Progress of Theoretical Physics, Vol. 71, No. 6, June 1984

A Self-Consistent Mean-Field Approach to the Dynamical Symmetry Breaking

—— *The Effective Potential of the Nambu and Jona-Lasinio Model* ——

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(Received August 4, 1983)

The dynamical symmetry breaking phenomena in the Nambu and Jona-Lasinio model are reexamined in the framework of a self-consistent mean-field (SCMF) theory. First, we formulate the SCMF theory in a lucid manner based on a successful decomposition of the Lagrangian into semiclassical and residual interaction parts by imposing a condition that "the dangerous term" in Bogoliubov's sense should vanish. Then, we show that the difference of the energy density between the super and normal phases, the correct expression of which the original authors failed to give, can be readily obtained by applying the SCMF theory. Furthermore, it is shown that the expression thus obtained is identical to that of the effective potential (E. P.) given by the path-integral method with an auxiliary field up to the one loop order in the loop expansion, then one finds a new and simple way to get the E. P. Some numerical results of the E. P. and the dynamically generated mass of fermion are also shown. As another demonstration of the powerfulness of the SCMF theory, we derive, in the Appendix, the energy density of the $O(N)$-ϕ^4 model including the higher order corrections in the sense of large N expansion.

§ 1. Introduction

The dynamical symmetry breaking (DSB) phenomena of the vacuum are essentially due to nonperturbative effects of interaction, as phase transitions of matter are. For example, one must include all order diagrams of some kind in the calculation of the so-called the effective potential (E. P.),[1] the minimum point of which gives the true vacuum. To calculate the E. P., the path-integral method has been adopted[2] so far. This method is systematic and elegant but not so easy to have an insight into the nature of the E.P. Moreover, actual calculations have been performed mainly up to the one-loop order in the loop expansion.[2),3] There is another and simple way to calculate the vacuum energy; to show that is the case is one of the main subjects of this report.

Our method is based on a self-consistent mean-field (SCMF) theory, which has been always a starting point to approach problems of phase transitions (at least of the second kind) of matter; the magnetic phase transitions, the superfluidity, the superconductivity and so on.[4] For instance, the BCS theory of the superconductivity turned out to be an SCMF theory as was demonstrated by Bogoliubov, Valatin and Gorkov. In this paper, by way of an SCMF approach, we examine the DSB in the Nambu and Jona-Lasinio (NJL)[5] model which is based on an analogy with the superconductivity. This model initiated the physics of the DSB in the elementary particle physics, and has been a useful theoretical laboratory in helping us to understand the DSB. Furthermore, the relevance of the NJL model to QCD (the quantum chromodynamics) has been discussed in recent literature.[6] Therefore, it is worthwhile to gain a deeper understanding of the model. It is certain that Nambu and Jona-Lasinio also intended to develop an SCMF theory of their model, however they failed in reducing the original Lagrangian to the proper one for the SCMF theory. As a result, they could not give the correct energy of the vacuum, which was given later by Suzuki[7] with the use of a variational method à la BCS. In this paper, we will give a correct formulation of the SCMF theory and discuss the vacuum of the NJL model.

In § 2, we develop the SCMF theory of the NJL model. In §3, we calculate the energy density of the vacuum in this framework and show that the result is identical to that given by Suzuki[7] and also the E.P. obtained by the path-integral method with an auxiliary field up to the one-loop order; hence one finds a new and simple way to get the E.P. It is also shown that the self-consistency condition to determine the mass (the gap equation) is nothing but the condition that the E.P. should take the extremum at the true vacuum. As is well known, the E.P. for a composite operator can be interpreted as the energy density of the vacuum only at its minimum point.[3] In Appendix B, we give the energy density of the state with a given condensation $\langle\bar{\psi}\psi\rangle$ by introducing an external field coupled to the composite field $\bar{\psi}\psi$, for completeness. In §4, we give some numerical results of the E.P. and the dynamically generated fermion mass. The final section is devoted to concluding remarks, where it is indicated that the present approach is applicable to a wide range of the many-body systems. As one of the applications, the energy density of the vacuum of the $O(N)$-ϕ^4 model[3),8)] is derived in Appendix C, where it is shown that the infinite series of higher order corrections in the sense of large N expansion can be readily obtained.

