# A SCHWARZ INEQUALITY FOR POSITIVE LINEAR MAPS ON $C^*$-ALGEBRAS[1]

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## 1. Introduction

Davis [5] has derived a Schwarz inequality for completely positive linear maps on $C^*$-algebras of operators. In this paper, we obtain the same inequality for positive linear maps, thus leading to better effect for 2-positive linear maps (in particular, for completely positive linear maps).

Herein, $C^*$-algebras possess an identity and are written in German type $\mathfrak{A}, \mathfrak{B}$. Capital letters $A, B$ stand for operators, Greek letters $\Phi, \Psi, \Omega$ for linear maps on $C^*$-algebras. $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded operators on the Hilbert space $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, we write $sp(T)$ for the spectrum of $T$, and $C^*(T)$ for the $C^*$-algebra generated by $T$. $C(\mathcal{S})$ stands for all continuous complex-valued functions defined on a compact Hausdorff space $\mathcal{S}$.

We denote by $\mathfrak{M}_n$ the collection of all $n \times n$ complex matrices. $\mathfrak{M}_n(\mathfrak{A}) = \mathfrak{A} \otimes \mathfrak{M}_n$ is the $C^*$-algebra of $n \times n$ matrices over $\mathfrak{A}$. A linear map $\Phi : \mathfrak{A} \rightarrow \mathfrak{B}$ is *positive* iff $\Phi(A)$ is positive for all positive $A$ in $\mathfrak{A}$. We define

$$\Phi \otimes 1_n : \mathfrak{M}_n(\mathfrak{A}) \rightarrow \mathfrak{M}_n(\mathfrak{B})$$

by

$$\Phi \otimes 1_n((A_{jk})_{j,k}) = (\Phi(A_{jk}))_{j,k}.$$

$\Phi$ is *n-positive* iff $\Phi \otimes 1_n : \mathfrak{M}_n(\mathfrak{A}) \rightarrow \mathfrak{M}_n(\mathfrak{B})$ is positive; the set of such $\Phi$ is denoted by $\mathbf{P}_n[\mathfrak{A}, \mathfrak{B}]$. (The suffix 1 is deleted if $n = 1$.) $\Phi$ is *completely positive* iff $\Phi$ is $n$-positive for all positive integers $n$.

We presume that all linear maps on $C^*$-algebras preserve the identity.

In §2, a Schwarz inequality (Theorem 2.1) is derived: If $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$, then $\Phi(f(A)) \geq f(\Phi(A))$ for any operator-convex function $f$ and Hermitian operator $A$ provided $f(A)$ is defined. An immediate consequence is the well-known inequality due to Kadison [12]: $\Phi(A^2) \geq \Phi(A)^2$ for all Hermitian $A$. Another useful inequality (Corollary 2.3) is $\Phi(A^{-1}) \geq \Phi(A)^{-1}$ for all positive invertible $A$.

Stinespring [15] and Arveson [1], [2] have established that completely positive linear maps, rather than positive linear maps, are the natural generalizations of positive functionals. From [4], we know that $\mathbf{P}[\mathfrak{A}, \mathfrak{B}] = \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ iff $\mathfrak{A}$ or $\mathfrak{B}$ is commutative. Hence, it is desirable to investigate 2-positive linear maps with special attention to completely positive linear maps.

A more delicate inequality is derived in Corollary 2.8: If $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$, then $\Phi(A^*A) \geq \Phi(A^*)\Phi(A)$ for all $A$ in $\mathfrak{A}$. As a consequence, every 2-positive linear map is 'locally' completely positive (Corollary 2.9).

Received October 3, 1973.

[1] The material of this paper constitutes a part of the author's Ph.D. thesis at the University of Toronto.

In §3, we relate any positive linear map with the 'multiplicative domain', an important subalgebra contained in the domain algebra.

