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A SCALING LIMIT FOR QUEUES IN SERIES

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We derive a law of large numbers for a tagged particle in the one-dimensional totally asymmetric simple exclusion process under a scaling different from the usual Euler scaling. By interpreting the particles as the servers of a series of queues we use this result to verify an open conjecture about the scaling behavior of the departure times from a long series of queues.

1. Introduction. Consider a sequence of n servers each with unlimited waiting space for customers. At time zero, k customers are in queue at the first server, and all the other servers have empty queues. The service discipline is first-in first-out at each server, and service times are i.i.d. Customers move through the system in an orderly fashion, joining the queue at server $i+1$ as soon as service with server i is completed. Let $T(k,n)$ denote the time when the kth customer leaves the nth server. We are interested in the scaling behavior of $T(k,n)$ when both k and n become large.

Let $V(i,j)$ be the service time of the ith customer at server j. Then it is not hard to see by induction that

$$T(k,n) = \max_{\pi\in\Pi(k,n)} \sum_{(i,j)\in\pi} V(i,j), \tag{1.1}$$

where $\Pi(k,n)$ is the collection of upright paths from $(1,1)$ to (k,n) along points of the lattice $\mathbf{N}^2$. This representation is suited for an application of the subadditive ergodic theorem. The result is that for all $x, y > 0$, the deterministic limit

$$\gamma(x,y) = \lim_{n\to\infty} \frac{1}{n} T([nx],[ny]) \tag{1.2}$$

exists in probability. Only in the case where $V(i,j)$ are Exp(1)-distributed has $\gamma(x,y)$ been explicitly computed: $\gamma(x,y) = x+y+2\sqrt{xy}$. This is a consequence of a hydrodynamic limit for the totally asymmetric simple exclusion process, first proved by Rost (1981). Bounds on $\gamma(x,y)$ for more general distributions have been derived by Glynn and Whitt (1991).

A limit in a different scaling is also due to Glynn and Whitt (1991): assume that the common distribution of the $V(i,j)$ has mean and variance one and an exponentially decaying tail. Let $a \in (0,1)$ be a fixed parameter. Then there is a constant $\alpha > 0$, independent of the distribution and of the particular value

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of the parameter a, such that for all $x, y > 0$,

$$\lim_{n\to\infty} \frac{T([nx],[n^a y]) - nx}{n^{(1+a)/2}} = \alpha\sqrt{xy} \quad \text{in probability.} \tag{1.3}$$

Greenberg, Schlunk, and Whitt (1993) have conjectured that $\alpha = 2$, on the basis of computer simulations and a related limit theorem. Our first result is a verification of this conjecture.

THEOREM 1.1. *The constant* $\alpha = 2$.

To prove this theorem, we convert the limit (1.3) into a statement about the position of a tagged particle in the totally asymmetric simple exclusion process. The key to the proof is a variational equality that couples the exclusion process with infinitely many copies of the queuing system (Lemma 3.1). This variational formula can be regarded as a particle version of the well-known Lax–Oleinik formula that gives viscosity solutions of Hamilton–Jacobi equations [Bardi and Evans (1984), Lions (1982)]. The variational coupling leads to a general scaling theorem for an infinite series of queues, stated in Section 2. This is a hydrodynamic scaling limit, but the scaling is the one determined by (1.3) which is different from the usual Euler scaling of the hydrodynamic limits of asymmetric particle systems. In the end, the value $\alpha = 2$ is identified by applying the scaling theorem to an equilibrium system with i.i.d. geometric queues.

The representation of the queues in terms of the exclusion process is explained in Section 2. The variational coupling is described in Section 3. Section 4 contains two technical lemmas and Section 5 the proofs of the theorems.

It is fruitful to look at the phenomena from both the particle system and the queuing point of view. Examples of this are the papers by Kipnis (1986) and Ferrari and Fontes (1994). Srinivasan (1991, 1993) provides a systematic interpretation of the hydrodynamic limits of the exclusion and zero-range processes in the language of queues. These results are originally due to Rost (1981), Andjel and Kipnis (1984), Andjel and Vares (1987) and Benassi and Fouque (1987, 1988). Recent extensions of these results, especially to more than one space dimension, appear in Landim (1991a, b) and Rezakhanlou (1991), and a review can be found in Ferrari (1994). Let us also mention that, even though the limit (1.3) requires exponentially decaying service time tails, the constant $\alpha = 2$ is relevant for more general distributions (with mean and variance one) via a functional central limit theorem for $\{T(k, n)\}$. The reader should consult Theorems 3.1 and 7.1 in Glynn and Whitt (1991).

2. Representations in terms of interacting particle systems. To represent the queuing system by Markovian interacting particle processes, from now on we assume that the service times $V(i, j)$ are i.i.d. Exp(1)-distributed. We generalize the problem described in the Introduction and work with a doubly infinite sequence of servers, indexed by $j \in \mathbb{Z}$. The servers are placed on the sites of the integer lattice Z and the location of server j is denoted by $\sigma(j)$. This is done so that

$$\sigma(j-1)+1 \leq \sigma(j) \quad \text{for all } j; \tag{2.1}$$

in other words, there is at most one server per site. The number of empty sites between $\sigma(j-1)$ and $\sigma(j)$ equals the number of customers in the queue at server j:

$$\eta(j) = \sigma(j) - \sigma(j-1) - 1 \quad \text{for all } j, \tag{2.2}$$

where $\eta(j) \in \{0, 1, 2, \ldots\}$ denotes the length of the queue at server j. If $\eta(j) \geq 1$, completion of a service at server j entails that a customer jumps from the queue at j to the queue at $j+1$, and the equivalent transition is that server j jumps one step to the left. Thus, to capture the queuing dynamics, the rule of evolution for the $(\sigma(j))_{j\in \mathrm{Z}}$ is the following.

Each server j waits an Exp(1)-distributed random amount of time, independently of the other servers, and then attempts to jump one step to the left. If the next site to the left is occupied [equivalently, $\sigma(j-1) = \sigma(j) - 1$], the jump is suppressed and $\sigma(j)$ does not move. But if the site is vacant [equivalently, $\sigma(j-1) \leq \sigma(j) - 2$], the jump is executed and $\sigma(j)$ decreases by one. In either case, the server resumes waiting for another Exp(1)-distributed random duration.

