(2^{j-1}r)^2}{m(B(x, 2^{j-1}r))} \qquad m\text{-a.e. in } B(x, 2^{j-1}r), \tag{6.20}$$

for $j = 1, \ldots, n$.

By a similar argument, if

$$u := G^x_{B(x,R)} - G^x_n \qquad \text{in } B(x, 2^n r),$$

we find

$$u(y) \simeq \frac{(2^n r)^2}{m(B(x, 2^n r))} m\text{-a.e. in } B(x, 2^n r). \tag{6.21}$$

Again by Section 2(*h*), $G^x_0 \equiv G^x_{B(x,r)} = 0$ q.e. on $\partial B(x, r)$, therefore

$$G^x_{B(x,R)}(y) = u(y) + \sum_{j=1}^{n} u_j(y), \qquad y \in \partial B(x, r),$$

hence, by (6.20), (6.21) and the duplication property of m

$$G^x_{B(x,R)}(y) \simeq \sum_{j=1}^{n} \frac{(2^j r)^2}{m(B(x, 2^j r))} \simeq \int_r^R \frac{s^2}{m(B(x, s))} \frac{ds}{s}.$$

REMARK 6.1. – By similar arguments as the beginning of this section, we find

$$\operatorname{cap}(B(x, r), B(x, R)) \simeq \left(\int_r^R \frac{s^2}{m(B(x, s))} \frac{ds}{s} \right)^{-1}$$

for $r \leqslant R/16$. Then the convergence of the integral at the right hand side as $r \downarrow 0$ is a necessary and sufficient condition for $\{x\}$ to have positive capacity.

7. – Energy's decay.

We first prove a *weighted Caccioppoli's inequality*.

PROPOSITION 7.1. – *Let v be a local solution in $B(x_0, 4r)$, i.e.*

$$a(v, w) = 0, \qquad \forall w \in D_0[B(x_0, 4r)],$$

$$v \in D_{\text{loc}}[B(x_0, 4r)].$$

We have

$$\int_{B(x_0,\, qr)} G^{x_0}\mu(v, v)(dx) + \frac{1}{2}\sup_{B(x_0,\, qr)} v^2 \leq c\frac{1}{m(B(x_0, r))}\int_{B(x_0,\, r) - \overline{B(x_0,\, qr)}} v^2 m(dx)$$

for every $q \in (0, q_0]$, for some $q_0 < 1$ and where G^{x_0} denotes the Green function with singularity at x_0 with respect to $B(x_0, 2r)$ and c is a structural constant depending on q_0.

PROOF. – By Theorem 5.4, v is bounded in $B(x_0, r)$. Let now $z \in B(x_0, r)$ with $B(z, sr) \subset B(z, tr) \subset B(x_0, r)$, $s < t < 1$, and denote by φ the cut-off function of $B(z, sr)$ with respect to $B(z, tr)$. We choose as test function $\varphi^2 v G_\rho^z$ where G_ρ^z denotes the regularized Green function relative to z and to the ball $B(z, 2r)$. Since φ, v, $G_\rho^z \in D_{\mathrm{loc}}[B(x_0, 2r)] \cap L^\infty(B(x_0, 2r), m)$, we have

$$0 = a(v, \varphi^2 v G_\rho^z) = \int_{B(z,\, tr)} \varphi^2 G_\rho^z \mu(v, v)(dx) +$$

$$+ \int_{B(z,\, tr)} 2\varphi v G_\rho^z \mu(v, \varphi)(dx) + \int_{B(z,\, tr)} v\varphi^2 \mu(v, G_\rho^z)(dx).$$

We observe that, by the estimate on G_ρ^z and the maximum principle, we have G_ρ^z, $(G_\rho^z)^{-1} \in L^\infty(B(z, tr) - B(z, sr), m)$, at least for ρ small enough. Therefore, by the Schwarz rule for every $\varepsilon > 0$ we have

$$\text{(7.1)} \quad \int_{B(z,\, tr)} \varphi^2 G_\rho^z \mu(v, v)(dx) + \frac{1}{2}\int_{B(z,\, tr)} \mu(v^2\varphi^2, G_\rho^z)(dx) \leq$$

$$\leq \frac{1}{4}\int_{B(z,\, tr)} \varphi^2 G_\rho^z \mu(v, v)(dx) + 8\int_{B(z,\, tr)} v^2 G_\rho^z \mu(\varphi, \varphi)\, m(dx) +$$

$$+ \frac{1}{4\varepsilon}\int_{B(z,\, tr) - B(z,\, sr)} v^2 G_\rho^z \mu(\varphi, \varphi)(dx) + \varepsilon\int_{B(z,\, tr) - B(z,\, sr)} \varphi^2 v^2 (G_\rho^z)^{-1}\mu(G_\rho^z, G_\rho^z)(dx).$$