Let $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$. The *multiplicative domain* of $\Phi$, in notation, $\mathfrak{A}_\Phi$, is defined as $\{A \in \mathfrak{A} \mid \Phi(XA) = \Phi(X)\Phi(A) \text{ for all } X \in \mathfrak{A}\}$. The main theorem (Theorem 3.1) deduced from the Schwarz inequality says that if $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ then $\mathfrak{A}_\Phi$ has just the simple form

$$\{A \in \mathfrak{A} \mid \Phi(A^*A) = \Phi(A^*)\Phi(A)\}.$$

The extremal behavior of multiplicative domains really governs the effect of 2-positive maps. In particular, we see that maps in $\mathbf{P}[C(\mathcal{S}), C(\mathfrak{I})]$ are decomposable canonically in terms of maps with trivial multiplicative domains (Remark 3.5).

The author would like to express his thanks to Professor Chandler Davis for many stimulating discussions which led to significant improvements in the paper.

## 2. A Schwarz inequality

A real-valued measurable function $f$ defined on an interval $(-a, a)$ may be considered as an operator-valued function defined on Hermitian operators with spectra contained in $(-a, a)$. Indeed, for a Hermitian operator $A$ with spectral resolution $E_\lambda$, $f(A)$ will mean $\int_{-a}^{a} f(\lambda)\, dE_\lambda$. $f$ is called an *operator-convex* function iff

$$\tfrac{1}{2}(f(A) + f(B)) \geq f(\tfrac{1}{2}(A + B))$$

for all Hermitian operators $A, B$ with spectra contained in $(-a, a)$.

Now we utilize the operator-valued functions to derive a Schwarz inequality.

THEOREM 2.1. *If $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$ and $f$ is an operator-convex function on $(-a, a)$; then $\Phi(f(A)) \geq f(\Phi(A))$ for all Hermitian $A \in \mathfrak{A}$ such that $sp(A) \subseteq (-a, a)$.*

*Proof.* We notice first that $f(\Phi(A))$ is well defined since

$$sp(A) \subseteq [-a + \varepsilon, a - \varepsilon]$$

for some positive $\varepsilon$ and $-a + \varepsilon = \Phi((-a + \varepsilon)I) \leq \Phi(A) \leq \Phi((a - \varepsilon)I) = a - \varepsilon$. $\Phi(f(A))$ is defined because ($f$ being continuous) $f(A)$ belongs to $C^*(A) \subseteq \mathfrak{A}$.

Now for Hermitian $A$, $C^*(A)$ is a commutative $C^*$-algebra. So $\Phi$ restricted to $C^*(A)$ is completely positive. By Davis's Theorem [5, p. 44], $\Phi(f(A)) \geq f(\Phi(A))$ as required. ∎

Bendat and Sherman [3, §3] (See also Davis [9, §4]) have shown that a real function is operator-convex iff it has an integral form

$$f(t) = \int_{-a}^{a} \frac{t^2}{a^2 - tx}\, d\mathrm{m}(x) + bt + c$$

where $\mathbf{m}$ is a regular Borel positive finite measure on $[-a, a]$. Hence, a lot of inequalities can be derived for Hermitian operators. Here we mention two important cases:

COROLLARY 2.2 (Kadison [12, p. 495]). *If* $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$, *then* $\Phi(A^2) \geq \Phi(A)^2$ *for all Hermitian* $A \in \mathfrak{A}$.

*Proof.* In Bendat and Sherman's formula, put $b = c = 0$, $\mathbf{m}$ = one point measure at the origin; then $f(t) = t^2$ is an operator-convex function. (In fact, it is straightforward to check by definition that $f(t) = t^2$ is operator-convex.) ∎

COROLLARY 2.3. *If* $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$, *then* $\Phi(A^{-1}) \geq \Phi(A)^{-1}$ *for all positive invertible* $A \in \mathfrak{A}$.