Additionally, it will be convenient to allow for the possibility that, for some j_0, $\sigma(j) = -\infty$ for all $j \leq j_0$. This simply means that the servers $(\sigma(j))_{j\leq j_0}$ do not take any customers and in no way influence the evolution of $(\sigma(j))_{j>j_0}$. This convention does not contradict (2.1), and (2.2) is irrelevant for servers at $-\infty$. Notice that if $\sigma(j_0) = -\infty < \sigma(j_0+1)$, then at all times there will be infinitely many customers at server j_0+1. In particular, this convention allows us to represent the situation described in the introduction, with infinitely many customers initially at server 1 and no customers at servers $j \geq 2$.

Write $\sigma(t) = (\sigma(j,t))_{j\in \mathrm{Z}}$ for the stochastic process thus defined, where $t \geq 0$ is the time variable. If we set $X_k = 1$ if site $k \in \mathrm{Z}$ is occupied by a server and $X_k = 0$ otherwise, then $X(t) = (X_k(t))_{k\in \mathrm{Z}}$ is commonly known as the totally asymmetric simple exclusion process. The individual servers can be regarded as tagged particles in this exclusion process.

The result that generalizes Theorem 1.1 is a law of large numbers for a tagged particle in the exclusion process. Let $v_0(x)$ be a nondecreasing $[-\infty, +\infty)$-valued function on R. Let $\sigma^n = (\sigma^n(j))_{j\in \mathrm{Z}}$, $n = 1, 2, 3, \ldots$, be a sequence of (possibly) random initial server configurations, and write $\sigma^n(t) = (\sigma^n(j,t))_{j\in \mathrm{Z}}$ for the process started at σ^n. We make the following two assumptions:

For each $y \in \mathrm{R}$, the limit

$$\lim_{n\to\infty} n^{-(1+a)/2} \sigma^n([n^a y]) = v_0(y) \tag{2.3}$$

holds in probability.

(2.4) There are constants $\beta_0 > 0$ and $C_0 > 0$ such that

$$\lim_{n\to\infty} P\{\sigma^n(k) \le \beta_0 n^{(1-a)/2} k \text{ for all } k \le -C_0 n^a\} = 1.$$

THEOREM 2.1. *Under assumptions* (2.3) *and* (2.4), *the limit*

$$\lim_{n\to\infty} \frac{\sigma^n([n^a x], nt) + nt}{n^{(1+a)/2}} = v(x,t) \tag{2.5}$$

holds in probability for all $t > 0$ *and all* $x \in \mathbf{R}$, *where the function* $v(x,t)$ *is defined by*

$$v(x,t) = \sup_{y\le x}\Big\{v_0(y) + 2\sqrt{t(x-y)}\Big\}. \tag{2.6}$$

Some remarks are in order: assumptions (2.3) and (2.4) imply that $v_0(y) \le \beta_0 y$ for $y \le -C_0$, and consequently (2.6) defines a number $v(x,t) \in [-\infty, +\infty)$. The function $v(\cdot, t)$ is nondecreasing for each fixed t. Example 5.1 in Section 5 shows that Theorem 2.1 may fail without an assumption such as (2.4). Here are two examples that satisfy assumptions (2.3) and (2.4).

EXAMPLE 2.1. Suppose v_0 is continuous and satisfies

$$\limsup_{y\to-\infty} \frac{v_0(y)}{|y|} < 0. \tag{2.7}$$

Let $\{\sigma^n(0)\}$ be any sequence of random variables that satisfies (2.3) for $y = 0$, and independently of $\sigma^n(0)$, let $\{\eta^n(i) = \sigma^n(i) - \sigma^n(i-1) - 1\colon i \in \mathbf{Z}\}$ be independent geometrically distributed random variables with expectations

$$E[\eta^n(i)] = n^{(1+a)/2}[v_0(in^{-a}) - v_0((i-1)n^{-a})].$$

EXAMPLE 2.2. Without the continuity assumption, the independent geometric queues of Example 2.1 may fail assumption (2.3). However, as long as (2.7) holds and v_0 is left-continuous at discontinuities, the deterministic initial locations $\sigma^n(i) = [n^{(1+a)/2} v_0(in^{-a})] + i$, $i \in \mathbf{Z}$, satisfy (2.3) and (2.4). The $+i$ is added to satisfy the exclusion rule (2.1) and does not affect (2.3) or (2.4).

To get an interpretation of Theorem 2.1 for a series of queues, notice that the number of customers served by server $[n^a x]$ in the nth system during the time interval $[0, nt]$ equals the displacement $\sigma^n([n^a x]) - \sigma^n([n^a x], nt)$. Then, if the hypotheses of Theorem 2.1 hold and n is large, this server will have served

$$nt - n^{(1+a)/2}[v(x,t) - v_0(x)] + o(n^{(1+a)/2})$$

customers in time $[0, nt]$, with high probability. Since the service rate is 1, the term $n^{(1+a)/2}[v(x,t) - v_0(x)] + o(n^{(1+a)/2})$ represents the total idle time of server $[n^a x]$ up to time nt.

Let us specialize this to the situation described in the introduction, with infinitely many customers initially at server 1 and no customers elsewhere. This initial queue situation is represented by the initial particle configuration $\sigma^n(i) = -\infty$ for $i \le 0$ and $\sigma^n(i) = \ell + i - 1$ for $i \ge 1$, for any fixed site ℓ. The corresponding initial macroscopic profile that satisfies (2.3) is $v_0(x) = -\infty$ for $x \le 0$ and $v_0(x) = 0$ for $x > 0$. By (2.6), $v(x, t) = -\infty$ for $x \le 0$ and $v(x, t) = 2\sqrt{tx}$ for $x > 0$, for all $t > 0$. Then server $[n^a x]$, $x > 0$, will have served

$$nt - 2n^{(1+a)/2}\sqrt{tx} + o(n^{(1+a)/2})$$

customers in time $[0, nt]$, with high probability. Compare this with the result under Euler scaling: for $0 < x < t$, server $[nx]$ will have served

$$nt - 2n\sqrt{tx} + nx + o(n) = nt - 2n\sqrt{xt}\left(1 - \sqrt{xt^{-1}}/2\right) + o(n)$$

customers in time $[0, nt]$, with high probability. [This can be inferred from Rost (1981) or from Section 4.2 in Srinivasan (1993).] The following rough conclusion emerges: for large times t, a server with label of order t^a $(0 < a < 1)$ has idle time of order $t^{(1+a)/2}$, while a server with label of order t has idle time of order t.