§ 2. Formulation of the SCMF theory

We start with the following Lagrangian density,[5]

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1 \tag{2·1a}$$

with

$$\mathcal{L}_0=\bar{\psi}[i\gamma\cdot\partial]\psi\,, \tag{2·1b}$$

$$\mathcal{L}_1=g[(\bar{\psi}\psi)^2+(\bar{\psi}i\gamma_5\psi)^2], \tag{2·1c}$$

where ψ represents the Dirac field (quark) with N_c colors. Flavours are ignored for simplicity. The coupling constant g has dimensions of $[\text{mass}]^{-2}$ and is assumed to be positive so that the attractive force between quark and anti-quark is guaranteed. Since this theory is not renormalizable, we must introduce a cutoff Λ. We will regard that masses or energies which will appear in the development of the theory have a physical significance only when they are smaller than Λ. Here, we note that the Lagrangian is invariant under the chiral transformation,

$$\psi\to e^{i\gamma_5\theta/2}\cdot\psi \quad\text{and}\quad \bar{\psi}\to\bar{\psi}\cdot e^{i\gamma_5\theta/2}\,. \tag{2·2}$$

In the SCMF theory, it is assumed that the system is well described as an assembly of the non-interacting particles moving in the mean-fields (MF) which are generated by the particles self-consistently. In the present case, four types of MF arise; scalar, pseudo-scalar, vector and axial-vector. Keeping this in mind, we first decompose the Lagrangian density (2·1) into two parts by introducing c-number fields σ, π, V_μ and A_μ:

$$\mathcal{L}=\mathcal{L}_{\text{MFA}}+\mathcal{L}_{\text{res}}\,, \tag{2·3a}$$

where

$$\mathcal{L}_{\rm MFA}=\bar{\psi}[i\gamma\cdot\partial-G(\sigma(x)+i\pi(x)\gamma_5+V_\mu(x)\gamma^\mu/\sqrt{N_c}+A_\mu(x)\gamma_5\gamma^\mu/\sqrt{N_c})]\psi$$
$$-\mu^2/2\cdot[\sigma^2(x)+\pi^2(x)+2V_\mu^2(x)+2A_\mu^2(x)], \tag{2·3b}$$

$$\mathcal{L}_{\rm res}=g[(\bar{\psi}\psi+\mu^2/G\cdot\sigma(x))^2+(\bar{\psi}i\gamma_5\psi+\mu^2/G\cdot\pi(x))^2]$$
$$+G/\sqrt{N_c}\cdot\bar{\psi}(V_\mu(x)\gamma^\mu+A_\mu(x)\gamma_5\gamma^\mu)\psi+\mu^2(V_\mu^2(x)+A_\mu^2(x)) \tag{2·3c}$$

with

$$(G/\mu)^2=2g\,.$$

Here, we have introduced a constant μ with mass dimension to make the c-number fields have the dimension of boson. G is a dimensionless constant. Note that this decomposition is merely an identity. The physics comes as follows: If one takes the interaction picture of $\mathcal{L}_{\rm res}$ and imposes proper conditions on the c-number fields, $\mathcal{L}_{\rm res}$ becomes a normal product of $\mathcal{L}_1$ with respect to the vacuum of $\mathcal{L}_{\rm MFA}$. Let us show this in the following.

The field equation of fermion in the interaction picture reads

$$i\gamma\cdot\partial\psi(x)=G[\sigma(x)+i\pi(x)\gamma_5+V_\mu(x)\gamma^\mu/\sqrt{N_c}+A_\mu(x)\gamma_5\gamma^\mu/\sqrt{N_c}]\psi(x). \tag{2·4}$$

Then, $\psi(x)$ can be decomposed as

$$\psi(x)=\sum_k[a_k u_k(x)+b_k^\dagger v_k(x)], \tag{2·5}$$

where u_k and v_k are the classical solutions to Eq. (2·4), which become the positive and negative energy solutions, respectively, when the MF are time-independent. The "vacuum" of $\mathcal{L}_{\rm MFA}$ which we denote by $|[\sigma,\pi,V_\mu,A_\mu]\rangle$ is naturally defined by

$$a_k|[\sigma,\pi,V_\mu,A_\mu]\rangle=b_k|[\sigma,\pi,V_\mu,A_\mu]\rangle=0\,. \tag{2·6}$$