Let us admit for the moment that the following result holds:

LEMMA 7.2. – *Let $f \in D_0[B(z, 2r), m]$ with $f = 0$ on $B(z, \rho)$. Then,*

$$\int_{B(z,\, 2r)} f^2 \mu(G_\rho^z, G_\rho^z)(dx) \leq 4\int_{B(z,\, 2r)} (G_\rho^z)^2 \mu(f, f)(dx).$$

We will now estimate the last term at the right hand side of (7.1). Let σ be the cut-off function of the annulus $B(z, tr) - B(z, sr)$ with respect to the balls $B(z, (s^2/t)r)$

and $B(z,(2t-s)r)$, Corollary 3.5. We apply Lemma 7.2 with $f=\sigma\varphi v$ and we find

$$\int_{B(z,\,tr)} \sigma^2\varphi^2 v^2 \mu(G_\rho^z, G_\rho^z)(dx) \le 4 \int_{B(z,\,tr)} (G_\rho^z)^2 \mu(\alpha\varphi v, \sigma\varphi v)(dx).$$

By the Schwarz rule and the property of σ, this inequality implies

$$\int_{B(z,\,tr)-B(z,\,sr)} \varphi^2 v^2 \mu(G_\rho^z, G_\rho^z)(dx) \le$$

$$\le \frac{40}{(s^2/t^2)(t-s)^2 r^2} \int_{B(z,\,tr)-B(z,\,s^*\tau)} v^2 (G_\rho^z)^2 m(dx) + 2 \int_{B(z,\,tr)-B(z,\,s^*\tau)} \varphi^2 (G_\rho^z)^2 \mu(v,v)(dx),$$

where $s^* = s^2/t$. If we now denote $\delta^* = (\sup G_\rho^z)/(\inf G_\rho^z)$, where the supremum and infimum are taken on $B(z,tr)-B(z,s^*r)$ and $\rho < s^*r$, then δ^* depends only on the ratio t/s, namely

$$0 < \delta^* \le c\left(\frac{t}{s}\right)^2 \frac{m(B(z,tr))}{m(B(z,sr))},$$

as it follows by the maximum principle satisfied by G_ρ^z in $B(z,tr)-\overline{B(z,s^*r)}$, by the estimates of Theorem 6.1 and the duplication property of m. Therefore from the previous inequality we also get

$$\text{(7.2)} \qquad \int_{B(z,\,tr)-B(z,\,s^*r)} \varphi^2 v^2 (G_\rho^z)^{-1} \mu(G_\rho^z, G_\rho^z)(dx) \le$$

$$\le \frac{40\delta^*}{(s^2/t^2)(t-s)^2 r^2} \int_{B(z,\,tr)-B(z,\,s^*r)} v^2 G_\rho^z m(dx) + 2\delta^* \int_{B(z,\,tr)} \varphi^2 G_\rho^z \mu(v,v)(dx).$$

By taking inequality (7.2) and the properties of φ into account, and by choosing ε such that $2\varepsilon\delta^* = 1/4$, we obtain from (7.1)

$$\int_{B(z,\,tr)} \varphi^2 G_\rho^z \mu(v,v)(dx) + \frac{1}{2} \int_{B(z,\,tr)} \mu(v^2\varphi^2, G_\rho^z)(dx) \le$$

$$\le \frac{c\delta}{(t-s)^2 r^2} \int_{B(z,\,tr)-B(z,\,s^*r)} v^2 G_\rho^z m(dx),$$

where δ is a constant which depends only on s/t. By the definition of G_ρ^z and since $\varphi \equiv 1$ on $B(z,sr)$:

$$\int_{B(z,\,tr)} \varphi^2 G_\rho^z \mu(v,v)(dx) + \frac{1}{2} \fint_{B(z,\,\rho)} v^2 m(dx) \le \frac{c\delta}{(t-s)^2 r^2} \int_{B(z,\,tr)-B(z,\,s^*r)} v^2 G_\rho^z m(dx).$$

Passing to the limit as $\rho \to 0+$ and taking into account that $G^z_\rho \to G^z$ uniformly in $B(z, tr) - B(z, \varepsilon r)$ for every fixed $\varepsilon > 0$, by the Lebesgue theorem in [7] we obtain, for m-a.e. z,

$$\int_{B(x_0, sr)} G^z \mu_x(v, v)(dx) + \frac{1}{2}\tilde{v}(z)^2 \leq \frac{c\delta}{(t-s)^2 r^2} \int_{B(z, tr) - B(z, s^* r)} v^2 G^z m(dx).$$

