Notre Dame Journal of Formal Logic
Volume XX, Number 2, April 1979
NDJFAM

A DEDUCTION THEOREM FOR REJECTION THESES IN ŁUKASIEWICZ'S SYSTEM OF MODAL LOGIC

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In this paper* I shall present a proof of a deduction theorem for rejection theses of Jan Łukasiewicz's system of modal logic.[1] By "rejection thesis" I mean the last line of any valid deduction in Łukasiewicz's system of the form $\dashv \Delta$. There are several such theses already proved by Łukasiewicz, e.g., theses 115-122,[2] but surely there are many more which are able to be proved. The theorem

$$\dashv \mathbf{A}_1, \ldots, \dashv \mathbf{A}_{m-1}, \dashv \mathbf{A}_m \vdash_{\text{Ł}} \dashv \mathbf{B} \Rightarrow \dashv \mathbf{A}_1, \ldots, \dashv \mathbf{A}_{m-1} \vdash_{\text{Ł}} \ulcorner \dashv NC\mathbf{B}\mathbf{A}_m \urcorner$$

is designed to facilitate the deduction of these latter theses, much as the standard deduction theorem facilitates the deduction of theses of classical logical systems.

My strategy to be used in proving this theorem will be similar to that used in proving the standard deduction theorem. I shall assume that the antecedent of the theorem is true, i.e., that there is a finite sequence of wffs which constitutes a demonstration of $\dashv \mathbf{B}$ from the premises $\dashv \mathbf{A}_1, \ldots, \dashv \mathbf{A}_m$, and then indicate how one may construct, from this initial sequence, a second sequence which will constitute a demonstration of $\ulcorner \dashv KN\mathbf{A}_m\mathbf{B} \urcorner$ from the premises $\dashv \mathbf{A}_1, \ldots, \dashv \mathbf{A}_{m-1}$. I shall then show how, once one has obtained this second sequence, one can add several steps to it and further deduce $\ulcorner \dashv NC\mathbf{B}\mathbf{A}_m \urcorner$. Before giving the details of my proof, I shall first introduce some conventions:

R_1 = Rule of substitution for assertions: from $\vdash \mathbf{A}$ one may infer the result of substituting a wff $\mathbf{B}$ for a sentential variable c throughout $\mathbf{A}$.
R_2 = Rule of detachment for assertions: from $\vdash \mathbf{A}$ and $\ulcorner \vdash C\mathbf{A}\mathbf{B} \urcorner$ one may infer $\mathbf{B}$.
R_3 = Rule of substitution for rejections: from $\dashv \mathbf{A}$, where $\mathbf{A}$ is a substitution instance of $\mathbf{B}$, one may infer $\dashv \mathbf{B}$.[3]

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*I wish to express my appreciation to Mr. Robert Wengert for his comments on the first version of this paper.

Received March 12, 1976

R_4 = Rule of detachment for rejections: from $\ulcorner\vdash C\mathbf{AB}\urcorner$ and $\dashv\mathbf{B}$ one may infer $\dashv\mathbf{A}$.[4]

Theorem 1 $\dashv\mathbf{A}_1, \ldots, \dashv\mathbf{A}_{m-1}, \dashv\mathbf{A}_m \vdash_{\mathfrak{L}} \dashv\mathbf{B} \Rightarrow \dashv\mathbf{A}_1, \ldots, \dashv\mathbf{A}_{m-1} \vdash_{\mathfrak{L}} \ulcorner\dashv KN\mathbf{A}_m\mathbf{B}\urcorner$.

Proof: Assume $\dashv\mathbf{A}_1, \ldots, \dashv\mathbf{A}_{m-1}, \dashv\mathbf{A}_m \vdash_{\mathfrak{L}} \dashv\mathbf{B}$ and let $\Sigma_1, \ldots, \Sigma_i, \ldots, \Sigma_k$ denote its demonstration. Construct the sequence which for every Σ_i of the form $\vdash\Gamma$ has a Σ_j^* which is the same as Σ_i and which for every Σ_i of the form $\dashv\Delta$ has a Σ_j^* of the form $\ulcorner\dashv KN\mathbf{A}_m\Delta\urcorner$. We now show how to use this latter sequence to construct a demonstration of $\dashv\mathbf{A}_1, \ldots, \dashv\mathbf{A}_{m-1} \vdash_{\mathfrak{L}} \ulcorner\dashv KN\mathbf{A}_m\mathbf{B}\urcorner$. Consider the following cases:

