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# A SECOND-ORDER ACCURATE LINEARIZED DIFFERENCE SCHEME FOR THE TWO-DIMENSIONAL CAHN-HILLIARD EQUATION

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ABSTRACT. The Cahn-Hilliard equation is a nonlinear evolutionary equation that is of fourth order in space. In this paper a linearized finite difference scheme is derived by the method of reduction of order. It is proved that the scheme is uniquely solvable and convergent with the convergence rate of order two in a discrete $L_2$-norm. The coefficient matrix of the difference system is symmetric and positive definite, so many well-known iterative methods (e.g. Gauss-Seidel, SOR) can be used to solve the system.

## 1. Introduction

We consider the Cahn-Hilliard equation

$$u_t + \Delta^2 u = \Delta\phi(u), \quad (x, y, t) \in \Omega \times (0, T] \tag{1.1}$$

for $u(x, y, t)$, subject to the boundary conditions

$$\frac{\partial u}{\partial \nu} = 0, \quad \frac{\partial}{\partial \nu}(\phi(u) - \Delta u) = 0 \quad \text{on} \quad \partial\Omega \times (0, T] \tag{1.2}$$

and the initial condition

$$u(x, y, 0) = u_0(x, y), \qquad (x, y) \in \overline{\Omega}, \tag{1.3}$$

where $\phi(\cdot) = \psi'(\cdot)$, $\psi(u) = \gamma(u^2 - \beta^2)^2/4$, $\gamma > 0$, $\Omega$ is the interior of the rectangle $[0, L_1]\times[0, L_2]$, and $\nu$ is the outward pointing normal to $\partial\Omega$. This initial-boundary value problem arises in the study of phase separation in binary mixtures [1 – 2]. In [3] a continuous in time Morley finite element Galerkin approximation for (1) is presented and an optimal-order error bound in $L_2$ derived. However, a nonlinear system of ordinary differential equations remains to be solved. The authors of [4] developed a completely discrete difference scheme for (1), which was also nonlinear. In this paper, a linearized finite difference scheme is derived for (1) by *the method of reduction of order* [5 – 7] (see §4 below). The coefficient matrix of the difference system is symmetric and positive definite, so many well-known iterative methods (e.g. Gauss-Seidel, SOR) can be used to solve the system. We prove that the difference scheme is uniquely solvable and second-order convergent in a discrete $L_2$-norm.

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Let $M_1, M_2, K$ be integers and $h_1 = L_1/M_1$, $h_2 = L_2/M_2$, $\tau = T/K$ such that $h_1 = \alpha_1 h$, $h_2 = \alpha_2 h$, $\tau = \alpha_3 h^{\epsilon+1/2}$, where $\alpha_1, \alpha_2, \alpha_3$ and $\epsilon$ are positive constants. The optimal choice for $\epsilon$ is $1/2$. We use the notations

$$\Omega_h = \{(x_i, y_j) | x_i = ih_1, y_j = jh_2, 0 \le i \le M_1, 0 \le j \le M_2\},$$

$$\Omega_\tau = \{t_k | t_k = k\tau, 0 \le k \le K\}.$$

Suppose $u = \{u_{ij} | 0 \le i \le M_1, 0 \le j \le M_2\}$ and $v = \{v_{ij} | 0 \le i \le M_1, 0 \le j \le M_2\}$ are two mesh functions on $\Omega_h$. Denote

$$D_{+x} u_{ij} = (u_{i+1,j} - u_{ij})/h_1, \quad D_{-x} u_{ij} = D_{+x} u_{i-1,j}, \quad \delta_x^2 u_{ij} = D_{+x} D_{-x} u_{ij};$$

$$D_{+y} u_{ij} = (u_{i,j+1} - u_{ij})/h_2, \quad D_{-y} u_{ij} = D_{+y} u_{i,j-1}, \quad \delta_y^2 u_{ij} = D_{+y} D_{-y} u_{ij};$$