We take the supremum for $z \in B(x_0, qr)$, by choosing $q \in (0, 1/3)$, $s = [2q(1-q)]^{1/2}$, $t = 1 - q$. Then, $B(z, tr) - B(z, s^* r) \subset B(x_0, r) - B(x_0, qr)$ for every $z \in B(x_0, qr)$. Therefore

$$\int_{B(x_0, (q/2)r)} G^{x_0}\mu(v, v)(dx) + \sup_{B(x_0, qr)} v^2 \leq \frac{c\delta}{(t-s)^2 r^2} \sup_{z \in B(x_0, q/2r)} \int_{B(z, r) - B(z, qr)} v^2 G^z m(dx) \leq$$

$$\leq \frac{c\delta}{r^2}\left(\frac{r^2}{m(B(z, qr))}\right) \int_{B(x_0, r) - B(x_0, qr)} v^2 m(dx) \leq c_q \frac{1}{m(B(x_0, r))} \int_{B(x_0, r) - B(x_0, (qr))} v^2 m(dx),$$

where c_q is a structural constant depending also on q. We have applied again the maximum principle satisfied by G^z on $B(x_0, r) - \overline{B(x_0, qr)} \subset B(z, (1 + q/2) r) - \overline{B(z, (q/2) r)}$, and the estimates of Theorem 6.1.

To complete the proof of Proposition 7.1, we now prove Lemma 7.2. By the definition of G^z_ρ, since $f \equiv 0$ on $B(z, \rho)$, we have

$$\int_{B(z, 2r)} \mu(G^z_\rho, G^z_\rho f^2)(dx) = 0.$$

Therefore,

$$\int_{B(z, 2r)} f^2 \mu(G^z_\rho, G^z_\rho)(dx) + 2 \int_{B(z, 2r)} f G^z_\rho \mu(G^z_\rho, f)(dx) = 0,$$

hence

$$\int_{B(z, 2r)} f^2 \mu(G^z_\rho, G^z_\rho)(dx) \leq \frac{1}{2} \int_{B(z, 2r)} f^2 \mu(G^z_\rho, G^z_\rho)(dx) + 2 \int_{B(z, 2r)} (G^z_\rho)^2 \mu(f, f)(dx).$$

Thus

$$\int_{B(z, 2r)} f^2 \mu(G^z_\rho, G^z_\rho)(dx) \leq 4 \int_{B(z, 2r)} (G^z_\rho)^2 \mu(f, f)(dx).$$

REMARK 7.2. - We have also proved the second part of Theorem 1.4.

Now we prove the potential estimate which is the main goal of our paper.

THEOREM 7.3. – *Let u be a local solution in X_0 and $B(x_0, 4R_0) \subset X_0$. Let*

$$\psi(r) = \int_{B(x_0, r)} G^{x_0}_{B(x_0, 2q^{-1}r)} \mu(u, u)(dx) + \operatorname*{osc}_{B(x_0, r)} u^2 .$$

Then, we have

$$\psi(r) \leq c\left(\frac{r}{R}\right)^{\beta} \psi(R)$$

for every $r \leq qR \leq q^2 R_0$ and $q \in (0, q_0]$, $q_0 = \min\{1/6, \kappa^{-1}\}$, where $\beta \in (0, 1)$ and c are structural constants depending also on q.

PROOF. – Let us consider the test function $w = (u-k) G^z_\rho \varphi$, where G^z_ρ is the regularized Green function relative to z and to the ball $B(z, tr)$, φ is the capacitary potential of $B(z, sr)$ with respect to $B(z, tr)$ as defined in Section 6 and k is a constant that will be specified later. ($\rho < sr$, $z \in B(x_0, qr)$, $s < t < 1$, q to be fixed). We have

$$0 = \int_{B(z, tr)} \mu(u, (u-k)\varphi G^z_\rho)(dx) = \int_{B(z, tr)} \varphi G^z_\rho \mu(u, u)(dx) + \int_{B(z, tr)} \varphi(u-k)\mu(u, G^z_\rho)(dx) +$$

$$+ \int_{B(z, tr)} G^z_\rho (u-k)\mu(u, \varphi)(dx) =$$

$$= \int_{B(z, tr)} \varphi G^z_\rho \mu(u, u)(dx) + \frac{1}{2}\int_{B(z, tr)} \mu((u-k)^2\varphi, G^z_\rho)(dx) -$$

$$- \frac{1}{2}\int_{B(z, tr)} (u-k)^2 \mu(\varphi, G^z_\rho)(dx) + \int_{B(z, tr)} G^z_\rho (u-k)\mu(u, \varphi)(dx).$$