Theorem 2.1 also implies that, macroscopically, the motion of a server is governed by a differential equation. Assume that v_0 is uniformly Lipschitz and $\inf v_0'(x) > 0$. Then by Theorem 2.1 in Bardi and Evans (1984), $v(x, t)$ defined by (2.6) is the unique viscosity solution of the Hamilton–Jacobi equation

$$v_t - (v_x)^{-1} = 0, \qquad v(x, 0) = v_0(x).$$

Several authors have treated laws of large numbers for tagged particles in the exclusion process under Euler scaling. Spitzer's (1970) paper contains remarks that amount to such a result [see Example 3.2 on page 281]. Kipnis (1986) and Saada (1987) prove results for equilibrium processes, but for exclusion processes more general than the totally asymmetric one. Rezakhanlou (1994) partially relaxes the equilibrium assumption. Completely general initial conditions are allowed in Seppäläinen (1996), but the results are valid only for the one-dimensional totally asymmetric process, just as in the present paper.

We conclude this section by stating a hydrodynamic limit for the zero-range process as a corollary of Theorem 2.1. The zero-range process provides an alternative natural way to represent the series of queues as an interacting particle system. The point of view is opposite to that used above: now servers are fixed in space and represented by the sites of the lattice Z, and customers are particles that jump from site to site. The process is $\eta(t) = (\eta(j, t))_{j \in Z}$ and is $(Z_+)^Z$-valued. The rule for the evolution is this: at each site j, after an Exp(1)-distributed random amount of time, if $\eta(j) \ge 1$, then $\eta(j)$ decreases by one and $\eta(j + 1)$ increases by one. After each jump or jump attempt the cycle is repeated, independently at each site and independently of the past.

To state a hydrodynamic scaling limit for this process, we follow the evolution of the empirical measure defined by

$$\mu^n(t, dx) = n^{-(1+a)/2} \sum_{j \in \mathbf{Z}} \eta(j, t) \delta_{jn^{-a}}(dx).$$

This is a random Radon measure on R, and the integral of a compactly supported test function ϕ is given by

$$\mu^n(t, \phi) = n^{-(1+a)/2} \sum_{j \in \mathbf{Z}} \phi(jn^{-a}) \eta(j, t).$$

COROLLARY 2.1. *Under assumptions* (2.3) *and* (2.4), *the measures* $\mu^n(nt, dx)$ *converge in probability to the Lebesgue–Stieltjes measure of the function* $v(x, t)$ *as* $n \to \infty$, *in the vague topology of Radon measures on* R. *Equivalently, the limit*

$$\lim_{n \to \infty} \mu^n(nt, \phi) = \int_{\mathbf{R}} \phi(x)\, dv(x, t)$$

holds in probability for any compactly supported, continuous test function ϕ *on* R.

This follows from Theorem 2.1 by noting that for an interval $(x, y]$,

$$\mu^n(nt, (x, y]) = \frac{\sigma^n([n^a y], nt) + nt}{n^{(1+a)/2}} - \frac{\sigma^n([n^a x], nt) + nt}{n^{(1+a)/2}}$$

which converges to $v(y, t) - v(x, t)$ in probability.

3. The variational formulation. The approach of this paper is based on a variational formula by which we couple the exclusion process $\sigma(t)$ with an infinite collection of queuing systems of the type described in the introduction. For this queuing system, where at time 0 infinitely many customers are in queue at server 1 and the queues for servers $2, 3, 4, \dots$ are empty, let $\xi(j, t)$ be the number of customers that have left server j by time t. In terms of the departure times $T(k, j)$,

$$\xi(j, t) = \min\{k \geq 0\colon T(k+1, j) > t\}. \tag{3.1}$$

Fix a site $\ell \in \mathbf{Z}$ for the moment. As explained in Section 2, this queuing system can be constructed in terms of a totally asymmetric simple exclusion process where initially the sites $\{\ell, \ell+1, \ell+2, \dots\}$ are all occupied, the sites $\{\dots, \ell-3, \ell-2, \ell-1\}$ are vacant, and particles jump to the left at exponential rate 1, respecting the exclusion rule. The particles are interpreted as servers jumping past customers, and are labeled by $j = 1, 2, 3, \dots$ from left to right. Write $\chi^\ell(j, t)$ for the position of server $j \in \mathbf{N}$ at time $t \geq 0$. We shall use the symbol χ^ℓ for the process in place of the σ of Section 2 when the process starts from the special initial configuration $\chi^\ell(j, 0) = \ell + j - 1$, $j \in \mathbf{N}$. Write

$\xi^\ell(j,t)$ for the number of customers that have left server j by time t in the χ^ℓ-system. We have the identity

$$\chi^\ell(j,t) = \ell + j - 1 - \xi^\ell(j,t). \tag{3.2}$$

Now assume given an initial configuration $(\sigma(i))_{i\in\mathbf{Z}}$ on $\{-\infty\}\cup\mathbf{Z}$ that satisfies the exclusion and ordering rule (2.1). The servers at $-\infty$ do not participate in the dynamics, so we simply ignore them in the following discussion. The process $\sigma(t) = (\sigma(i,t))_{i\in\mathbf{Z}}$ is constructed by the standard graphical representation [see Griffeath (1979) or Liggett (1985), especially the notes on page 176 of Liggett (1985) for historical references]. Pictorially, the construction takes place on $\mathbf{Z}\times[0,\infty)$. Put independent Poisson(1) processes on the time lines $\{m\}\times[0,\infty)$, $m\in\mathbf{Z}$. The points of the Poisson processes are event times. The rule of evolution is this: suppose (m,τ) is an event time and $\sigma(j,\tau-) = m$. If there is no server at site $m-1$ at time τ, server j jumps to site $m-1$: $\sigma(j,\tau) = m-1$. Otherwise the jump is suppressed and server j remains at site m: $\sigma(j,\tau) = m$. All event times are distinct with probability 1 so conflicts do not arise. From the queuing perspective, the event times represent potential service completions, and a server who encounters an event time jumps if there is a customer in his queue, whose service is then completed.

For each initial server location $\sigma(i) > -\infty$ construct a process $\chi^{\sigma(i)}(t)$, whose server particles start off from the locations $\chi^{\sigma(i)}(j,0) = \sigma(i)+j-1$, $j=1,2,3,\dots$. All these processes are coupled through a common realization of Poisson event times. However, they are invisible to each other in the sense that the exclusion interaction (2.1) is valid only between servers of the same process. In other words, each site contains at most one σ-server, and at most one $\chi^{\sigma(i)}$-server for each i. The key fact is described in the following lemma.