$$u_{i+1/2,j} = (u_{i+1,j} + u_{ij})/2, \quad u_{i,j+1/2} = (u_{i,j+1} + u_{ij})/2$$

and define the inner product

$$\begin{aligned}(u, v) = h_1 h_2 \Bigg[ & \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} u_{ij} v_{ij} \\ & + \frac{1}{2} \sum_{i=1}^{M_1-1} (u_{i0} v_{i0} + u_{i,M_2} v_{i,M_2}) + \frac{1}{2} \sum_{j=1}^{M_2-1} (u_{0j} v_{0j} + u_{M_1,j} v_{M_1,j}) \\ & + \frac{1}{4} (u_{00} v_{00} + u_{M_1,0} v_{M_1,0} + u_{0,M_2} v_{0,M_2} + u_{M_1,M_2} v_{M_1,M_2}) \Bigg]\end{aligned}$$

and the discrete $L_2$-norm

$$\|u\| = \sqrt{(u, u)}.$$

In addition, if $w = \{w^k | 0 \le k \le K\}$ is a mesh function on $\Omega_\tau$, we use the notation

$$w^{\bar{k}} = (w^{k+1} + w^{k-1})/2, \quad \Delta_t w^k = (w^{k+1} - w^{k-1})/(2\tau).$$

It is obvious that

$$(u, v) = h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} (u_{ij} v_{ij} + u_{i-1,j} v_{i-1,j} + u_{i,j-1} v_{i,j-1} + u_{i-1,j-1} v_{i-1,j-1})/4$$

and

$$\Delta_t w^k = (w^{\bar{k}} - w^{k-1})/\tau.$$

Let

$$v = \phi(u) - \Delta u;$$

then (1) is equivalent to

$$u_t = \Delta v, \quad (x, y, t) \in \Omega \times (0, T], \tag{2.1}$$

$$v = \phi(u) - \Delta u, \quad (x, y, t) \in \Omega \times (0, T], \tag{2.2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L_1} = \left.\frac{\partial u}{\partial y}\right|_{y=0} = \left.\frac{\partial u}{\partial y}\right|_{y=L_2} = 0, \quad 0 \le x \le L_1, 0 \le y \le L_2, 0 \le t \le T, \tag{2.3}$$

$$\left.\frac{\partial v}{\partial x}\right|_{x=0} = \left.\frac{\partial v}{\partial x}\right|_{x=L_1} = \left.\frac{\partial v}{\partial y}\right|_{y=0} = \left.\frac{\partial v}{\partial y}\right|_{y=L_2} = 0, \quad 0 \le x \le L_1, 0 \le y \le L_2, 0 \le t \le T, \tag{2.4}$$

$$u(x, y, 0) = u_0(x, y), \qquad (x, y) \in \overline{\Omega}. \tag{2.5}$$

Our difference scheme for (2) is as follows:

$$u_{ij}^0 = u_0(x_i, y_j), \quad u_{ij}^1 = u_0(x_i, y_j) + \tau u_1(x_i, y_j), \quad 0 \le i \le M_1, 0 \le j \le M_2; \tag{3.1}$$

for $1 \le k \le K-1$,

$$\Delta_t u_{ij}^k = \delta_x^2 v_{ij}^k + \delta_y^2 v_{ij}^k, \quad 1 \le i \le M_1 - 1, 1 \le j \le M_2 - 1, \tag{3.2}$$

$$\Delta_t u_{i0}^k = \delta_x^2 v_{i0}^k + 2D_{+y} v_{i0}^k / h_2, \quad 1 \le i \le M_1 - 1, \tag{3.3}$$

$$\Delta_t u_{i,M_2}^k = \delta_x^2 v_{i,M_2}^k - 2D_{-y} v_{i,M_2}^k / h_2, \quad 1 \le i \le M_1 - 1, \tag{3.4}$$

$$\Delta_t u_{0j}^k = 2D_{+x} v_{0j}^k / h_1 + \delta_y^2 v_{0j}^k, \quad 1 \le j \le M_2 - 1, \tag{3.5}$$