Then, for $\rho < sr$:

$$\int_{B(z, tr)} \varphi G^z_\rho \mu(u, u)(dx) + \frac{1}{2} \fint_{B(z, \rho)} (u-k)^2 m(dx) \leq$$

$$\leq \frac{1}{2}\int_{B(z, tr)} (u-k)^2 \mu(\varphi, G^z_\rho)(dx) - \int_{B(z, tr)} G^z_\rho (u-k)\mu(u, \varphi)(dx) \leq$$

$$\leq \frac{1}{2}\int_{B(z, tr)} \mu(\varphi, G^z_\rho (u-k)^2)(dx) - 2\int_{B(z, tr)} G^z_\rho (u-k)\mu(u, \varphi)(dx) =$$

$$= \frac{1}{2}\int_{B(z, tr)} G^z_\rho (\tilde{u}-k)^2 d\nu_{B(z, sr)} - 2\int_{B(z, tr)} G^z_\rho (u-k)\mu(u, \varphi)(dx) \leq$$

$$\leqslant \frac{1}{2} \sup_{B(z,\,tr)} (u-k)^2 \int_{B(z,\,tr)} G_\rho^z \, \mathrm{d}\nu_{B(z,\,sr)} - 2 \int_{B(z,\,tr)} G_\rho^z (u-k)\mu(u,\varphi)(\mathrm{d}x) =$$

$$= \frac{1}{2} \sup_{B(z,\,tr)} (u-k)^2 a(\varphi, G_\rho^z) - 2 \int_{B(z,\,tr)} G_\rho^z (u-k)\mu(u,\varphi)(\mathrm{d}x) \leqslant$$

$$\leqslant \frac{1}{2} \sup_{B(z,\,tr)} (u-k)^2 + \frac{1}{\eta} \int_{B(z,\,tr)-B(z,\,sr)} G_\rho^z \mu(u,u)(\mathrm{d}x) +$$

$$+\eta \int_{B(z,\,tr)-B(z,\,sr)} G_\rho^z (u-k)^2 \mu(\varphi,\varphi)(\mathrm{d}x) \leqslant$$

$$\leqslant \frac{1}{2} \sup_{B(z,\,tr)} (u-k)^2 + \frac{1}{\eta} \int_{B(z,\,tr)-B(z,\,sr)} G_\rho^z \mu(u,u)(\mathrm{d}x) +$$

$$+\eta \sup_{B(z,\,tr)} (u-k)^2 \sup_{B(z,\,tr)-B(z,\,sr)} G_\rho^z \cdot \operatorname{cap}(B(z,sr), B(z,tr)),$$

where $\nu_{B(z,\,sr)}$ is the equilibrium measure associated with the equilibrium potential φ and we have taken into account [18] Lemma 3.3.1, Theorem 3.2.2 and Theorem 3.1.5. In particular, $a(\varphi, G_\rho^z) = \fint_{B(z,\,\rho)} \varphi m(\mathrm{d}x) = 1$ because $\varphi = 1$ m-a.e. on $B(x, sr)$ and

$$a(\varphi, G_\rho^z(u-k)^2) = \int_{B(z,\,tr)} G_\rho^z (\tilde{u}-k)^2 \, \mathrm{d}\nu_{B(z,\,sr)} .$$

Then, by the maximum principle and Theorem 6.1, we have for arbitrary $\eta > 0$

$$\int_{B(z,\,tr)} \varphi G_\rho^z \mu(u,u)(\mathrm{d}x) + \frac{1}{2} \fint_{B(z,\,\rho)} (u-k)^2 m(\mathrm{d}x) \leqslant \tag{7.3}$$

$$\leqslant (1/2 + c\eta) \sup_{B(z,\,tr)} (u-k)^2 + \frac{1}{\eta} \int_{B(z,\,tr)-B(z,\,sr)} G_\rho^z \mu(u,u)(\mathrm{d}x),$$

where c, is a structural constant. Passing to the limit as $\rho \to 0$ we obtain by Lebesgue theorem in [7], for m-a.e. z

$$\int_{B(z,\,tr)} G_{B(x,\,tr)}^z \mu(u,u)(\mathrm{d}x) + \frac{1}{2}(u(z)-k)^2 \leqslant (1/2 + c\eta) \sup_{B(z,\,tr)} (u-k)^2 +$$

$$+ \frac{1}{\eta} \int_{B(z,\,tr)-B(z,\,sr)} G_{B(x,\,tr)}^z \mu(u,u)(\mathrm{d}x).$$