*Proof.* In Bendat and Sherman's formula, put $a = b = c = 1$, $\mathbf{m}$ = one point measure such that $\mathbf{m}(\{1\}) = 1$; so $f(t) = (1 - t)^{-1}$ is operator-convex on $(-1, 1)$. By Theorem 2.1, $\Phi((I - X)^{-1}) \geq (I - \Phi(X))^{-1} = (\Phi(I - X))^{-1}$ for Hermitian $X$ with spectrum contained in $(-1, 1)$. Replacing $I - X$ by $\varepsilon A$, we get $\Phi(A^{-1}) \geq \Phi(A)^{-1}$ for positive $A$ such that $sp(A) \subseteq (0, \varepsilon^{-1})$, hence for all positive invertible $A$. ∎

The inequality in Corollary 2.3 gives some non-vacuous information about positive linear maps. Indeed if $A \geq \varepsilon > 0$, the naïve definition says that $\Phi(A) \geq \varepsilon$ while the derived inequality says that

$$\Phi(A) \geq \Phi(A^{-1})^{-1} \geq \varepsilon.$$

We remark that Corollary 2.3 is not true for an arbitrary invertible Hermitian operator. For example; let $\mathfrak{A}$ = the commutative $C^*$-algebra of ordered pairs

$$\{(\alpha, \beta) \mid \alpha, \beta \text{ are complex numbers}\};$$

$\Phi$ = the linear functional such that $\Phi(\alpha, \beta) = \frac{1}{2}(\alpha + \beta)$; $A = (1, -1)$. Then $A = A^{-1}$, and $\Phi(A) = \Phi(A^{-1}) = 0$, so the inequality $\Phi(A^{-1}) \geq \Phi(A)^{-1}$ does not hold.

Referring to Theorem 2.1, the inequality may become equality for all Hermitian $A$. We will see that such will happen only in the extraordinary cases: $f$ is *affine* (i.e., $f$ is of the form $f(t) = a_1 t + a_0$) or $\Phi$ is extreme. We recall that for $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$, $\Phi$ is a *$C^*$-homomorphism* iff $\Phi(A^2) = \Phi(A)^2$ for every Hermitian $A$ in $\mathfrak{A}$, and Størmer [16, p. 242] has proved that every $C^*$-homomorphism is extreme. The following lemma gives an alternative characterization of a $C^*$-homomorphism.

LEMMA 2.4. *Let* $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$. *Then* $\Phi$ *is a $C^*$-homomorphism iff* $\Phi(A^{-1}) = \Phi(A)^{-1}$ *for all positive invertible* $A$ *in* $\mathfrak{A}$.

*Proof.* Assume $\Phi$ preserves the inverse for every positive invertible operator. Then for any positive invertible $A$, we apply Kadison's inequality (Corollary 2.2) and get

$$\Phi(A^2)^{-1} = \Phi(A^{-2}) \geq \Phi(A^{-1})^2 = \Phi(A)^{-2}, \qquad \Phi(A^2) \leq \Phi(A)^2.$$

Applying Kadison's inequality again, $\Phi(A^2) = \Phi(A)^2$.

To extend this to an arbitrary Hermitian operator $A$, replace $A$ by $A + nI$ for a sufficiently large $n$. Hence $\Phi$ is a $C^*$-homomorphism.

The converse follows from the fact a $C^*$-homomorphism restricted to $C^*(A)$, for any Hermitian $A$, is a *homomorphism. ∎

THEOREM 2.5. *Let $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$. If $f$ is a non-affine operator-convex function on $(-a, a)$, and $\Phi(f(A)) = f(\Phi(A))$ for all Hermitian $A$ in $\mathfrak{A}$ such that $sp(A) \subseteq (-a, a)$, then $\Phi$ is a $C^*$-homomorphism.*

*Proof.* By Bendat and Sherman's formula,

$$f(t) = \int_{-a}^{a} \frac{t^2}{a^2 - tx}\, d\mathbf{m}(x) + bt + c.$$

Since $f$ is non-affine, the *carrier* of $\mathbf{m}$ (the smallest closed subset $\mathcal{S}$ of $[-a, a]$ such that $\mathbf{m}(\mathcal{S}) = \mathbf{m}([-a, \ a]))$ is nonvoid.