$$\Delta_t u_{M_1,j}^k = -2D_{-x} v_{M_1,j}^k / h_1 + \delta_y^2 v_{M_1,j}^k, \quad 1 \le j \le M_2 - 1, \tag{3.6}$$

$$\Delta_t u_{00}^k = 2D_{+x} v_{00}^k / h_1 + 2D_{+y} v_{00}^k / h_2, \tag{3.7}$$

$$\Delta_t u_{M_1,0}^k = -2D_{-x} v_{M_1,0}^k / h_1 + 2D_{+y} v_{M_1,0}^k / h_2, \tag{3.8}$$

$$\Delta_t u_{0,M_2}^k = 2D_{+x} v_{0,M_2}^k / h_1 - 2D_{-y} v_{0,M_2}^k / h_2, \tag{3.9}$$

$$\Delta_t u_{M_1,M_2}^k = -2D_{-x} v_{M_1,M_2}^k / h_1 - 2D_{-y} v_{M_1,M_2}^k / h_2, \tag{3.10}$$

$$v_{ij}^k = \phi(u_{ij}^k) - (\delta_x^2 u_{ij}^{\bar{k}} + \delta_y^2 u_{ij}^{\bar{k}}), \quad 1 \le i \le M_1 - 1, 1 \le j \le M_2 - 1, \tag{3.11}$$

$$v_{i0}^k = \phi(u_{i0}^k) - (\delta_x^2 u_{i0}^{\bar{k}} + 2D_{+y} u_{i0}^{\bar{k}} / h_2), \quad 1 \le i \le M_1 - 1, \tag{3.12}$$

$$v_{i,M_2}^k = \phi(u_{i,M_2}^k) - (\delta_x^2 u_{i,M_2}^{\bar{k}} - 2D_{-y} u_{i,M_2}^{\bar{k}} / h_2), \quad 1 \le i \le M_1 - 1, \tag{3.13}$$

$$v_{0j}^k = \phi(u_{0j}^k) - (2D_{+x} u_{0j}^{\bar{k}} / h_1 + \delta_y^2 u_{0j}^{\bar{k}}), \quad 1 \le j \le M_2 - 1, \tag{3.14}$$

$$v_{M_1,j}^k = \phi(u_{M_1,j}^k) - (-2D_{-x} u_{M_1,j}^{\bar{k}} / h_1 + \delta_y^2 u_{M_1,j}^{\bar{k}}), \quad 1 \le j \le M_2 - 1, \tag{3.15}$$

$$v_{00}^k = \phi(u_{00}^k) - (2D_{+x} u_{00}^{\bar{k}} / h_1 + 2D_{+y} u_{00}^{\bar{k}} / h_2), \tag{3.16}$$

$$v_{M_1,0}^k = \phi(u_{M_1,0}^k) - (-2D_{-x} u_{M_1,0}^{\bar{k}} / h_1 + 2D_{+y} u_{M_1,0}^{\bar{k}} / h_2), \tag{3.17}$$

$$v_{0,M_2}^k = \phi(u_{0,M_2}^k) - (2D_{+x} u_{0,M_2}^{\bar{k}} / h_1 - 2D_{-y} u_{0,M_2}^{\bar{k}} / h_2), \tag{3.18}$$

$$v_{M_1,M_2}^k = \phi(u_{M_1,M_2}^k) - (-2D_{-x} u_{M_1,M_2}^{\bar{k}} / h_1 - 2D_{-y} u_{M_1,M_2}^{\bar{k}} / h_2), \tag{3.19}$$

where $u_1 = \Delta(\phi(u_0) - \Delta u_0)$.