In our interaction picture, using Wick's theorem, $\mathcal{L}_{\rm res}$ can be expanded as

$$\mathcal{L}_{\rm res}=g[:(\bar{\psi}\psi)^2:+:(\bar{\psi}i\gamma_5\psi)^2:]+R_0+R_1\,, \tag{2·7a}$$

where

$$R_0=\Sigma^2(x)+\Pi^2(x)+\frac{1}{2}(\beta_\mu^2(x)+\alpha_\mu^2(x)) \tag{2·7b}$$

and

$$R_1=2(\Sigma(x):\bar{\psi}\psi:+\Pi(x):\bar{\psi}i\gamma_5\psi:)-(\beta_\mu(x):\bar{\psi}\gamma^\mu\psi:+\alpha_\mu(x):\bar{\psi}\gamma_5\gamma^\mu\psi:) \tag{2·7c}$$

with

$$\begin{bmatrix} \Sigma(x) \\ \Pi(x) \\ \beta_\mu(x) \\ \alpha_\mu(x) \end{bmatrix} = \langle[\sigma, \pi, V_\mu, A_\mu]|\bar{\psi}(x) \begin{bmatrix} 1 \\ i\gamma_5 \\ \frac{1}{\sqrt{N_c}}\gamma_\mu \\ \frac{1}{\sqrt{N_c}}\gamma_5\gamma_\mu \end{bmatrix} \psi(x)|[\sigma, \pi, V_\mu, A_\mu]\rangle + \mu^2/G\cdot \begin{bmatrix} \sigma(x) \\ \pi(x) \\ 2V_\mu(x) \\ 2A_\mu(x) \end{bmatrix}. \tag{2·7d}$$

We note that the vector and axial-vector terms in (2·7b, c) come out from the Fock terms of $\mathcal{L}_{\rm res}$, which indicate that they are next to leading terms in the sense of large N_c expansion. Now we impose the condition that R_1 (the dangerous term in Bogoliubov's sense) should vanish,

$$\Sigma(x) = \Pi(x) = \beta_\mu(x) = \alpha_\mu(x) = 0\,. \tag{2·8}$$

Then, the residual interaction $\mathcal{L}_{\rm res}$ becomes a normal product of $\mathcal{L}_1$,

$$\begin{aligned}\mathcal{L}_{\rm res} &= g[:(\bar{\psi}\psi)^2: + :(\bar{\psi}i\gamma_5\psi)^2:] \\ &= :\mathcal{L}_1:\,,\end{aligned} \tag{2·9a}$$

and

$$\langle[\sigma, \pi, V_\mu, A_\mu]|\mathcal{L}_{\rm res}|[\sigma, \pi, V_\mu, A_\mu]\rangle = 0 \tag{2·9b}$$

is concluded. If one remembers that the state $|[\sigma, \pi, V_\mu, A_\mu]\rangle$ is defined in terms of the fields σ, π, V_μ and A_μ, one can see that Eq. (2·7d) with (2·8) becomes a self-consistency condition (SCC), which determines the c-number fields as the self-consistent mean fields generated by the fermions,

$$-\mu^2/G\cdot \begin{bmatrix} \sigma(x) \\ \pi(x) \\ 2V_\mu(x) \\ 2A_\mu(x) \end{bmatrix} = \langle[\sigma, \pi, V_\mu, A_\mu]|\bar{\psi}(x) \begin{bmatrix} 1 \\ i\gamma_5 \\ \frac{1}{\sqrt{N_c}}\gamma_\mu \\ \frac{1}{\sqrt{N_c}}\gamma_5\gamma_\mu \end{bmatrix} \psi(x)|[\sigma, \pi, V_\mu, A_\mu]\rangle. \tag{2·10}$$

The SCC will be found to play an essential role in the SCMF theory: It turns out to be the gap equation (mass equation) for the ground state and reduces to the field equations of the mean-fields for the excited states.[9]