LEMMA 3.1. *For almost all realizations of the Poisson event times, the following holds for all $k\in\mathbf{Z}$ and $t\ge 0$:*

$$\sigma(k,t) = \sup_{i\le k} \chi^{\sigma(i)}(k-i+1,t). \tag{3.3}$$

PROOF. Fix $t_1<\infty$, arbitrarily large. With probability 1, we can pick sites $m_0 << 0 << m_1$ arbitrarily far away so that the intervals $\{m_0\}\times[0,t_1]$ and $\{m_1+1\}\times[0,t_1]$ contain no event times. Thus the evolution of the server particles in the block $\{m_0,\dots,m_1\}$ of sites is isolated from the evolution of the rest of the process, up to time t_1. Equation (3.3) holds at time $t=0$ [the supremum is attained at $i=k$ and at any other $i<k$ such that $\sigma(k)-\sigma(i) = k-i$]. Since a.s. there are only finitely many event times in the space–time block

$$\{m_0,\dots,m_1\}\times[0,t_1], \tag{3.4}$$

it suffices to prove inductively that (3.3) holds right after each event time.

Thus assume that (m,τ) is an event time in the space–time block (3.4), and that (3.3) holds for all servers $\sigma(k,t)$ in the block, with $0\le t<\tau$. If there is no σ-server at site m at time $\tau-$, the event time (m, τ) has no effect on (3.3). So let $\sigma(k, \tau-) = m$. Two cases need to be considered.

Suppose first that $\sigma(k)$ does not jump at time τ. Then it must be that $\sigma(k-1, \tau-) = m-1$. By the induction assumption (3.3) holds for $k-1$ at time $\tau-$, and the supremum is attained at some $i \leq k-1$. Then by the exclusion rule and the induction assumption again,

$$m = \chi^{\sigma(i)}(k-i, \tau-) + 1 \leq \chi^{\sigma(i)}(k-i+1, \tau-) \leq \sigma(k, \tau-) = m.$$

It follows that the supremum in (3.3) for $\sigma(k, \tau-)$ is attained at i, and that $\chi^{\sigma(i)}(k-i+1)$ cannot jump at time τ. Consequently (3.3) continues to hold for $\sigma(k, \tau)$, with supremum still attained at i.

Suppose then that $\sigma(k)$ does jump at time τ, which happens if $\sigma(k-1, \tau-) \leq m-2$. We wish to argue that $\chi^{\sigma(i)}(k-i+1)$ also jumps for any $i \leq k$ such that $\chi^{\sigma(i)}(k-i+1, \tau-) = m$. For $i = k$, this is clear because by construction, nothing obstructs the jumps of the first server $\chi^{\sigma(k)}(1)$. For $i \leq k-1$, the induction assumption gives

$$\chi^{\sigma(i)}(k-i, \tau-) \leq \sigma(k-1, \tau-) \leq m-2.$$

Thus no $\chi^{\sigma(i)}(k-i+1)$ located at m at time $\tau-$ is obstructed by its neighbor, so they all jump and equation (3.3) continues to hold at time τ. □

We shall use (3.3) in the form

$$\sigma(k, t) = \sup_{i \leq k}\{\sigma(i) + k - i - \xi^{\sigma(i)}(k-i+1, t)\}, \tag{3.5}$$

got by combining (3.2) and (3.3). Under scaling, this identity turns into (2.6); see (5.2). The interpretation of (3.5) is that the $\chi^{\sigma(i)}$-processes are defined with the graphical representation, and the random variables $\xi^{\sigma(i)}(j, t)$ are then defined via (3.2) in terms of the $\chi^{\sigma(i)}$-processes. For a fixed i, the process $(\xi^{\sigma(i)}(j, t)\colon j \in \mathbf{N},\ t \geq 0)$ is equal in distribution to the process $(\xi(j, t)\colon j \in \mathbf{N},\ t \geq 0)$ defined by (3.1), where the departure times $\{T(k, n)\}$ are defined by (1.1) in terms of the i.i.d. Exp(1) variables $\{V(i, j)\}$. Both definitions of ξ are needed in the sequel.

4. Properties of $\boldsymbol{\xi}$.

Throughout the proofs, we make repeated use of the fact that $0 < a < 1$, without alerting the reader. The first step is to convert the limit (1.3) into a statement about ξ.

LEMMA 4.1. *For all $t, y > 0$,*

$$\lim_{n\to\infty} \frac{\xi([n^a y], nt) - nt}{n^{(1+a)/2}} = -\alpha\sqrt{ty} \quad \textit{in probability.} \tag{4.1}$$

PROOF. While one can certainly conjecture this statement from (1.3), it does not follow without some explicit estimates. Fortunately, Glynn and Whitt (1991) have provided a strong approximation result adequate for our purposes.

Theorems 3.1, 4.1 and 7.1 of Glynn and Whitt (1991) give us the following picture: There is a probability space that supports the departure times $\{T(k, j)\}$, a collection of stochastic processes $\{\hat{D}_j(t): t \geq 0\}$, $j \in \mathrm{N}$, and a finite random variable Y such that

$$\max_{\substack{1 \leq j \leq n^a \\ 1 \leq k \leq n}} |T(k, j) - k - \sqrt{n}\hat{D}_j(k/n)| \leq Y n^a \log n \quad \text{a.s.} \tag{4.2}$$

holds for all n. Furthermore, the processes $\hat{D}_j(\cdot)$ satisfy

$$\{\hat{D}_j(nt): t \geq 0\} =_{\mathscr{D}} \{\sqrt{n}\hat{D}_j(t): t \geq 0\}, \tag{4.3}$$

and

$$\lim_{n\to\infty} n^{-1/2}\hat{D}_{[nx]}(1) = \alpha\sqrt{x} \quad \text{in probability, for all } x > 0. \tag{4.4}$$

That $\alpha > 0$ follows from the superadditivity of $\hat{D}_n(n)$ utilized in the proof of Theorem 7.1 in Glynn and Whitt (1991). We shall employ the following corollary of (4.2): for a fixed $\kappa \in \mathrm{N}$, there is a finite random variable Y such that

$$\left|T([nx], [n^a y]) - [nx] - \sqrt{\kappa n}\hat{D}_{[n^a y]}\big([nx]/\kappa n\big)\right| \leq Y n^a \log n \tag{4.5}$$

holds a.s. for all n, uniformly over $x \leq \kappa$ and $y \leq \kappa^a$.