We now choose $t = 1 - q$, $s = 2q$, with $q \in (0, q_0]$, $q_0 = \min\{1/6, \kappa^{-1}\}$, κ being the constant occuring in Assusmption II (j). By taking the supremum for $z \in B(x_0, qr)$, if $\gamma := \; := 2c\eta$ we find, by Theorem 6.1,

$$\sup_{B(x_0, qr)} (u-k)^2 \leq (1+\gamma) \sup_{B(x_0, r)} (u-k)^2 +$$

$$+ \frac{2c}{\gamma} \sup_{B(x_0, qr)} \left(\frac{r^2}{m(B(z, 2qr))} \right) \int_{B(x_0, r) - B(x_0, qr)} \mu(u, u)(dx) \leq$$

$$\leq (1+\gamma) \sup_{B(x_0, r)} (u-k)^2 + \frac{c}{\gamma} \frac{r^2}{m(B(x_0, qr))} \int_{B(x_0, r) - B(x_0, qr)} \mu(u, u)(dx) \leq$$

$$\leq (1+\gamma) \sup_{B(x_0, r)} (u-k)^2 + \frac{C}{\gamma} \int_{B(x_0, r) - B(x_0, qr)} G^{x_0}_{B(x_0, 2r)} \mu(u, u)(dx),$$

where $\gamma > 0$ is arbitrary and C is a structural constant which depends also on q.

By Proposition 7.1, we have

$$\int_{B(x_0, qr)} G^{x_0}_{B(x_0, 2r)} \mu(u, u)(dx) \leq c \sup_{B(x_0, r)} (u-k)^2$$

for some structural constant c depending also on q. Then, by (7.3)

$$\int_{B(x_0, qr)} G^{x_0}_{B(x_0, 2r)} \mu(u, u)(dx) + \sup_{B(x_0, qr)} (u-k)^2 \leq$$

$$\leq (c+\gamma) \sup_{B(x_0, r)} (u-k)^2 + \frac{C}{\gamma} \int_{B(x_0, r) - B(x_0, qr)} G^{x_0}_{B(x_0, 2r)} \mu(u, u)(dx).$$

By «hole filling» after multiplication by γ, we obtain

$$(7.4) \qquad (C+\gamma) \int_{B(x_0, qr)} G^{x_0}_{B(x_0, 2r)} \mu(u, u)(dx) + \gamma \sup_{B(x_0, qr)} (u-k)^2 \leq$$

$$\leq \gamma(c+\gamma) \sup_{B(x_0, r)} (u-k)^2 + C \int_{B(x_0, r)} G^{x_0}_{B(x_0, 2r)} \mu(u, u)(dx).$$

We now study the last term at the right hand side of (7.4):

$$\int_{B(x_0,r)} G^{x_0}_{B(x_0,2r)}\mu(u,u)(dx) = \int_{B(x_0,r)} G^{x_0}_{B(x_0,2q^{-1}r)}\mu(u,u)(dx) -$$

$$- \int_{B(x_0,r)} (G^{x_0}_{B(x_0,2q^{-1}r)} - G^{x_0}_{B(x_0,r)})\mu(u,u)(dx) \leq$$

$$\leq \int_{B(x_0,r)} G^{x_0}_{B(x_0,2q^{-1}r)}\mu(u,u)(dx) - c\frac{r^2}{m(B(x_0,r))}\int_{B(x_0,r)} \mu(u,u)(dx),$$

where c is a structural constant depending also on q. Here we have taken into account that

$$F = G^{x_0}_{B(x_0,2q^{-1}r)} - G^{x_0}_{B(x_0,2r)}$$

is a solution of the problem

$$a(F,v) = 0, \qquad \forall v \in D_0[B(x_0,2r)],$$

therefore, by the maximum principle and Theorem 6.1

$$\inf_{B(x_0,r)} F \geq \inf_{\partial B(x_0,2r)} \tilde{F} = \inf_{\partial B(x_0,2r)} G^{x_0}_{B(x_0,2q^{-1}r)} \geq c\frac{r^2}{m(B(x,r))}.$$

Therefore, by Poincaré inequality (j), we also have for arbitrary $\bar{q} \in (0,1)$

$$\int_{B(x_0,r)} G^{x_0}_{B(x_0,2r)}\mu(u,u)(dx) \leq \int_{B(x_0,r)} G^{x_0}_{B(x_0,2q^{-1}r)}\mu(u,u)(dx) -$$

$$-c\frac{1}{m(B(x_0,r))}\int_{B(x_0,\kappa^{-1}\bar{q}r)} |u-\bar{u}|^2 m(dx),$$

where $\bar{u}$ denotes the average of u on $B(x_0,\kappa^{-1}\bar{q}r)$.