Now suppose $\Phi(f(A)) = f(\Phi(A))$ for all Hermitian $A$ such that $sp(A) \subseteq (-a, a)$. For each $s$ in the carrier of $\mathbf{m}$, $g(t) = t^2/(a^2 - st)$ is operator-convex, hence $\Phi(g(A)) = g(\Phi(A))$ by virtue of Theorem 2.1. In case $s = 0$, we get immediately that $\Phi(A^2) = \Phi(A)^2$. In case $s \neq 0$,

$$g(t) = t^2/(a^2 - st) = (a^4(a^2 - st)^{-1} - st - a^2)/s^2;$$

by a transformation as in the proof of Corollary 2.3, we deduce that $\Phi(A^{-1}) = \Phi(A)^{-1}$ for all positive invertible $A$. Therefore, $\Phi$ is a $C^*$-homomorphism in both cases. ∎

*Remark* 2.6. An operator-convex function plays an essential role in the above results. The following example shows that Theorem 2.1 would be false if we replace an *operator-convex* function by a general *convex* function:

The function $f(t) = t^4$ is convex but not operator-convex. Let $\Phi : \mathfrak{M}_3 \to \mathfrak{M}_2$ be the compression map

$$\Phi((a_{jk})_{1 \leq j,k \leq 3}) = (a_{jk})_{1 \leq j,k \leq 2},$$

and

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then

$$\Phi(A)^4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \not\geq \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix} = \Phi(A^4).$$

Theorem 2.5 is of interest when referring to Theorem 2.1. However, certain facts reveal that a more general case may be true. We conjecture that

Theorem 2.5 remains true if we require $f$ to be a general non-affine real continuous function instead of an operator-convex function.

The Schwarz inequality derived in Theorem 2.1 is in some respects unsatisfactory. For example, it does not govern non-Hermitian operators. We will achieve this effect for 2-positive linear maps. A function $f$ on $(-a, a)$ is *even* iff $f(t) = f(-t)$ for all $t$. Indeed, every operator-convex function $f$ induces an even operator-convex function $f(t) + f(-t)$. Following is a modified Schwarz inequality.

THEOREM 2.7. *Let $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$. If $f$ is an even operator-convex function on $(-a, a)$, then for every $A \in \mathfrak{A}$ with the norm less than $a$,*

$$\Phi(f(|A|)) \geq f(|\Phi(A)|).$$

(Here $|X|$ stands for $(X^*X)^{1/2}$.)

*Proof.* Applying Theorem 2.1 to $\Phi \otimes 1_2 \in \mathbf{P}[\mathfrak{M}_2(\mathfrak{A}), \mathfrak{M}_2(\mathfrak{B})]$, we get

$$(*) \qquad \Phi \otimes 1_2(f(T)) \geq f(\Phi \otimes 1_2(T))$$

for all Hermitian $T \in \mathfrak{M}_2(\mathfrak{A})$ with $sp(T) \subseteq (-a, a)$. Now, let

$$T = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix};$$

then

$$|T| = \begin{bmatrix} |A| & 0 \\ 0 & |A^*| \end{bmatrix}.$$

As $f$ is even, $f(t) = f(|t|)$, so

$$f(T) = f(|T|) = \begin{bmatrix} f(|A|) & 0 \\ 0 & f(|A^*|) \end{bmatrix}.$$

Similarly,

$$f(\Phi \otimes 1_2(T)) = \begin{bmatrix} f(|\Phi(A)|) & 0 \\ 0 & f(|\Phi(A^*)|) \end{bmatrix}.$$

By $(*)$, we obtain the required inequality. ∎

Putting $f(t) = t^2$ in the above theorem, we get the important result:

COROLLARY 2.8. *If $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$, then $\Phi(A^*A) \geq \Phi(A^*)\Phi(A)$ for all $A$ in $\mathfrak{A}$.* ∎