Now fix $y, t > 0$, let $0 < \varepsilon < \alpha\sqrt{ty}$, and pick an integer $\kappa > \max\{t, y^{1/a}\}$. Let

$$s_n = t - n^{(a-1)/2}(\alpha\sqrt{yt} - \varepsilon)$$

and

$$X_n = [ns_n] + \sqrt{\kappa n}\hat{D}_{[n^a y]}\big([ns_n]/\kappa n\big) - Y n^a \log n,$$

a random quantity. By (4.5)

$$T([ns_n], [n^a y]) \geq X_n \quad \text{a.s.}$$

while (4.3) and (4.4), together with some straightforward but lengthy calculations, show that

$$\lim_{n\to\infty} P\{X_n > nt\} = 1.$$

This and (3.1) imply that, with probability tending to 1 as $n \to \infty$,

$$\xi([n^a y], nt) \leq [ns_n] \leq nt - n^{(1+a)/2}(\alpha\sqrt{yt} - \varepsilon),$$

which in turn implies that, with probability tending to 1 as $n \to \infty$,

$$\frac{\xi([n^a y], nt) - nt}{n^{(1+a)/2}} \leq -\alpha\sqrt{ty} + \varepsilon.$$

Since $\varepsilon > 0$ can be made arbitrarily small, this is half the proof of the lemma. The argument for the other half proceeds along exactly the same lines and we leave the details to the reader. □

The second fact we need is an estimate on the size of ξ.

LEMMA 4.2. *Let $\sigma^n = (\sigma^n(i))_{i\in Z}$ be any sequence of initial server locations. For any $x \in R$ and $r_0, r_1 > 0$, the following holds almost surely: for large enough n,*

$$\xi^{\sigma^n(i)}([n^a x] - i, nt) \geq \big(1 - n^{(a-1)/2}\big)^2\big(\sqrt{nt} - \sqrt{[n^a x] - i}\,\big)^2 \tag{4.6}$$

for all i such that $n^a x - r_1 n^{(1+a)/2} \leq i \leq (x - r_0)n^a$.

PROOF. For simpler notation, let $j = [n^a x] - i$ and

$$J(n) = \{j \in N\colon r_0 n^a \leq j \leq r_1 n^{(1+a)/2}\}$$

be the range of j. Let $k = k(n, j)$ be the smallest integer that satisfies

$$k \geq \big(1 - n^{(a-1)/2}\big)^2\big(\sqrt{nt} - \sqrt{j}\big)^2.$$

Since the distribution of $\xi^{\sigma^n(i)}(j, nt)$ is independent of the base point $\sigma^n(i)$, the lemma follows from proving

$$\sum_n \sum_{j\in J(n)} P\{T(k, j) > nt\} < \infty. \tag{4.7}$$

We shall use a coupling argument but describe the coupling only informally. Pick a parameter $\nu \in (0, 1)$ and consider the equilibrium queuing system where customers arrive in a Poisson($1 - \nu$) stream, the queues are i.i.d. Geom(ν)-distributed, and the waiting times of an individual customer are i.i.d. Exp(ν)-distributed. [See Section 2.1 and 2.2 in the monograph of Kelly (1979) for more details on this.] In the equilibrium system the time when customer k leaves server j can be represented as $S_k^{1-\nu} + S_j^{\nu}$. Here $S_k^{1-\nu}$ is a sum of k i.i.d. Exp($1-\nu$) variables and represents the time when customer k enters the system, and S_j^{ν} is a sum of j i.i.d. Exp(ν) variables and represents the waiting time of customer k in the j first queues. Compared to our original system, the customers are slowed down in the equilibrium system. Thus $T(k, j)$ is stochastically dominated by $S_k^{1-\nu} + S_j^{\nu}$. This is proved rigorously by coupling the two systems through common exponential clocks that signal completions of services. By an elementary scaling property of exponential distributions, we write

$$\begin{aligned} P\{T(k, j) > nt\} &\leq P\{S_k^{1-\nu} + S_j^{\nu} > nt\} \\ &\leq P\{S_k^{1-\nu} > (1 - \nu)nt\} + P\{S_j^{\nu} > \nu nt\} \\ &= P\{S_k^1 > (1 - \nu)^2 nt\} + P\{S_j^1 > \nu^2 nt\}, \end{aligned} \tag{4.8}$$

where S_m^1 stands for the sum of m i.i.d. Exp(1) variables.

These probabilities will be treated with a simple large deviation bound. For $\alpha, \beta > 0$, Chebyshev's inequality yields

$$P\{S_m^1 \geq m + \beta\} \leq \exp(-\alpha(m + \beta))E\big[\exp(\alpha S_m^1)\big]$$

from which optimizing over α gives, for $\beta \le m$,

$$P\{S_m^1 \ge m + \beta\} \le \exp(-\beta^2/6m). \tag{4.9}$$

Pick and fix γ so that

$$\frac{1}{2} < \gamma < \frac{1}{2}\left(\frac{1+3a}{1+a}\right), \tag{4.10}$$

and set

$$\nu = \nu(j, n) = \left(\frac{j + j^\gamma}{nt}\right)^{1/2}. \tag{4.11}$$

Fix a number $\theta \in (0, 1)$. For large enough n,

$$k = k(n, j) \le (1 + \theta n^{(a-1)/2})\big(1 - n^{(a-1)/2}\big)^2\big(\sqrt{nt} - \sqrt{j}\,\big)^2$$

holds for all $j \in J(n)$. To bound the first probability on the last line of (4.8), first check that, by the second inequality in (4.10) and the bound $j \le r_1 n^{(1+a)/2}$, the inequality

$$(1 - \nu)^2 \ge (1 + \theta n^{(a-1)/2})\big(1 - n^{(a-1)/2}\big)\big(1 - \sqrt{j/nt}\,\big)^2$$

holds for large enough n (no square in the middle factor on the right). From this conclude that

$$\begin{aligned} &(1-\nu)^2 nt - k \\ &\quad \ge k\big[(1-\nu)^2(1+\theta n^{(a-1)/2})^{-1}\big(1 - n^{(a-1)/2}\big)^{-2}\big(1 - \sqrt{j/nt}\,\big)^{-2} - 1\big] \\ &\quad \ge k\big[\big(1 - n^{(a-1)/2}\big)^{-1} - 1\big] \\ &\quad \ge n^{(a-1)/2}k/2, \end{aligned} \tag{4.12}$$

also for large n.