Now let us rewrite the SCC in a convenient form for evaluation. If we define the Green's function by

$$S_F(x, y;[\sigma, \pi, V_\mu, A_\mu]) = -i\langle[\sigma, \pi, V_\mu, A_\mu]|T(\psi(x)\,\bar{\psi}(y))|[\sigma, \pi, V_\mu, A_\mu]\rangle, \tag{2·11}$$

the SCC becomes

$$-\mu^2/G\cdot\begin{bmatrix}\sigma(x)\\ \pi(x)\\ 2V_\mu(x)\\ 2A_\mu(x)\end{bmatrix}=-i\lim_{y\to x+}\mathrm{Tr}\left\{\begin{bmatrix}1\\ i\gamma_5\\ \frac{1}{\sqrt{N_c}}\gamma_\mu\\ \frac{1}{\sqrt{N_c}}\gamma_5\gamma_\mu\end{bmatrix}\cdot S_F(x,\ y;\ [\sigma,\ \pi,\ V_\mu,\ A_\mu])\right\}. \quad (2\cdot12)$$

Equation (2·12) is highly non-linear equations for the MF and include nonperturbative effects of the interaction. With the solution of (2·12), the state $|[\sigma,\ \pi,\ V_\mu,\ A_\mu]\rangle$ is determined through Eq. (2·6); however, some remarks are needed for the ground state (true vacuum), which is the subject of the next section.

Although our formalism is applicable both for the true vacuum and the excited states, we will concentrate on the former in this article: The excited states will be treated in a separate paper.[9]

§ 3. Determination of the vacuum

If the state $|[\sigma,\ \pi,\ V_\mu,\ A_\mu]\rangle$ is the vacuum, on account of the symmetry properties of the state, we can put

$$\sigma(x)=\text{constant}\equiv\sigma_0 \quad\text{and}\quad \pi(x)=V_\mu(x)=A_\mu(x)=0\,. \quad (3\cdot1)$$

In the following, we represent the vacuum state by $|\sigma_0\rangle$ for simplicity. The SCC in this case reads

$$-\mu^2\sigma_0/G=-iN_c\lim_{y\to x+}\mathrm{Tr}\ S_F(x-y;\ M), \quad (3\cdot2\text{a})$$

$$=-iN_c\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}\frac{1}{\not p-M}, \quad (3\cdot2\text{b})$$

$$=2N_c/\Omega\cdot\sum_{\boldsymbol p}\frac{-M}{\sqrt{\boldsymbol p^2+M^2}}, \quad (3\cdot2\text{c})$$

$$=-N_c\Lambda^3/(2\pi^2)\cdot x\left\{\sqrt{1+x^2}-x^2\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right|\right\}_{x=M/\Lambda}, \quad (3\cdot2\text{d})$$

where Ω is the volume of the system, $M=G\sigma_0$, $S_F(x-y;\ M)$ denotes the Feynman propagator with mass M, and Λ is a cutoff of the three-momentum. Equation (3·2) is essentially identical to the mass equation obtained by NJL who used the ladder approximation for the Bethe-Salpeter equation to derive it. As long as $g>\pi^2/(N_c\cdot\Lambda^2)\equiv g_c$, Eq. (3·2d) has a nonvanishing solution for σ_0, although always is there a trivial solution $\sigma_0=0$; there is no compelling reason to exclude one of the two. One can determine the true σ_0 only by comparing the energy of the states corresponding to the two solutions; the true vacuum has the lowest energy. Provided that σ_0 satisfies the SCC Eq. (3·2), the Hamiltonian H can be decomposed as follows in accordance with Eq. (2·3):

$$H=H_{\mathrm{MFA}}+H_{\mathrm{res}}\,, \quad (3\cdot3\text{a})$$

where

$$H_{\rm MFA}=\int d^3x[\bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+M)\psi+\mu^2/2\cdot\sigma_0^2] \tag{3·3b}$$

and

$$H_{\rm res}=-g\int d^3x[:(\bar{\psi}\psi)^2:+:(\bar{\psi}i\gamma_5\psi)^2:], \tag{3·3c}$$

so that

$$\langle\sigma_0|H_{\rm res}|\sigma_0\rangle=0\,. \tag{3·3d}$$

Here, $H_{\rm MFA}$ and $H_{\rm res}$ correspond to $\mathcal{L}_{\rm MFA}$ and $\mathcal{L}_{\rm res}$, respectively. Note that the last term in $H_{\rm MFA}$, by which a double counting of the interaction is avoided has appeared naturally in our formulation but was missed in the original paper: They included this term in the residual interaction $H_{\rm res}$ in our notation, then their residual interaction has a nonvanishing vacuum expectation value; hence they failed in obtaining the correct expression for the vacuum energy.