COROLLARY 2.9. *Every 2-positive linear map is locally completely positive. This means that, if $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$, then for any $x$ in $\mathfrak{H}$ the underlying space of $\mathfrak{B}$, there exists a completely positive linear map $\Psi_x : \mathfrak{A} \to \mathfrak{B}(\mathfrak{H})$ (which need not preserve identity) with $\|\Psi_x\| \leq 1$, such that $\Phi(\cdot)x = \Psi_x(\cdot)x$.*

*Proof.* With the Schwarz inequality of Corollary 2.8 in hand, we are ready to refer to Størmer [16, p. 268] and obtain the required result.

For completeness, we sketch the short proof.

We may assume $\| x \| = 1$. Starting from the positive functional $\langle \Phi(\cdot)x, x\rangle$ on $\mathfrak{A}$, we construct the 'associated representation' $\Pi$ of $\mathfrak{A}$ on a Hilbert space $\mathcal{K}$; i.e., $\Pi$ is a cyclic representation with a cyclic vector $v \in \mathcal{K}$ such that

$$\langle \Pi(A)v, v\rangle = \langle \Phi(A)x, x\rangle \quad \text{for all} \quad A \in \mathfrak{A}.$$

Define $V : \mathcal{K} \to \mathcal{K}$ by $\Pi(A)v \mapsto \Phi(A)x$; the Schwarz inequality of Corollary 2.8 guarantees that $V$ is well defined. Then $\Psi_x = V\Pi(\cdot)V^*$ is the required completely positive map for $\Phi$ at $x$. ∎

## 3. Multiplicative domains

In Corollary 2.8, we showed that if $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ then

$$\Phi(A^*A) \geq \Phi(A^*)\Phi(A) \quad \text{for all} \quad A \in \mathfrak{A};$$

now, we examine the subset of $\mathfrak{A}$ for which equality holds:

THEOREM 3.1. *If $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$, then the set $\{A \in \mathfrak{A} \mid \Phi(A^*A) = \Phi(A^*)\Phi(A)\}$ is a closed subalgebra of $\mathfrak{A}$. In fact, it is just the multiplicative domain,*

$$\mathfrak{A}_\Phi \equiv \{A \in \mathfrak{A} \mid \Phi(XA) = \Phi(X)\Phi(A) \quad \text{for all} \quad X \in \mathfrak{A}\}.$$

*Proof.* It is straightforward to see that $\mathfrak{A}_\Phi$ is a closed algebra. It remains to show that if $\Phi(A^*A) = \Phi(A^*)\Phi(A)$, then $\Phi(XA) = \Phi(X)\Phi(A)$ for all $X$ in $\mathfrak{A}$.

Let $H$ be a Hermitian operator in $\mathfrak{A}$. By Kadison's inequality,

$$\Phi \otimes 1_2\left(\begin{bmatrix} 0 & A^* \\ A & H \end{bmatrix}^2\right) \geq \left(\Phi \otimes 1_2\begin{bmatrix} 0 & A^* \\ A & H \end{bmatrix}\right)^2,$$

i.e.,

$$\begin{bmatrix} \Phi(A^*A) & \Phi(A^*H) \\ \Phi(HA) & \Phi(AA^* + H^2) \end{bmatrix} \geq \begin{bmatrix} \Phi(A^*)\Phi(A) & \Phi(A^*)\Phi(H) \\ \Phi(H)\Phi(A) & \Phi(A)\Phi(A^*) + \Phi(H)^2 \end{bmatrix}.$$

That $\Phi(A^*A) = \Phi(A^*)\Phi(A)$ forces $\Phi(HA) = \Phi(H)\Phi(A)$. Now for arbitrary $X$ in $\mathfrak{A}$, $X = \operatorname{re}X + i\operatorname{im}X$. Thus the desired result is immediate. ∎

The preceding theorem does not hold for a general positive linear map. For example, let $\Phi$ be the transpose map $\mathfrak{M}_n \to \mathfrak{M}_n$ $(n > 1)$. Then

$$\{A \in \mathfrak{M}_n \mid \Phi(A^*A) = \Phi(A^*)\Phi(A)\} = \{\text{normal matrices}\},$$

which is not an algebra; while $(\mathfrak{M}_n)_\Phi$ consists of scalars only.