The relations (3.2)–(3.19) can be rewritten in vector-matrix form as

$$(u^{\bar{k}} - u^{k-1})/\tau = -Av^k, \tag{4.1}$$

$$v^k = \phi(u^k) + Au^{\bar{k}}, \tag{4.2}$$

where

$$\begin{aligned}
&(Aw)_{ij} = -(\delta_x^2 w_{ij} + \delta_y^2 w_{ij}), \quad 1 \le i \le M_1 - 1,\, 1 \le j \le M_2 - 1, \\
&(Aw)_{i0} = -(\delta_x^2 w_{i0} + 2D_{+y} w_{i0}/h_2), \quad 1 \le i \le M_1 - 1, \\
&(Aw)_{i,M_2} = -(\delta_x^2 w_{i,M_2} - 2D_{-y} w_{i,M_2}/h_2), \quad 1 \le i \le M_1 - 1, \\
&(Aw)_{0j} = -(2D_{+x} w_{0j}/h_1 + \delta_y^2 w_{0j}), \quad 1 \le j \le M_2 - 1, \\
&(Aw)_{M_1,j} = -(-2D_{-x} w_{M_1,j}/h_1 + \delta_y^2 w_{M_1,j}), \quad 1 \le j \le M_2 - 1, \\
&(Aw)_{00} = -(2D_{+x} w_{00}/h_1 + 2D_{+y} w_{00}/h_2), \\
&(Aw)_{M_1,0} = -(-2D_{-x} w_{M_1,0}/h_1 + 2D_{+y} w_{M_1,0}/h_2), \\
&(Aw)_{0,M_2} = -(2D_{+x} w_{0,M_2}/h_1 - 2D_{-y} w_{0,M_2}/h_2), \\
&(Aw)_{M_1,M_2} = -(-2D_{-x} w_{M_1,M_2}/h_1 - 2D_{-y} w_{M_1,M_2}/h_2); \\
&(\phi(u^k))_{ij} = \phi(u_{ij}^k), \qquad 0 \le i \le M_1,\; 0 \le j \le M_2.
\end{aligned} \tag{5}$$

Substituting (4.2) into (4.1), we obtain

$$(I + \tau A^2) u^{\bar{k}} = u^{k-1} - \tau A \phi(u^k), \quad 1 \le k \le K - 1, \tag{6}$$

where $I$ is an $(M_1 + 1) \times (M_2 + 1)$ unit matrix. If $u^{\bar{k}}$ is determined, then $u^{k+1} = 2u^{\bar{k}} - u^{k-1}$. We construct the difference scheme (3.1) and (6) for (1.1-3).

The main result of this paper is the following theorem, which is proved in §3.

**Theorem 1.1.** (I) *The difference scheme* (3.1) *and* (6) *is uniquely solvable.*

(II) *If* (1.1-3) *has solution* $u(x, y, t) \in C^6(\overline{\Omega} \times [0, T])$, *then the solution of the difference scheme* (3.1) *and* (6) *converges to the solution of* (1.1-3) *in the discrete* $L_2$*-norm, and the rate of convergence is* $O(h_1^2 + h_2^2 + \tau^2)$.

## 2. Some anxiliary lemmas

**Lemma 2.1.** *Let* $A$ *be defined in* (5); *then* $A$ *is symmetric and positive semidefinite.*

*Proof*. Through simple and trivial calculations, we may obtain $(Au, v) - (u, Av) = 0$ and $(Au, u) \ge 0$ for any mesh functions $u, v$ on $\Omega_h$. So $A$ is symmetric and positive semidefinite.

**Lemma 2.2.** *If* $f \in C^4[a, b]$ *and*

$$\frac{df}{dx}(a) = \frac{d^3 f}{dx^3}(a) = \frac{df}{dx}(b) = \frac{d^3 f}{dx^3}(b) = 0, \tag{7}$$

*then*

$$\frac{d^2 f}{dx^2}(a) = \frac{2}{h^2}[f(a+h) - f(a)] + O(h^2), \tag{8.1}$$

$$\frac{d^2 f}{dx^2}(b) = -\frac{2}{h^2}[f(b) - f(b-h)] + O(h^2) \tag{8.2}$$

*for small* $h$.