Case 1: $\vdash\Sigma_i$ is an assertion axiom (i.e., an axiom of the form $\vdash\Gamma$) or a variant of such an axiom. Use the following proof to demonstrate $\vdash\Sigma_i$:

1. $\vdash\Sigma_i$
2. $\vdash\Sigma_i$ — 1; immediate inference

Case 2: $\vdash\Sigma_i$ is generated by the use of R_2 and two earlier steps in the sequence, $\vdash\Sigma_a$ and $\ulcorner\vdash C\Sigma_a\Sigma_i\urcorner$. Use the following proof to demonstrate $\vdash\Sigma_i$:

1. $\vdash\Sigma_a$
2. $\vdash C\Sigma_a\Sigma_i$
3. $\vdash\Sigma_i$ — 1; 2; R_2

Case 3: $\dashv\Sigma_i$ is a rejection axiom or $\dashv\mathbf{A}_j$ $(1 \leqslant j \leqslant m-1)$. Use the following proof to demonstrate $\ulcorner\dashv KN\mathbf{A}_m\Sigma_i\urcorner$:

1. $\vdash CKpqq$ — **PC**[5]
2. $\vdash CKN\mathbf{A}_m\Sigma_i\Sigma_i$ — 1; R_1, $p/N\mathbf{A}_m$, q/Σ_i
3. $\dashv\Sigma_i$
4. $\dashv KN\mathbf{A}_m\Sigma_i$ — 2, 3; R_4

Case 4: Σ_a is a substitution instance of Σ_i, and $\dashv\Sigma_a$ is either a rejection axiom or $\dashv\mathbf{A}_j$ $(1 \leqslant j \leqslant m-1)$. Use the following proof to demonstrate $\ulcorner\dashv KN\mathbf{A}_m\Sigma_i\urcorner$:

1. $\vdash CKpqq$ — **PC**
2. $\vdash CKN\mathbf{A}_m\Sigma_i\Sigma_i$ — 1; R_1, $p/N\mathbf{A}_m$, q/Σ_i
3. $\dashv\Sigma_a$
4. $\dashv\Sigma_i$ — 3; R_3
5. $\dashv KN\mathbf{A}_m\Sigma_i$ — 2; 4; R_4

Case 5: $\dashv\Sigma_i$ is $\dashv\mathbf{A}_m$. Use the following proof to demonstrate $\ulcorner\dashv KN\mathbf{A}_m\Sigma_i\urcorner$:

1. $\vdash CKNppq$ — **PC**
2. $\vdash CKN\mathbf{A}_m\mathbf{A}_m\Delta p$ — 1; R_1, $p/\mathbf{A}_m$, $q/\Delta p$[6]
3. $\dashv\Delta p$ — axiom 4 in Łukasiewicz's system[7]
4. $\dashv KN\mathbf{A}_m\mathbf{A}_m$ — 2; 3; R_4
5. $\dashv KN\mathbf{A}_m\Sigma_i$ — 4; R_3

Case 6: Σ_a is a substitution instance of Σ_i and $\dashv\Sigma_a$ is $\dashv\mathbf{A}_m$. Use the following proof to demonstrate $\ulcorner\dashv KN\mathbf{A}_m\Sigma_i\urcorner$:

1. $\vdash CKNppq$	PC
2. $\vdash CKN\mathbf{A}_m\mathbf{A}_m\Delta p$	1; R_1, $p/\mathbf{A}_m$, $q/\Delta p$
3. $\dashv \Delta p$	axiom 4 in Łukasiewicz's system
4. $\dashv KN\mathbf{A}_m\mathbf{A}_m$	3; 4; R_4
5. $\dashv KN\mathbf{A}_m\Sigma_a$	4; R_3
6. $\dashv KN\mathbf{A}_m\Sigma_i$	4; R_4

Case 7: $\dashv\Sigma_i$ is obtained from Σ_j's of the form $\vdash C\Sigma_i\Gamma$ and $\dashv\Gamma$ by use of R_4. Use the following steps to obtain $\ulcorner\dashv KN\mathbf{A}_m\Sigma_i\urcorner$:

1. $\vdash CCpqCCrsCKprKqs$	PC
2. $\vdash CCN\mathbf{A}_mN\mathbf{A}_mCC\Sigma_i\Gamma CKN\mathbf{A}_m\Sigma_iKN\mathbf{A}_m\Gamma$	1; R_1, $p/N\mathbf{A}_m$, $q/N\mathbf{A}_m$, r/Σ_i, s/Γ
3. $\vdash Cpp$	PC
4. $\vdash CN\mathbf{A}_mN\mathbf{A}_m$	3; R_1, $p/N\mathbf{A}_m$
5. $\vdash CC\Sigma_i\Gamma CKN\mathbf{A}_m\Sigma_iKN\mathbf{A}_m\Gamma$	4; 2; R_2
6. $\vdash C\Sigma_i\Gamma$	
7. $\vdash CKN\mathbf{A}_m\Sigma_iKN\mathbf{A}_m\Gamma$	5; 6; R_2
8. $\dashv KN\mathbf{A}_m\Gamma$	
9. $\dashv KN\mathbf{A}_m\Sigma_i$	7; 8; R_4

We shall thus be able to construct a new sequence the last line of which is $\ulcorner\dashv KN\mathbf{A}_m\mathbf{B}\urcorner$. This sequence constitutes a demonstration of this last line from the premises $\dashv\mathbf{A}_1, \ldots, \dashv\mathbf{A}_{m-1}$.

Theorem 2 $\dashv\mathbf{A}_1, \ldots, \dashv\mathbf{A}_{m-1}, \dashv\mathbf{A}_m \vdash_{Ł} \dashv\mathbf{B} \Rightarrow \dashv\mathbf{A}_1, \ldots, \dashv\mathbf{A}_{m-1} \vdash_{Ł} \ulcorner\dashv NC\mathbf{B}\mathbf{A}_m\urcorner$.

Proof: By Theorem 1 we know that, given $\dashv\mathbf{A}_1, \ldots, \dashv\mathbf{A}_{m-1}, \dashv\mathbf{A}_m \vdash_{Ł} \dashv\mathbf{B}$, we can construct a demonstration of $\ulcorner\dashv KNA\mathbf{A}_m\mathbf{B}\urcorner$ from the premises $\dashv\mathbf{A}_1, \ldots,$ $\dashv\mathbf{A}_{m-1}$. We shall begin with the last line of that demonstration and from it demonstrate $\ulcorner\dashv NC\mathbf{B}\mathbf{A}_m\urcorner$.

Σ_k	$\dashv KN\mathbf{A}_m\mathbf{B}$	
Σ_{k+1}	$\vdash CKpqKqp$	PC
Σ_{k+2}	$\vdash CK\mathbf{B}N\mathbf{A}_mKN\mathbf{A}_m\mathbf{B}$	Σ_{k+1}; R_1, $p/\mathbf{B}$, $q/N\mathbf{A}_m$
Σ_{k+3}	$\dashv K\mathbf{B}N\mathbf{A}_m$	Σ_k; Σ_{k+2}; R_4
Σ_{k+4}	$\vdash CNCpqKpNq$	PC
Σ_{k+5}	$\vdash CNC\mathbf{B}\mathbf{A}_mK\mathbf{B}N\mathbf{A}_m$	Σ_{k+4}; R_1, $p/\mathbf{B}$, $q/\mathbf{A}_m$
Σ_{k+6}	$\dashv NC\mathbf{B}\mathbf{A}_m$	Σ_{k+5}; Σ_{k+3}; R_4

NOTES

1. I refer to the system developed by Łukasiewicz in [1].
2. *Ibid.*, p. 145.
3. *Ibid.*, p. 114.
4. *Ibid.*
5. I use '**PC**' to stand for 'propositional calculus'. I feel free to make use of theses of the propositional calculus, since all of the axioms, and *a fortiori* all of the theses, of that calculus are demonstrable in Łukasiewicz's system. (*cf.* page 124 in [1])

6. 'Δ' is Łukasiewicz's possibility functor.

7. Łukasiewicz, [1], p. 137.

REFERENCE

[1] Łukasiewicz, J., "A system of modal logic," *The Journal of Computing Systems*, vol. 1 (1953), pp. 111-149.

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