The energy density of the vacuum which we represent by $E(\sigma_0)$ are defined as

$$E(\sigma_0)\equiv\langle\sigma_0|H|\sigma_0\rangle/\Omega=\langle\sigma_0|H_{\rm MFA}|\sigma_0\rangle/\Omega\,, \tag{3·4a}$$

$$E(0)=\langle 0|H|0\rangle/\Omega\,. \tag{3·4b}$$

They are easily evaluated as

$$E(\sigma_0)=-2N_c/\Omega\cdot\sum_{\boldsymbol{p}}E_{\boldsymbol{p}}^{(M)}+\mu^2/2\cdot\sigma_0^2\,, \tag{3·5a}$$

$$E(0)=-2N_c/\Omega\cdot\sum_{\boldsymbol{p}}E_{\boldsymbol{p}}^{(0)} \tag{3·5b}$$

with

$$E_{\boldsymbol{p}}^{(M)}=\sqrt{\boldsymbol{p}^2+M^2}\,. \tag{3·6}$$

These expressions can be understood easily from the hole-theoretical point of view: $E(0)$ and the first term of $E(\sigma_0)$ are the energy density of the Dirac sea composed of massless and massive fermions, respectively; with the second term of $E(\sigma_0)$, as is already stated, the double counting of the interaction energy is avoided. The energy density difference $\mathcal{V}(\sigma_0)$ of the two vacua $|\sigma_0\rangle$ and $|0\rangle$ becomes

$$\mathcal{V}(\sigma_0)=-2N_c/\Omega\cdot\sum_{\boldsymbol{p}}(\sqrt{\boldsymbol{p}^2+M^2}-|\boldsymbol{p}|)+\mu^2/2\cdot\sigma_0^2\,, \tag{3·7a}$$

$$=-N_c\Lambda^4/(4\pi^2)\cdot\left\{\left(1+\frac{x^2}{2}\right)\sqrt{1+x^2}-1-\frac{x^4}{2}\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right|\right\}+\mu^2/2\cdot\sigma_0^2 \tag{3·7b}$$

with $x=M/\Lambda$. It is noteworthy that $\mathcal{V}(\sigma_0)$ can be rewritten in an invariant form (see Appendix A),

$$\mathcal{V}(\sigma_0)=iN_c{\rm Tr}\int\frac{d^4p}{(2\pi)^4}\ln\left(\frac{\not{p}-M}{\not{p}}\right)+\mu^2/2\cdot\sigma_0^2\,. \tag{3·7c}$$

Although σ_0 in the above equation is a solution of the SCC (3·2), if we vary σ_0 freely and regard $\mathcal{V}(\sigma_0)$ as a function of σ_0, $\mathcal{V}(\sigma_0)$ is identical to the expression of the effective potential up to the one-loop order obtained by the path-integral method with an auxiliary field. We note that the E.P. for composite field can be interpreted as the energy density of the vacuum only at the minimum point.[3] The energy density with a given condensation $\langle\bar{\psi}\psi\rangle$ can be obtained by introducing an external field coupled to the composite field $\bar{\psi}\psi$; see Appendix B. In the effective potential approach, the optimum value of σ_0 is determined by

$$\frac{d\mathcal{V}(\sigma_0)}{d\sigma_0}=0\,, \tag{3·8}$$

which we call the energy extremum condition (EEC). For the E.P.- and SCMF-approach to be consistent, the EEC Eq. (3·8) must coincide with the SCC Eq. (3·2); as can be easily verified, that is the case.

Note that we must use the same cutoff scheme and cutoff Λ to calculate $\mathcal{V}(\sigma_0)$ and the r.h.s. of Eq. (3·2), as is evident from the derivation. Inserting Eq. (3·2c) into Eq. (3·7a), we get the difference of the energy density between the normal and the true vacuum,

$$\mathcal{V}(\sigma_M)=-2N_c/\Omega\cdot\sum_{p}\left(\sqrt{p^2+M^2}-\frac{M^2}{2\sqrt{p^2+M^2}}-|p|\right), \tag{3·9}$$

where we define σ_M as the value of σ_0 at the true vacuum. This expression was first given by Suzuki,[7] who used the reduced Hamiltonian of the NJL model.