By choosing $\bar{q}$ such that $\kappa^{-1}\bar{q} = q$ and taking the doubling property of m into account, we then find

$$\int_{B(x_0,r)} G^{x_0}_{B(x_0,2r)}\mu(u,u)(dx) \leq \int_{B(x_0,r)} G^{x_0}_{B(x_0,2q^{-1}r)}\mu(u,u)(dx) - c'\sup_{B(x_0,qr)}|u-\bar{u}|^2, \tag{7.5}$$

with c' a structural constant depending also on q. Taking into account (7.5) and choosing $k = \bar{u}$ in (7.4), we obtain

$$(C+\gamma)\int_{B(x_0,qr)} G^{x_0}_{B(x_0,2r)}\mu(u,u)(dx) + (c'+\gamma)\sup_{B(x_0,qr)}|u-\bar{u}|^2 \leqslant$$

$$\leqslant \gamma(c+\gamma)\sup_{B(x_0,r)}|u-\bar{u}|^2 + C\int_{B(x_0,r)} G^{x_0}_{B(x_0,2q^{-1}r)}\mu(u,u)(dx)$$

We observe that

$$\sup_{B(x_0,qr)}|u-\bar{u}|^2 \geqslant \frac{1}{4}(\operatorname*{osc}_{B(x_0,qr)} u)^2\,,$$

$$\sup_{B(x_0,r)}|u-\bar{u}|^2 \leqslant \frac{1}{4}(\operatorname*{osc}_{B(x_0,r)} u)^2,$$

therefore

$$\int_{B(x_0,qr)} G^{x_0}_{B(x_0,2r)}\mu(u,u)(dx) + \frac{1}{4}\frac{c'+\gamma}{C+\gamma}(\operatorname*{osc}_{B(x_0,qr)} u)^2 \leqslant \tag{7.6}$$

$$\leqslant \gamma\frac{c+\gamma}{C+\gamma}(\operatorname*{osc}_{B(x_0,r)} u)^2 + \frac{C}{C+\gamma}\int_{B(x_0,r)} G^{x_0}_{B(x_0,2q^{-1}r)}\mu(u,u)(dx)\,.$$

We denote now

$$\Psi(r) = \int_{B(x_0,r)} G^{x_0}_{B(x_0,2q^{-1}r)}\mu(u,u)(dx) + \frac{1}{4}\frac{c'+\gamma}{C+\gamma}(\operatorname*{osc}_{B(x_0,r)} u)^2$$

and we choose $\gamma > 0$ such that $4\gamma((c+\gamma)/(c'+\gamma)) \leqslant C(C+\gamma) = \nu < 1$.

From (7.6), we have

$$\Psi(qr) \leqslant \nu\Psi(r)$$

then, see [23],

$$\Psi(r) \leqslant c\left(\frac{r}{R}\right)^{\beta}\Psi(R)\,,$$

where c and $\beta > 0$ are structural constants depending also on q, hence the result follows.

From Theorem 7.3 we prove easily the Saint-Venant principle in Theorem 1.4:

PROOF OF THEOREM 1.4. - Let $\bar{q} \in (0, 1/6)$. From Proposition 7.1, by choosing

$v = u - \bar{u}$ where $\bar{u}$ is the average of u on $B(x_0, \bar{u}^{-1}R)$, we have

$$(\operatorname*{osc}_{B(x_0, R)} u)^2 \leq \frac{c}{m(B(x_0, \bar{q}^{-1}R))} \int_{B(x_0, \bar{q}^{-1}R)} |u - \bar{u}|^2 m(\mathrm{d}x).$$

Then, by the Poincaré inequality, Theorem 6.1 and the maximum principle

$$(\operatorname*{osc}_{B(x_0, R)} u)^2 \leq \frac{cR^2}{m(B(x_0, q^{-1}R))} \int_{B(x_0, \bar{q}^{-1}\kappa R)} \mu(u, u)(\mathrm{d}x) \leq$$

$$\leq c \int_{B(x_0, \bar{q}^{-1}\kappa R)} G^{x_0}_{B(x_0, \bar{q}^{-1}\kappa R)} \mu(u, u)(\mathrm{d}x)$$

and the result follows from Theorem 7.3, by choosing $\bar{q}$ such that $\kappa^{-1}\bar{q} = q$, for a given $q \in (0, q_0]$, $q_0 = \min\{1/6, \kappa^{-1}\}$.