Applying (4.9) and (4.12) leads to

$$\begin{aligned} P\{S_k^1 > (1-\nu)^2 nt\} &\le P\{S_k^1 > k + n^{(a-1)/2}k/2\} \\ &\le \exp[-kn^{a-1}/24], \end{aligned}$$

from which, again for large n,

$$\begin{aligned} \sum_{j \in J(n)} P\{S_k^1 > (1-\nu)^2 nt\} &\le \sum_{k \ge nt/2} \exp[-kn^{a-1}/24] \\ &\le c_0 n^{1-a} \exp[-c_1 n^a]. \end{aligned} \tag{4.13}$$

Above, c_0 and c_1 are constants.

For the second probability on the last line of (4.8),

$$\begin{aligned}\sum_{j\in J(n)} P\{S_j^1 > \nu^2 nt\} &\le \sum_{j\ge r_0 n^a} P\{S_j^1 > j + j^\gamma\}\\ &\le \sum_{j\ge r_0 n^a} \exp[-j^{2\gamma-1}/6]\\ &\le c_2 \exp[-c_3 n^{a(2\gamma-1)}].\end{aligned} \tag{4.14}$$

This is a summable sequence over n, by virtue of (4.10). Combining (4.8), (4.13), and (4.14) establishes (4.7) and proves the lemma. □

5. Proof of Theorems 1 and 2. We begin by proving an intermediate version of Theorem 2.1, before we identify the value of α.

Lemma 5.1. *Under assumptions* (2.3) *and* (2.4), *the limit*

$$\lim_{n\to\infty} \frac{\sigma^n([n^a x], nt) + nt}{n^{(1+a)/2}} = v(x,t) \equiv \sup_{y\le x}\Big\{v_0(y) + \alpha\sqrt{t(x-y)}\Big\} \tag{5.1}$$

holds in probability for all $t > 0$ *and all* $x \in$ R.

The proof of Lemma 5.1 is essentially this: rewrite (3.5) in the form

$$\begin{aligned}&\frac{\sigma^n([n^a x], nt) + nt}{n^{(1+a)/2}}\\ &\qquad= \sup_{i:\, i\le n^a x}\left\{\frac{\sigma^n(i)}{n^{(1+a)/2}} + \frac{[n^a x] - i}{n^{(1+a)/2}} - \frac{\xi^{\sigma^n(i)}([n^a x] - i, nt) - nt}{n^{(1+a)/2}}\right\},\end{aligned} \tag{5.2}$$

take $i = [n^a y]$, let $n \to \infty$, and utilize assumption (2.3) and Lemma 4.1. The technical fact needed for carrying out this approach is that, by virtue of assumption (2.4), we can suitably restrict the range of i in (5.2) without affecting the supremum, with high probability. This restriction is done in two steps, in Lemmas 5.2 and 5.3.

Notice from (2.6) that for all $x \in$ R and $t > 0$, $v(x,t) > -\infty$ if and only if $v_0(x) > -\infty$. The case $v(x,t) = -\infty$ will be dealt with separately after the finite case.

Lemma 5.2. *Fix* $x \in$ R *and* $t > 0$, *and assume* $v_0(x) > -\infty$. *Under assumptions* (2.3) *and* (2.4), *there exists a number* $r_1 > 0$ *such that the following holds with probability tending to 1 as* $n \to \infty$:

$$\begin{aligned}\sigma^n([n^a x], nt) = \max\big\{&\sigma^n(i) + [n^a x] - i - \xi^{\sigma^n(i)}([n^a x] - i, nt):\\ &n^a x - r_1 n^{(1+a)/2} < i \le n^a x\big\}.\end{aligned} \tag{5.3}$$

Proof. Let $r_1 > t/\beta_0$ where β_0 is the constant appearing in assumption (2.4), and then pick r_2 and r_3 so that

$$\beta_0 r_1 > r_2 > r_3 > t. \tag{5.4}$$

By assumptions (2.3) and $v_0(x) > -\infty$,

$$\lim_{n\to\infty} P\{\sigma^n([n^a x]) \le -(r_2 - r_3)n\} = 0. \tag{5.5}$$

Since $\xi^{\sigma^n([n^a x])}(1, nt)$ is Poisson(nt) distributed,

$$\lim_{n\to\infty} P\{\xi^{\sigma^n([n^a x])}(1, nt) > r_3 n\} = 0. \tag{5.6}$$

Equations (5.5) and (5.6) imply, via (3.5), that

$$\lim_{n\to\infty} P\{\sigma^n([n^a x], nt) > -r_2 n\} = 1. \tag{5.7}$$

Set $i_0^n = [n^a x - r_1 n^{(1+a)/2} + 1]$. Assumption (2.4) implies, via (5.4), that

$$\lim_{n\to\infty} P\{\sigma^n(i_0^n) + [n^a x] - i_0^n < -r_2 n\} = 1. \tag{5.8}$$

Combining (5.7) and (5.8) establishes the lemma, because for $i \le i_0^n$,

$$\begin{aligned}\sigma^n(i) + [n^a x] - i - \xi^{\sigma^n(i)}([n^a x] - i, nt) &\le \sigma^n(i) + [n^a x] - i \\ &\le \sigma^n(i_0^n) + [n^a x] - i_0^n. \qquad \square\end{aligned}$$

LEMMA 5.3. *For any fixed $x \in \mathrm{R}$ and $t > 0$, assumptions $v_0(x) > -\infty$,* (2.3) *and* (2.4) *imply this: There exists a number $s < 0$ such that, with probability tending to 1 as $n \to \infty$,*

$$\begin{aligned}\sigma^n([n^a x], nt) = \max\{\sigma^n(i) + [n^a x] - i - \xi^{\sigma^n(i)}([n^a x] - i, nt)&: \\ n^a s < i \le n^a x\}&.\end{aligned} \tag{5.9}$$