§ 4. Numerical results

For a summarizing illustration, we evaluate the E.P. and give its approximate form in the case where the symmetry breaking occurs. For this sake, let us rewrite $\mathcal{V}(\sigma_0)$ (Eq. (3·7b)) and SCC (Eq. (3·2d)) by using a parameter $C\equiv g/g_c=gN_c\cdot\Lambda^2/\pi^2$ (the ratio of the original and critical coupling constant) and $\bar{x}\equiv G\sigma_M/\Lambda$,

$$\frac{4g}{\Lambda^2}\mathcal{V}(\sigma_0)=x^2-C\left\{\left(1+\frac{x^2}{2}\right)\sqrt{1+x^2}-1-\frac{x^4}{2}\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right|\right\}, \tag{4·1}$$

$$\bar{x}/C=\bar{x}\left(\sqrt{1+\bar{x}^2}-\bar{x}^2\ln\left|\frac{1+\sqrt{1+\bar{x}^2}}{\bar{x}}\right|\right). \tag{4·2}$$

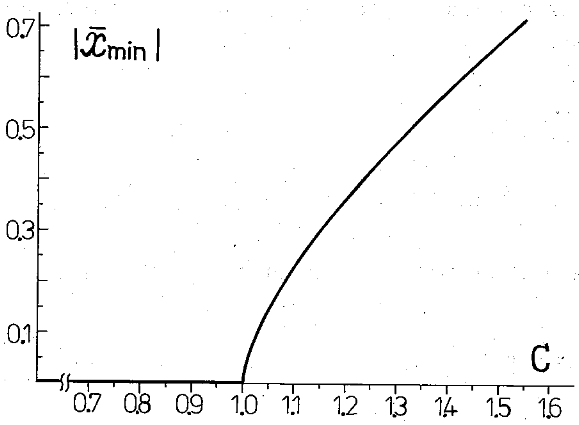

Fig. 1. $|\bar{x}_{min}|$ at which the E.P. has the minimum value is plotted as a function of the coupling strength $C=g/g_c$.

The nature of the vacuum is characterized by C,

the normal phase ($\sigma_M=0$) for $C\leq1$,

the super phase ($\sigma_M\neq0$) for $C>1$.

This fact can be directly seen in Fig. 1 where $|\bar{x}_{min}|$ (at which the E.P. has the minimum value) is plotted as a function of C. We show in Fig. 2 the shape of the E.P. for the two typical values of $C(C=0.85$ and $C=1.25)$. In the case $C>1$ (DSB occurring case),

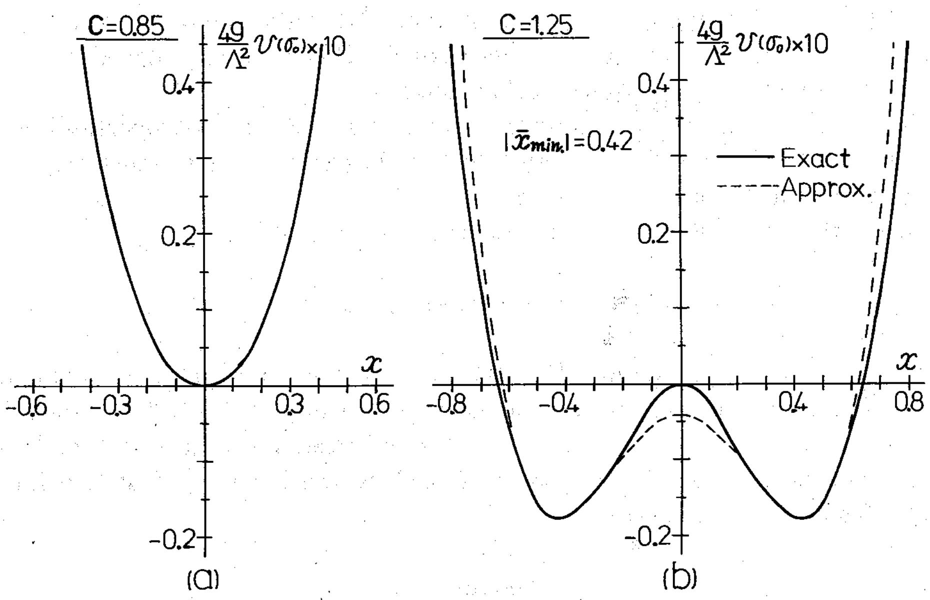

Fig. 2. The shape of the effective potentials (a) for $C=0.85$ and (b) for $C=1.25$. The solid lines are for the exact formula (Eq. (4·1)) and the dashed line is for the approximate one (Eq. (4·3)).