Concerning the nonhomogeneous case, we have the following result, that can be obtained by repeating the preceding proofs.

THEOREM 7.4. – *Let u be a solution of the problem*

$$a(u, v) = \int_{B(x_0, R_0)} fv m(\mathrm{d}x),$$

$$\forall v \in D_0[B(x_0, R_0)], \qquad u \in D_{\mathrm{loc}}[B(x_0, R_0)],$$

where $f \in L^p(B(x_0, R_0), m)$ and p is as in Section 3. Then

$$\psi(r) \leq c\left(\frac{r}{R}\right)^{\beta} \psi(R) + k\|f\|_{L^p(B(x_0, R_0), m)} R^{\gamma}$$

for every $q \in (0, q_0]$, for some $q_0 < 1$ depending on κ, and where ψ is as in Theorem 7.3, $r \leq qR \leq q^2 R_0$, $B(x_0, R_0) \subset X_0$, c and $\beta \in (0, 1)$ are structural constants depending also on $q_0 \in (0, 1/6\kappa^{-1})$, $\gamma \in (0, 1)$ depends on p and k depends on $\|u\|_{L^2(B(x_0, 2/3R_0), m)}$. Moreover

$$\int_{B(x_0, r)} G^{x_0}_{B(x_0, 2q^{-1}r)} \mu(u, u)(\mathrm{d}x) \leq$$

$$\leq c'\left(\frac{r}{R}\right)^{\beta}\left(\int_{B(x_0, R)} G^{x_0}_{B(x_0, 2q^{-1}R)} \mu(u, u)(\mathrm{d}x) + k'\|f\|_{L^p(B(x_0, R_0), m)} R^{\gamma}\right.$$

where $r \leq 2q^2 R \leq 2q^2 R_0$, c' is a structural constant which depends on q and k' is a structural constant which depends on q, $\|u\|_{L^2(B(x_0, 2/3R_0), m)}$ and $\inf_{x \in X_0} m(B(x, 1))$.

REFERENCES

[1] A. BEURLING - J. DENY, *Dirichlet spaces*, Proc. Nat. Acad. Sc. U.S.A., **45** (1959), pp. 208-215.

[2] M. BIROLI - U. MOSCO, *Formes de Dirichlet et estimations structurelles dans les milieux discontinus*, C. R. Acad. Sci. Paris A, **313** (1991), pp. 593-598.

[3] M. BIROLI - U. MOSCO, *Sobolev inequalities for Dirichlet forms on homogeneous spaces*, in *Bounary Value Problems for Partial Differential Equations and Applications, Essais for 70th Birthday of E. Magenes*, edited by C. Baiocchi e J. L. Lions, Masson, Paris (1993), pp. 305-311.

[4] M. BIROLI - U. MOSCO, *Wiener criterion and potential estimates for obstacle problems relative to degenerate elliptic operators*, Ann. Mat. Pura Appl., (IV), **159** (1991), pp. 255-281.

[5] J. M. BONY, *Opérateurs elliptiques dégénérés associés aux axiomatiques de la théorie du potentiel*, Corso C.I.M.E. Potential Theories, Ed. Cremonese, Roma (1970), pp. 69-120.

[6] N. BURGER, *Espace des fonctions à variation moyenne bornée sur un espace de nature homogène*, C.R. Acad. Sc. Paris A, **286** (1978), pp. 139-142.

[7] A. P. CALDERON, *Inequalities for the maximal function relative to a metric*, Studia Math., **57** (1975), pp. 297-306.

[8] R. R. COIFMAN - G. WEISS, *Analyse harmonique non commutative sur certains espaces homogènes*, Lecture Notes in Math., **242**, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[9] J. DENY, *Méthodes hilbertiennes et théorie du potentiel*, C.I.M.E., Ed. Cremonese, Roma (1970), pp. 121-203.

[10] E. FABES - D. JERISON - C. KENIG, *The Wiener test for degenerate elliptic equations*, Ann. Inst. Fourier, 3 (1982), pp. 151-183.

[11] E. FABES - C. KENIG - R. SERAPIONI, *The local regularity of solutions of degenerate elliptic equations*, Comm. Part. Diff. Eq., 7 (1982), pp. 77-116.

[12] C. L. FEFFERMAN - D. PHONG, *Subelliptic eigenvalue problems, Conference on Harmonic analysis, Chicago*, edited by W. Becker et al., Wadsworth (1981), pp. 590-606.