PROOF. Let $K^n(t)$ count the jump attempts of server $\sigma^n([n^a x])$ up to time t, both those actually executed and those suppressed by the exclusion rule (2.1). Then trivially $\sigma^n([n^a x], nt) \ge \sigma^n([n^a x]) - K^n(nt)$. By assumptions (2.3) and $v_0(x) > -\infty$, and since $K^n(nt)$ is Poisson(nt) distributed, there is a constant $c_0 > -\infty$ such that

$$\lim_{n\to\infty} P\{\sigma^n([n^a x], nt) > c_0 n^{(1+a)/2} - nt\} = 1. \tag{5.10}$$

Next we claim that for some $s < 0$, the inequality

$$\beta_0 i n^{(1-a)/2} - \left(1 - n^{(a-1)/2}\right)^2 \left(\sqrt{nt} - \sqrt{[n^a x] - i}\,\right)^2 < (c_0 - r_1) n^{(1+a)/2} - nt \tag{5.11}$$

holds for all $i \le n^a s$, where r_1 is the constant appearing in (5.3). To justify (5.11), let $j = [n^a x] - i$ and divide (5.11) by nt to get

$$c_1 n^{(a-1)/2} + 1 < \beta_0 t^{-1} n^{-(1+a)/2} j + \left(1 - n^{(a-1)/2}\right)^2 \left(1 - \sqrt{j/nt}\,\right)^2 \tag{5.12}$$

with $c_1 = t^{-1}(x\beta_0 - c_0 + r_1)$. Next use

$$\left(1 - n^{(a-1)/2}\right)^2 \left(1 - \sqrt{j/nt}\,\right)^2 \ge 1 - 2n^{(a-1)/2} - 2\sqrt{j/nt}$$

to argue that (5.12) follows from

$$(c_1 + 2) n^{(a-1)/2} < \sqrt{j/n}\left(\beta_0 t^{-1} \sqrt{j n^{-a}} - 2/\sqrt{t}\,\right).$$

This last inequality is true for $j \geq r_0 n^a$ for a suitable $r_0 > 0$, so (5.11) holds for $i \leq n^a s$ for any choice of s such that $s < x - r_0$. Pick and fix s to satisfy this and the inequality $s < -C_0$ where C_0 is the constant appearing in assumption (2.4).

The intersection of the events in (2.4), (4.6), (5.3) and (5.10) occurs with probability tending to 1 as n increases. The deterministic inequality (5.11) implies that the event in (5.9) contains this intersection. This proves the lemma. □

PROOF OF LEMMA 5.1. Let us abbreviate

$$Z_n = \frac{\sigma^n([n^a x], nt) + nt}{n^{(1+a)/2}}. \tag{5.13}$$

(i) *The case* $v(x,t) > -\infty$ *for all* x. Equivalently, we are assuming that $v_0(x) > -\infty$ for all x. Given $\varepsilon > 0$, pick $y \leq x$ so that $v_0(y) + 2\sqrt{t(x-y)} > v(x,t) - \varepsilon$. Then taking $i = [n^a y]$ on the right-hand side of (5.2) shows, by assumption (2.3) and Lemma 4.1, that

$$\lim_{n\to\infty} P\big\{Z_n \geq v(x,t) - \varepsilon\big\} = 1. \tag{5.14}$$

For the converse, pick a partition

$$s = r_0 < r_1 < \cdots < r_m = x$$

with mesh $\delta = \max\{r_{\ell+1} - r_\ell\}$, where s is the number given by Lemma 5.3. Since each i appearing on the right-hand side of (5.9) lies in one of the partition intervals $\{[n^a r_\ell], \dots, [n^a r_{\ell+1}]\}$, (5.9) implies that

$$\begin{aligned} Z_n \leq \max_{0\leq \ell\leq m-1}&\left\{\frac{\sigma^n([n^a r_{\ell+1}])}{n^{(1+a)/2}} - \frac{\xi^{\sigma^n([n^a r_{\ell+1}])}\big([n^a x] - [n^a r_\ell], nt\big) - nt}{n^{(1+a)/2}}\right\} \\ &+ O(n^{(a-1)/2}). \end{aligned} \tag{5.15}$$

The random variable on the right-hand side of (5.15) converges in probability, again by (2.3) and Lemma 4.1, to the quantity

$$\max_{0\leq\ell\leq m-1}\Big\{v_0(r_{\ell+1}) + 2\sqrt{t(x-r_\ell)}\Big\}.$$

This in turn is bounded above by

$$v(x,t) + \omega(\delta),$$

where $\omega(\delta)$ is the modulus of continuity of the function $y \mapsto 2\sqrt{t(x-y)}$ on $s \leq y \leq x$, and satisfies $\lim_{\delta\to 0}\omega(\delta) = 0$. These facts together show that

$$\lim_{n\to\infty} P\big\{Z_n \leq v(x,t) + \varepsilon\big\} = 1. \tag{5.16}$$

This completes the proof of Lemma 5.1 for the case $v_0 > -\infty$.

(ii) *The general case*. Now suppose $v_0(x)$ is a $[-\infty, +\infty)$-valued nondecreasing function, not identically equal to $-\infty$. By (i), we may assume that for some

$x_0 > -\infty$, $v_0(x) = -\infty$ for $x < x_0$. Let $\{\sigma^n\} \subset (\{-\infty\} \cup \mathbb{Z})^{\mathbb{Z}}$ be a sequence of random initial configurations that satisfy (2.3) and (2.4). Equation (5.14) is still valid, so we only have to worry about deducing (5.16). For this we use a sequence of approximating functions $u_0 = u_0^B$, indexed by a parameter $B << 0$ that tends to $-\infty$. We set

$$u_0(x) = \begin{cases} x + B, & \text{if } y + B \geq v_0(y) \text{ for all } y \leq x, \\ v_0(x), & \text{otherwise.} \end{cases}$$

Let

$$b = \sup\{x\colon\ y + B \geq v_0(y) \text{ for all } y \leq x\}$$

and consider values of B small enough so that b is finite. Notice that $y + B$ is never below $v_0(y)$ on the entire real line because $v_0(y) = -\infty$ for small enough y. One can see that $u_0(x) = x + B$ for $x < b$, $u_0(b) = \max\{b + B, v_0(b)\}$, and $u_0(x) = v_0(x)$ for $x > b$. For a fixed B, define initial configurations $\zeta^n = \zeta^{n,B}$ by

$$\zeta^n(i) = \begin{cases} \sigma^n(i), & \text{for } i > [n^a b], \\ \max\{\sigma^n(i), [n^{(1+a)/2}(in^{-a} + B)]\}, & \text{for } i \leq [n^a b]. \end{cases} \tag{5.17}$$