*Proof*. Using Taylor expansion, we have

$$f(a+h) = f(a) + \frac{df}{dx}(a)h + \frac{1}{2}\frac{d^2f}{dx^2}(a)h^2 + \frac{1}{6}\frac{d^3f}{dx^3}(a)h^3 + O(h^4).$$

Noticing (7), we obtain

$$f(a+h) = f(a) + \frac{1}{2}\frac{d^2f}{dx^2}(a)h^2 + O(h^4).$$

It follows that

$$\frac{d^2f}{dx^2}(a) = \frac{2}{h^2}[f(a+h) - f(a)] + O(h^2).$$

This is (8.1). The other relation (8.2) can be obtained similarly. □

**Lemma 2.3.** *Let* $c_1, c_2$ *and* $a_k$, $k = 1, 2, 3, \ldots$, *be positive and satisfy*

$$a_{k+1} \le (1 + c_1\tau)a_k + c_2\tau, \quad k = 1, 2, 3, \ldots ;$$

*then*

$$a_{k+1} \le \exp(c_1 k\tau)(a_1 + c_2/c_1), \quad k = 1, 2, 3, \ldots .$$

*Proof*. We have

$$\begin{aligned} a_{k+1} &\le (1 + c_1\tau)a_k + c_2\tau \\ &\le (1 + c_1\tau)[(1 + c_1\tau)a_{k-1} + c_2\tau] + c_2\tau \\ &= (1 + c_1\tau)^2 a_{k-1} + [(1 + c_1\tau) + 1]c_2\tau \\ &\le \cdots \\ &\le (1 + c_1\tau)^k a_1 + [(1 + c_1\tau)^{k-1} + (1 + c_1\tau)^{k-2} + \cdots + (1 + c_1\tau) + 1]c_2\tau \\ &= (1 + c_1\tau)^k a_1 + \{[(1 + c_1\tau)^k - 1]/[(1 + c_1\tau) - 1]c_2\}\tau \\ &\le \exp(c_1 k\tau)(a_1 + c_2/c_1), \quad k = 1, 2, 3, \ldots . \quad \square \end{aligned}$$

## 3. The analysis of the difference scheme

We now come to the proof of Theorem 1.1. From Lemma 2.1 we see that the coefficient matrix of the system of linear algebraic equations (6) is symmetric and positive definite. So the difference scheme (3.1) and (6) is uniquely solvable. This completes the proof of the first part of the theorem.

Since $\phi(u) = \frac{1}{4}\frac{d}{du}[\gamma(u^2 - \beta^2)^2]$, we have

$$\frac{\partial\phi(u)}{\partial x} = \gamma(3u^2 - \beta^2)\frac{\partial u}{\partial x}, \quad \frac{\partial\phi(u)}{\partial y} = \gamma(3u^2 - \beta^2)\frac{\partial u}{\partial y}.$$

Noticing (2.3), we have

$$\text{(9)} \qquad \left.\frac{\partial\phi(u)}{\partial x}\right|_{x=0} = \left.\frac{\partial\phi(u)}{\partial x}\right|_{x=L_1} = \left.\frac{\partial\phi(u)}{\partial y}\right|_{y=0} = \left.\frac{\partial\phi(u)}{\partial y}\right|_{y=L_2} = 0,$$
$$0 \le x \le L_1, \, 0 \le y \le L_2, \, 0 \le t \le T.$$

Differentiating (2.1) with respect to $x$, we have

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

or

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^3 v}{\partial x^3} + \frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x}\right).$$

Noticing (2.3) and (2.4), we obtain

$$\left.\frac{\partial^3 v}{\partial x^3}\right|_{x=0} = \left.\frac{\partial^3 v}{\partial x^3}\right|_{x=L_1} = 0, \quad 0 \le y \le L_2, \, 0 < t \le T. \tag{10.1}$$

Differentiating (2.1) with respect to $y$, we obtain

$$\left.\frac{\partial^3 v}{\partial y^3}\right|_{y=0} = \left.\frac{\partial^3 v}{\partial y^3}\right|_{y=L_2} = 0, \quad 0 \le x \le L_1, \, 0 < t \le T. \tag{10.2}$$