[13] C. L. FEFFERMAN - A. SANCHEZ CALLE, *Fundamental solution for second order subelliptic operators*, Ann. Math., **124** (1986), pp. 247-272.

[14] G. FICHERA, *Sulle equazioni differenziali lineari del 2° ordine ellittico-paraboliche*, Atti Acc. Naz. Lincei, Mem. Sez. I (8), **5** (1956), pp. 1-30.

[15] G. FICHERA, *Premessa ad una teoria generale dei problemi al contorno per le equazioni differenziali*, Corso Ist. Alta Matematica, Libreria Eredi V. Veschi, Roma (1958).

[16] B. FRANCHI - E. LANCONELLI, *An embedding theorem for Sobolev spaces related to non smooth vector fields and Harnack inequality*, Comm. Part. Diff. Eq., **9** (1984), pp. 1237-1264.

[17] B. FRANCHI - R. SERAPIONI, *Pointwise estimates for a class of degenerate elliptic operators: a geometrical approach*, Ann. Sc. Norm. sup. Pisa, **14**, 4 (1987), pp. 527-569.

[18] M. FUKUSHIMA, *Dirichlet Forms and Markov Processes*, North-Holland Math., **23**, North-Holland and Kodansha, Amsterdam (1980).

[19] D. GILBARG - N. S. TRUDINGER, *Elliptic partial differential equations of second order*, Grund. Math. Wiss, **234**, Springer-Verlag, Berlin-Heidelberg-New York (1977).

[20] D. JERISON, *The Poincaré inequality for vector fields satisfying an Hörmander's condition*, Duke Math. J., **53**, 2 (1986), pp. 503-523.

[21] D. JERISON - A. SANCHEZ CALLE, *Subelliptic second order differential operators*, Lecture Notes in Math., **1277**, Springer-Verlag, Berlin-Heidelberg-New York (1987), pp. 46-77.

[22] J. J. KOHN - L. NIRENBERG, *Degenerate elliptic equations of second order*, Comm. Pure Appl. Math., **20** (1967), pp. 797-872.

[23] S. KOSUOKA - D. STROOCK, *Applications of Malliavin calculus, III*, J. Fac. Sc. Univ. Tokyo, Sec. IA Math., **34** (1987), pp. 391-442.

[24] Y. LEJEAN, *Measure associées à une forme de Dirichlet: Applications*, Bull. Soc. Math. France, **106** (1978), pp. 61-112.

[25] U. MOSCO, *Wiener criterion and potential estimates for the obstacle problem*, Indiana Univ. Math. J., **3**, 36 (1987), pp. 455-494.

[26] U. MOSCO, *Composite media and asymptotic Dirichlet forms*, J. Func. An., **123**, 2 (1994), pp. 368-421.

[27] J. MOSER, *On Harnack's inequality for elliptic differential equations*, Comm. Pure Appl. Math., **14** (1961), pp. 377-591.

[28] M. K. V. MURTHY - G. STAMPACCHIA, *Boundary value problems for some degenerate elliptic operators*, Ann. Mat. Pura Appl., **80** (1968), pp. 1-122.

[29] A. NAGEL - E. STEIN - S. WEINGER, *Balls and metrics defined by vector fields I: Basic properties*, Acta Math., **155** (1985), pp. 103-147.

[30] O. A. OLEINIK - E. V. RADKEVIC, *Second Order Equations with Nonnegative Characteristic Form*, A.M.S., Plenum Press, New York, London (1979).

[31] L. P. ROTSCHILD - E. STEIN, *Hypoelliptic differential operators and nilpotent groups*, Acta Math., **137** (1976), pp. 247-320.

[32] L. SALOFF-COSTE, *A note on Poincaré, Sobolev and Harnack inequalities*, Duke Math. J., Int. Math. Res. Notices, **2** (1992), pp. 27-38.

[33] A. SANCHEZ CALLE, *Fundamental solutions and geometry of square of vector fields*, Inv. Math., **78** (1984), pp. 143-160.

[34] M. L. SILVERSTEIN, *Symmetric Markov Processes*, Lecture Notes in Math., **426**, Springer-Verlag, Berlin-Heidelberg-New York (1974).

[35] M. L. SILVERSTEIN, *Boundary Theory for Symmetric Markov Processes*, Lecture Notes in Math., **516**, Springer-Verlag, Berlin-Heidelberg-New York (1976).

[36] G. STAMPACCHIA, *Équations elliptiques du second ordre à coéfficients discontinus*, Les Presses de l'Université de Montréal, Montréal (1966).