Assumptions (2.3) and (2.4) hold again with σ^n, v_0 and β_0 replaced by ζ^n, u_0 and $\widetilde{\beta}_0 = \min\{\beta_0, 1\}$, respectively. Consequently the part proved above gives

$$\lim_{n\to\infty} \frac{\zeta^n([n^a x], nt) + nt}{n^{(1+a)/2}} = u(x, t) \equiv \sup_{y \leq x}\Big\{u_0(y) + \alpha\sqrt{t(x-y)}\Big\}. \tag{5.18}$$

We can couple $\sigma^n(\cdot)$ and $\zeta^n(\cdot)$ through a common graphical representation, and then the inequality

$$\sigma^n(i, t) \leq \zeta^n(i, t) \quad \text{for all } i \in \mathbb{Z}, \qquad t > 0, \tag{5.19}$$

which holds at time $t = 0$ by construction (5.17), continues to hold at all successive times. Thus we get, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\big\{Z_n \leq u(x, t) + \varepsilon\big\} = 1.$$

It remains to observe that $u(x, t) \searrow v(x, t)$ as $B \searrow -\infty$. This completes the proof of Lemma 5.1. □

For the proof of Theorem 1 we need these large deviation bounds on geometric random variables:

LEMMA 5.4. *Suppose $S_n = X_1 + \cdots + X_n$ is a sum of n i.i.d. Geom(p) variables, in other words, $P\{X_i = k\} = pq^k$ for $k = 0, 1, 2, \ldots$, with $q = 1 - p$. Then for $0 < \varepsilon < 1$,*

$$P\{S_n \leq (1 - \varepsilon)nq/p\} \leq \exp(-nq\varepsilon^2/2)$$

and

$$P\{S_n \geq (1 + \varepsilon)nq/p\} \leq \exp(-nq\varepsilon^2/6).$$

PROOF. For the first probability, Chebyshev's inequality gives for $\alpha < 0$,

$$P\{S_n \le (1-\varepsilon)nq/p\} = P\{\exp(\alpha S_n) \ge \exp(\alpha(1-\varepsilon)nq/p)\}$$
$$\le \exp\{-n[\alpha(1-\varepsilon)q/p - \log p/(1-qe^{\alpha})]\}.$$

Set $\alpha = \log[(1-\varepsilon)(p+(1-\varepsilon)q)^{-1}]$ and do a Taylor expansion in the exponent. A similar argument proves the second inequality of the statement. □

PROOF OF THEOREM 1.1. Define a sequence $\sigma^n = (\sigma^n(j))_{j\in\mathbb{Z}}$, $n = 1, 2, 3, \dots,$ of random initial server configurations as follows: $P\{\sigma^n(0) = 0\} = 1$, and $\{\eta^n(i) = \sigma^n(i) - \sigma^n(i-1) - 1 \colon i \in \mathbb{Z}\}$ are i.i.d. Geom$(1-\nu)$ distributed, with

$$\nu = \nu_n = \frac{n^{(1-a)/2}}{1+n^{(1-a)/2}}.$$

Note that $E[\sigma^n(k)] = n^{(1-a)/2}k$. By Lemma 5.4, assumptions (2.3) and (2.4) hold with $v_0(y) = y$, $\beta_0 < 1$, and $C_0 > 0$. So by Lemma 5.1,

$$\lim_{n\to\infty} \frac{\sigma^n(0, nt) + nt}{n^{(1+a)/2}} = \sup_{y\le 0}\{y + \alpha\sqrt{-ty}\} = \alpha^2 t/4 \tag{5.20}$$

holds in probability.

On the other hand, the i.i.d. geometric distributions are invariant for the zero-range process $(\eta^n(i))_{i\in\mathbb{Z}}$ [Andjel (1982)]. It is well known that, with service rate 1 and queues in Geom$(1-\nu)$ equilibrium, the departure process of a queue is a rate ν Poisson process [see page 34 in Kelly (1979)]. Since $\sigma^n(0,0) = 0$ and server 0 jumps left whenever a customer leaves the queue, $-\sigma^n(0, nt)$ is Poisson distributed with parameter

$$nt\nu = nt - \frac{nt}{1+n^{(1-a)/2}}.$$

Consequently

$$\lim_{n\to\infty} \frac{\sigma^n(0, nt) + nt}{n^{(1+a)/2}} = t. \tag{5.21}$$

Comparison of (5.20) and (5.21) gives $\alpha = 2$. □

For the proof of Theorem 2.1, combine Lemma 5.1 and Theorem 1.1.

The following simple example demonstrates how Theorem 2.1 can fail without assumption (2.4).

EXAMPLE 5.1. Let $\{c_j\}$ be any sequence of numbers greater than or equal to 1 increasing to $+\infty$. By Lemma 4.1, choose an increasing sequence $\{n_j\}$ of integers such that $n_j > c_j^{3/2(1-a)}$ and

$$\sum_j P\left\{\frac{\xi([c_j n_j], n_j t) - n_j t}{n_j^{(1+a)/2}} > -2\sqrt{c_j t} + 1\right\} < +\infty.$$

For each n_j, define an initial configuration by

$$\sigma^{n_j}(-[c_j n_j^a]+k) = -[c_j^{1/4} n^{(1+a)/2}] + \max\{k, [kn^{(1-a)/2} + (c_j^{1/4} - c_j) n^{(1+a)/2}]\}$$

for all $k \in \mathrm{Z}$, and for $n \neq n_j$ put $\sigma^n(k) = k[n^{(1-a)/2}]$ for all $k \in \mathrm{Z}$. Then (2.3) holds with $v_0(x) = x$, and by (2.6), $v(x,t) = x+t$. But the conclusion (2.5) of Theorem 2.1 fails because with probability 1, for large enough n_j,

$$\frac{\sigma^{n_j}(0, n_j t) + n_j t}{n_j^{(1+a)/2}} \geq \frac{\sigma^{n_j}(-[c_j n_j^a])}{n_j^{(1+a)/2}} - \frac{\xi^{\sigma^{n_j}(-[c_j n_j^a])}([c_j n_j], n_j t) - n_j t}{n_j^{(1+a)/2}} \geq -c_j^{1/4} + 2\sqrt{c_j t}$$

which tends to $+\infty$ as $j \to \infty$.

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