Similarly, differentiating (2.2) with respect to $x$ and $y$, respectively, and using (2.3), (2.4) and (9), we get

$$\begin{gathered}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x=0} = \left.\frac{\partial^3 u}{\partial x^3}\right|_{x=L_1} = \left.\frac{\partial^3 u}{\partial y^3}\right|_{y=0} = \left.\frac{\partial^3 u}{\partial y^3}\right|_{y=L_2} = 0, \\ 0 \le x \le L_1, \, 0 \le y \le L_2, \, 0 < t \le T.\end{gathered} \tag{11}$$

Define the following mesh functions on $\Omega_h \times \Omega_\tau$ :

$$U_{ij}^k = u(x_i, y_j, t_k), \quad V_{ij}^k = v(x_i, y_j, t_k), \quad \tilde{u}_{ij}^k = U_{ij}^k - u_{ij}^k, \quad \tilde{v}_{ij}^k = V_{ij}^k - v_{ij}^k.$$

Using Lemma 2.2 and Taylor expansion, noticing (2.3), (2.4), (10) and (11), we obtain the error equations of the difference scheme (3.1) and (6) as follows:

$$\tilde{u}^0 = 0, \quad \tilde{u}^1 = R, \tag{12.1}$$

$$\Delta_t \tilde{u}^k = -A\tilde{v}^k + F^k, \quad 1 \le k \le K-1, \tag{12.2}$$

$$\tilde{v}^k = \phi(U^k) - \phi(u^k) + A\tilde{u}^{\bar{k}} + G^k, \quad 1 \le k \le K-1, \tag{12.3}$$

where

$$(\Delta_t \tilde{u}^k)_{ij} = (\tilde{u}_{ij}^{k+1} - \tilde{u}_{ij}^{k-1})/(2\tau), \quad (\phi(U^k) - \phi(u^k))_{ij} = \phi(U_{ij}^k) - \phi(u_{ij}^k),$$

$$R = (r_{ij}), \quad F^k = (f_{ij}^k), \quad G^k = (g_{ij}^k)$$

and there exists a constant $c_1$ such that

$$|r_{ij}| \le c_1 \tau^2, \tag{13.1}$$

$$|f_{ij}^k| \le c_1(h_1^2 + h_2^2 + \tau^2), \quad |g_{ij}^k| \le c_1(h_1^2 + h_2^2 + \tau^2), \tag{13.2}$$

because of the assumption that the solution $u(x, y, t)$ belongs to $C^6(\overline{\Omega} \times [0, T])$.

Denote

$$c_2 = \max_{0 \le x \le L_1, \, 0 \le y \le L_2, \, 0 \le t \le T} |u(x, y, t)|, \qquad c_3 = \max_{c_2 - 1 \le z \le c_2 + 1} |d\phi(z)/dz|.$$

We will prove that

$$\left\|\tilde{u}^k\right\| \le c_4(h_1^2 + h_2^2 + \tau^2), \tag{14}$$

where

$$c_4 = \exp\left(\frac{3}{2}(1 + c_3{}^2)T\right) c_1 \sqrt{L_1 L_2 (1 + \frac{2}{1 + c_3{}^2})}.$$

From (13) and (12.1), we have

$$\left\|\tilde{u}^0\right\| = 0, \qquad \left\|\tilde{u}^1\right\| \leq \sqrt{L_1L_2}c_1\tau^2, \tag{15.1}$$

$$\left\|f^k\right\| \leq \sqrt{L_1L_2}c_1(h_1^2+h_2^2+\tau^2), \qquad \left\|g^k\right\| \leq \sqrt{L_1L_2}c_1(h_1^2+h_2^2+\tau^2), \qquad 1 \leq k \leq K-1. \tag{15.2}$$