COROLLARY 3.2. *Every 2-positive $C^*$-homomorphism is a $^*$homomorphism.*

*Proof.* Let $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ be a $C^*$-homomorphism, i.e., for all Hermitian $A$ in $\mathfrak{A}$, $\Phi(A^2) = \Phi(A)^2$. By Theorem 3.1, $\mathfrak{A}_\Phi$ is an algebra containing all Hermitian operators in $\mathfrak{A}$; so $\mathfrak{A}_\Phi = \mathfrak{A}$. Hence we conclude that $\Phi$ is a $^*$homomorphism. ∎

An alternative proof of Corollary 3.2 without using Theorem 3.1 is to combine Corollary 2.8 with Størmer [16, Corollary 3.6, p. 446]. This, however, involves a much deeper structure theorem of $C^*$-homomorphisms.

If $\mathfrak{A}$, $\mathfrak{B}$ are commutative, and $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$, then $\mathfrak{A}_\Phi$ is a $C^*$-algebra and it represents the amount of 'extremeness' that $\Phi$ possesses, in fact, $\mathfrak{A}_\Phi = \mathfrak{A}$ iff $\Phi$ is extreme.

In the general case, $\Phi$ may be extreme while $\mathfrak{A}_\Phi \subsetneq \mathfrak{A}$. Nevertheless $\mathfrak{A}_\Phi$ has a great deal to do with the 'extremal behaviour' of $\Phi$.

THEOREM 3.3. *If $\Phi, \Psi, \Omega \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ and $\Phi = \frac{1}{2}(\Psi + \Omega)$, then*

$$\mathfrak{A}_\Phi = \mathfrak{A}_\Psi \cap \mathfrak{A}_\Omega \cap \{A \in \mathfrak{A} \mid \Phi(A) = \Psi(A) = \Omega(A)\}.$$

*Proof.* For any $A$ in $\mathfrak{A}$,

$$\begin{aligned}\Phi(A^*A) &= \tfrac{1}{2}(\Psi(A^*A) + \Omega(A^*A)) \\ &\geq \tfrac{1}{2}(\Psi(A^*)\Psi(A) + \Omega(A^*)\Omega(A)) \\ &= \tfrac{1}{4}(\Psi(A^*) + \Omega(A^*))(\Psi(A) + \Omega(A)) \\ &\quad + \tfrac{1}{4}(\Psi(A^*) - \Omega(A^*))(\Psi(A) - \Omega(A)) \\ &\geq \tfrac{1}{4}(\Psi(A^*) + \Omega(A^*))(\Psi(A) + \Omega(A)) \\ &= \Phi(A^*)\Phi(A).\end{aligned}$$

If $A \in \mathfrak{A}_\Phi$, then $\Phi(A^*A) = \Phi(A^*)\Phi(A)$ and all of the above inequalities become equalities. Hence

$$\Psi(A^*A) = \Psi(A^*)\Psi(A), \qquad \Omega(A^*A) = \Omega(A^*)\Omega(A) \quad \text{and} \quad \Psi(A) = \Omega(A).$$

Thus we conclude that $\mathfrak{A}_\Phi \subseteq \mathfrak{A}_\Psi \cap \mathfrak{A}_\Omega \cap \{A \in \mathfrak{A} \mid \Phi(A) = \Psi(A) = \Omega(A)\}$.

The opposite inclusion is trivial. ∎

The set $\mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ is convex. If $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ is not extreme, then there is an open line-segment in $\mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ passing through $\Phi$. Theorem 3.3 says that every map lying in the open segment has the same multiplicative domain and agrees with $\Phi$ on the multiplicative domain.