It follows from (15.1) that (14) is valid for $k = 0$ and $k = 1$. Now suppose (14) is valid for $1 \leq k \leq l$. Then, for small $h$,

$$|\tilde{u}_{ij}^k| \leq 2c_4(h_1^2+h_2^2+\tau^2)/\sqrt{h_1h_2} \leq 1, \quad 0 \leq i \leq M_1, 0 \leq j \leq M_2, 1 \leq k \leq l,$$

and therefore

$$\left|\phi(U_{ij}^k) - \phi(u_{ij}^k)\right| \leq c_3\left|\tilde{u}_{ij}^k\right|, \quad 0 \leq i \leq M_1, 0 \leq j \leq M_2, 1 \leq k \leq l.$$

For $1 \leq k \leq l$, taking the inner product of (12.2) with $2\tilde{u}^{\bar{k}}$, and (12.3) with $2\tilde{v}^k$, then adding the results and using Lemma 2.1, we obtain

$$\begin{aligned}
&2(\tilde{u}^{\bar{k}}, \Delta_t\tilde{u}^k) + 2\left\|\tilde{v}^k\right\|^2 \\
&\quad \doteq 2[-(\tilde{u}^{\bar{k}}, A\tilde{v}^k) + (\tilde{v}^k, A\tilde{u}^{\bar{k}})] + 2(\tilde{v}^k, \phi(U^k) - \phi(u^k)) \\
&\qquad + 2(\tilde{u}^{\bar{k}}, F^k) + 2(\tilde{v}^k, G^k) \\
&\quad = 2(\tilde{v}^k, \phi(U^k) - \phi(u^k)) + 2(\tilde{u}^{\bar{k}}, F^k) + 2(\tilde{v}^k, G^k) \\
&\quad \leq \left\|\tilde{v}^k\right\|^2 + \left\|\phi(U^k) - \phi(u^k)\right\|^2 + \left\|\tilde{u}^{\bar{k}}\right\|^2 + \left\|\tilde{F}^k\right\|^2 + \left\|\tilde{v}^k\right\|^2 + \left\|\tilde{G}^k\right\|^2 \\
&\quad \leq 2\left\|\tilde{v}^k\right\|^2 + \left\|\tilde{u}^{\bar{k}}\right\|^2 + {c_3}^2\left\|\tilde{u}^k\right\|^2 + (\left\|\tilde{F}^k\right\|^2 + \left\|\tilde{G}^k\right\|^2)
\end{aligned}$$

or,

$$\begin{aligned}
&(\left\|\tilde{u}^{k+1}\right\|^2 - \left\|\tilde{u}^{k-1}\right\|^2)/(2\tau) \\
&\quad \leq \left\|\tilde{u}^{\bar{k}}\right\|^2 + {c_3}^2\left\|\tilde{u}^k\right\|^2 + (\left\|\tilde{F}^k\right\|^2 + \left\|\tilde{G}^k\right\|^2) \\
&\quad \leq (\left\|\tilde{u}^{k+1}\right\|^2 + \left\|\tilde{u}^{k-1}\right\|^2)/2 + {c_3}^2\left\|\tilde{u}^k\right\|^2 + (\left\|\tilde{F}^k\right\|^2 + \left\|\tilde{G}^k\right\|^2).
\end{aligned}$$

Thus,

$$(1-\tau)\left\|\tilde{u}^{k+1}\right\|^2 \leq (1+\tau)\left\|\tilde{u}^{k-1}\right\|^2 + 2{c_3}^2\tau\left\|\tilde{u}^k\right\|^2 + 2\tau(\left\|\tilde{F}^k\right\|^2 + \left\|\tilde{G}^k\right\|^2).$$

When $\tau \leq 1/3$,

$$\left\|\tilde{u}^{k+1}\right\|^2 \leq (1+3\tau)\left\|\tilde{u}^{k-1}\right\|^2 + 3{c_3}^2\tau\left\|\tilde{u}^k\right\|^2 + 3\tau(\left\|\tilde{F}^k\right\|^2 + \left\|\tilde{G}^k\right\|^2).$$

From the above inequality and (15.2), we have

$$\begin{aligned}
&\max(\left\|\tilde{u}^{k+1}\right\|^2, \left\|\tilde{u}^k\right\|^2) \\
&\quad \leq [1 + 3(1+{c_3}^2)\tau]\max(\left\|\tilde{u}^k\right\|^2, \left\|\tilde{u}^{k-1}\right\|^2) + 3\tau(\left\|F^k\right\|^2 + \left\|G^k\right\|^2) \\
&\quad \leq [1 + 3(1+{c_3}^2)\tau]\max(\left\|\tilde{u}^k\right\|^2, \left\|\tilde{u}^{k-1}\right\|^2) \\
&\qquad + 6\tau L_1L_2{c_1}^2(h_1^2+h_2^2+\tau^2)^2, \ 1 \leq k \leq l.
\end{aligned}$$

Utilizing Lemma 2.3 and (15.1), we have

$$\begin{aligned}
&\max(\left\|\tilde{u}^{l+1}\right\|^2, \left\|\tilde{u}^{l}\right\|^2) \\
&\le \exp\left(3(1+c_3{}^2)l\tau\right)[\max(\left\|\tilde{u}^{1}\right\|^2, \left\|\tilde{u}^{0}\right\|^2) + \frac{6L_1L_2c_1{}^2}{3(1+c_3{}^2)}(h_1^2+h_2^2+\tau^2)^2] \\
&\le \exp\left(3(1+c_3{}^2)l\tau\right)[L_1L_2c_1{}^2\tau^4 + \frac{2L_1L_2c_1{}^2}{1+c_3{}^2}(h_1^2+h_2^2+\tau^2)^2] \\
&\le \exp\left(3(1+c_3{}^2)T\right)L_1L_2(1+\frac{2}{1+c_3{}^2})c_1{}^2(h_1^2+h_2^2+\tau^2)^2.
\end{aligned}$$

or

$$\left\|\tilde{u}^{l+1}\right\| \le \exp\left(\frac{3}{2}(1+c_3{}^2)T\right)c_1\sqrt{L_1L_2(1+\frac{2}{1+c_3{}^2})}\,(h_1^2+h_2^2+\tau^2).$$

By the induction principle, (14) is true. This completes the proof of Theorem 1.1. □

## 4. Comments

In this paper we use the method of reduction of order to derive the linearized difference scheme (3.1) and (6) for (1.1-3). First, a new variable $v$ is introduced to reduce the original problem into an equivalent system of second-order differential equations (2.1-5), and a difference scheme (3.1-19) is constructed for the latter. Then, the discrete variables are separated to obtain the difference scheme (3.1) and (6) containing only the original variable $u$. The aim of introducing the intermediate variable $v$ is to prove the solvability and convergence of the difference scheme (3.1) and (6).

A difference scheme similar to (3.1) and (6) may be constructed [6] on nonuniform meshes, and similar results hold if we rewrite (1.1-3) as the following equivalent system of first-order differential equations:

$$\begin{aligned}
&u_t = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}, \quad v_1 = \frac{\partial v}{\partial x}, \quad v_2 = \frac{\partial v}{\partial y}, \qquad (x, y, t) \in \Omega \times (0, T], \\
&v = \phi(u) - (\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}), \quad u_1 = \frac{\partial u}{\partial x}, \quad u_2 = \frac{\partial u}{\partial y}, \qquad (x, y, t) \in \Omega \times (0, T], \\
&u_1|_{x=0} = u_1|_{x=L_1} = u_2|_{y=0} = u_2|_{y=L_2} = 0, \quad 0 \le x \le L_1, 0 \le y \le L_2, 0 \le t \le T, \\
&v_1|_{x=0} = v_1|_{x=L_1} = v_2|_{y=0} = v_2|_{y=L_2} = 0, \quad 0 \le x \le L_1, 0 \le y \le L_2, 0 \le t \le T, \\
&u(x, y, 0) = u_0(x, y), \qquad (x, y) \in \overline{\Omega}.
\end{aligned}$$

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