*Remark* 3.4. Let $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$. The *left kernel* of $\Phi$ is the set

$$\{A \in \mathfrak{A} \mid \Phi(A^*A) = 0\}.$$

From the Schwarz inequality, $\Phi(A^*A) \geq \Phi(A^*)\Phi(A) \geq 0$, it follows that $\Phi(A^*A) = 0$ iff $\Phi(A^*A) = \Phi(A^*)\Phi(A)$ and $\Phi(A) = 0$; that is, the left kernel is the intersection of the kernel and the multiplicative domain. Alternatively, the left kernel is the largest left ideal contained in the kernel. Furthermore, $\Phi$ restricted to $\mathfrak{A}_\Phi$ is an algebraic homomorphism; the kernel of the restricted map is the left kernel of $\Phi$.

$\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ is *faithful* iff the left kernel of $\Phi$ is trivial. Equivalently, $\Phi$ is faithful iff $\Phi|_{\mathfrak{A}_\Phi}$ is an algebraic isomorphism.

*Remark* 3.5. Multiplicative domains of $\mathbf{P}[C(\mathcal{S}), C(\mathfrak{I})]$. The significance of multiplicative domains can be best revealed by the tractable structure of positive linear maps on continuous functions. A thorough description is divided into four parts as follows.

(i) *Suppose* $\Phi \in \mathbf{P}[C(\mathcal{S}), C(\mathfrak{I})]$. *Then* $C(\mathcal{S})_\Phi$ *is a* $C^*$*-algebra. So the left kernel is an ideal contained in* $C(\mathcal{S})_\Phi$. *By factoring out the left kernel, we may assume* $\Phi$ *is faithful* (Remark 3.4). (To be precise, we should say that there exists a faithful

$$\Phi_0 \in \mathbf{P}[C(\mathcal{S}_0), C(\mathfrak{I})],$$

where $\mathcal{S}_0$ is a closed subset of $\mathcal{S}$, such that $\Phi(f) = \Phi_0(f\,|_{\mathcal{S}_0})$ for all $f \in C(\mathcal{S})$.)

(ii) *Suppose* $\Phi \in \mathbf{P}[C(\mathcal{S}), C(\mathfrak{I})]$ *is faithful. Let* $g \in C(\mathcal{S})$. *We write* $\mathfrak{X} =$ *Range* $g$, *and* $\mathcal{S}_x = \{s \in \mathcal{S} \mid g(s) = x\}$, $\mathfrak{I}_x = \{t \in \mathfrak{I} \mid \Phi(g)(t) = x\}$, *for each* $x \in \mathfrak{X}$. *Then the following are equivalent*:

(a) $g \in C(\mathcal{S})_\Phi$.

(b) $\mathfrak{I} = \bigcup \mathfrak{I}_x$, *and there exist* $\Phi_x \in \mathbf{P}[C(\mathcal{S}_x), C(\mathfrak{I}_x)]$ *such that*

$$\Phi(f) = \oplus \Phi_x(f\,|_{\mathcal{S}_x}) \quad \textit{for all} \quad f \in C(\mathcal{S}).$$

(Roughly, we say that $\mathcal{S}$, $\mathfrak{I}$ are broken into the same number of slices, and $\Phi$ sends each slice of $\mathcal{S}$ to the corresponding slice of $\mathfrak{I}$.)

*Proof.* (b) $\Rightarrow$ (a). As $g$ assumes scalar value $x$ on $\mathcal{S}_x$, by the presumed decomposition formula $\Phi(f) = \oplus\Phi_x(f\,|_{\mathcal{S}_x})$, it is immediate that

$$\Phi(g^*g) = \Phi(g^*)\Phi(g) = \oplus \mid x \mid^2 I_x$$

where $I_x$ is the identity of $C(\mathfrak{I}_x)$. Hence $g \in C(\mathcal{S})_\Phi$.

(a) $\Rightarrow$ (b). As $C^*(g) \subseteq C(\mathcal{S})_\Phi$ and $\Phi$ is faithful, so

$$\Phi(C^*(g)) \simeq C^*(g) \simeq C(\mathfrak{X});$$

they are related in such a manner that for any $x \in \mathfrak{X}$ and $g' \in C^*(g)$, both $g'\,|_{\mathcal{S}_x}$ and $\Phi(g')\,|_{\mathfrak{I}_x}$ assume a common constant value. Evidently, $\mathfrak{I} = \bigcup \mathfrak{I}_x$ is the disjoint union of a class of non-void closed subsets.

Now for each fixed $a \in \mathfrak{X}$, define $\Phi_a : C(\mathcal{S}_a) \to C(\mathfrak{I}_a)$ with

$$\Phi_a(f\,|_{\mathcal{S}_a}) = \Phi(f)\,|_{\mathfrak{I}_a} \quad \text{for all} \quad f \in C(\mathcal{S}).$$

It is well defined since if $f\,|_{\mathcal{S}_a} = 0$, then there exist $g_n \in C^*(g)$ such that $g_n\,|_{\mathcal{S}_a} = I$ and $\| fg_n \| \to 0$ (as $n \to \infty$); thus

$$\Phi(f)\,|_{\mathfrak{I}_a} = \Phi(f)\Phi(g_n)\,|_{\mathfrak{I}_a} = \Phi(fg_n)\,|_{\mathfrak{I}_a}$$

must be zero. (An example to construct $g_n$: First pick up $h_n \in C(\mathfrak{X})$ such that $\| h_n \| = 1$, $h_n(a) = 1$ and $h_n$ restricted to

$$\{x \in \mathfrak{X} : \| f\,|_{\mathcal{S}_x} \| \geq 1/n\}$$

is zero. Then define $g_n(s) = h_n(x)$ whenever $s \in \mathcal{S}_x$.) Hence

$$\Phi(f) = \oplus \Phi_x(f|_{\mathcal{S}_x})$$

as required. ∎

(iii) *Let $\Phi \in \mathbf{P}[C(\mathcal{S}), C(\mathfrak{J})]$ be faithful. Suppose $C(\mathcal{S})_\Phi$ is equivalent to $C(\mathfrak{X})$. Then there exist continuous surjections*

$$\sigma : \mathcal{S} \rightarrow \mathfrak{X} \quad \textit{and} \quad \tau : \mathfrak{J} \rightarrow \mathfrak{X};$$

*and for each $x \in \mathfrak{X}$, there corresponds $\Phi_x \in \mathbf{P}[C(\sigma^{-1}\{x\}), C(\tau^{-1}\{x\})]$ such that $\Phi_x$ has the trivial multiplicative domain and*

$$\Phi(f) = \oplus \Phi_x(f|_{\sigma^{-1}\{x\}})$$

*for all $f \in C(\mathcal{S})$.*

*Proof.* As $\Phi(C(\mathcal{S})_\Phi) \simeq C(\mathcal{S})_\Phi \simeq C(\mathfrak{X})$, there exist continuous surjections

$$\sigma : \mathcal{S} \rightarrow \mathfrak{X} \quad \text{and} \quad \tau : \mathfrak{J} \rightarrow \mathfrak{X}$$

such that for each $x \in \mathfrak{X}$ and $g \in C(\mathcal{S})_\Phi$, $g|_{\sigma^{-1}\{x\}}$ and $\Phi(g)|_{\tau^{-1}\{x\}}$ assume a common constant value. The rest of the proof is similar to (ii). ∎

(iv) *Let $\Phi \in \mathbf{P}[C(\mathcal{S}), C(\mathfrak{J})]$ be faithful. Then $C(\mathcal{S})_\Phi = \{scalars\}$ iff $\Phi$ is 'indecomposable' in the sense that $\mathcal{S}$, $\mathfrak{J}$ cannot be further 'sliced'* (see (ii)).

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