we can expand Eq. (4·1) to the power-series of $x^2-\bar{x}^2$, which becomes up to the second order,

$$4g/\Lambda^2\cdot\mathcal{V}(\sigma_0)=A+B(x^2-\bar{x}^2)^2\,, \tag{4·3}$$

where

$$A=C(1-\sqrt{1+\bar{x}^2}+\bar{x}^2/(2C)) \text{ and } B=(C/\sqrt{1+\bar{x}^2}-1)/(2\bar{x}^2).$$

The dashed line in Fig. 2 is obtained by using this approximate expression. As is seen from the figure, this approximation is very good around $x=\bar{x}$ and not so bad in the whole range of x.

Of course, the shape of the E.P. depends on the cutoff scheme: For example, if an invariant cutoff at $p_0^2+|\boldsymbol{p}|^2=\Lambda^2$ is adopted, resulting E.P. has the minimum at a finite x only for $1<C\lesssim 1.21$.

§ 5. Concluding remarks

The present method to examine the DSB phenomena is applicable to bose systems, as the superfluidity of He^4 can be treated by using the Bogoliubov transformation for bosons.[4)] The energy density of the vacuum for bose systems can be expressed by a sum of the zero point energy, instead of the energy of the Dirac sea for fermi systems. This change corresponds to the difference between the Fredholm determinants of bosons and fermions in the path-integral method.[10)] To demonstrate the powerfulness of our method and the applicability to bose systems, we derive the energy density of the $O(N)$-ϕ^4 model[3),8)] in Appendix C; we show that the higher order corrections in the sense of large N expansion can be readily obtained in our framework.

Our method is also applicable to the system with finite temperature and chemical potential. In this case, the sum of the energy of the Dirac sea is replaced by that of the occupied states multiplied by a distribution function.

Furthermore, we can treat not only the ground state (vacuum) properties but also the collective modes (including the Nambu-Goldstone bosons) in the framework of the self-consistent mean-field theory.[9]

Details of the above will be reported elsewhere.

Acknowledgements

We thank Dr. H. Kuratsuji and Professor K. Matsuyanagi for their comments. We also acknowledge the members of the nuclear theory group of Kyoto University for their interest. This work is done as a part of the annual research project on "the Degrees of Freedom of Quarks in Nuclei" in 1983 organized by the Research Institute for Fundamental Physics, Kyoto University.

Appendix A

Here, we show that Eq. (3·7c) can be derived starting from Eq. (3·7a) as follows:

$$\begin{aligned}
-\frac{2N_c}{\Omega}\sum_p(\sqrt{\boldsymbol{p}^2+M^2}-|\boldsymbol{p}|) &= -2N_c\int\frac{d^3p}{(2\pi)^3}(\sqrt{\boldsymbol{p}^2+M^2}-|\boldsymbol{p}|)\\
&=2N_c\cdot i\cdot\int\frac{d^3p}{(2\pi)^3}\int_0^{M^2}dm^2\frac{i}{2\sqrt{\boldsymbol{p}^2+m^2}}\\
&=2N_c\cdot i\cdot\int\frac{d^3p}{(2\pi)^3}\int_0^{M^2}dm^2\int\frac{dp_0}{2\pi}\frac{1}{p^2-m^2}\\
&=N_c\cdot i\cdot\int\frac{d^3p}{(2\pi)^3}\int_0^{M}dm\int\frac{dp_0}{2\pi}\mathrm{Tr}\frac{1}{\not p-m}\\
&=N_c\cdot i\cdot\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}\int_0^{M}dm\frac{-1}{\not p-m}\\
&=N_c\cdot i\cdot\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}\ln\left(\frac{\not p-M}{\not p}\right).
\end{aligned}$$

The last expression of the above equation is nothing but the first term of the r.h.s. of Eq. (3·7c).

Appendix B

In this appendix, we give the energy density of the state with a given condensation $\langle\bar\psi\psi\rangle$. To construct the state in which the condensation $\langle\bar\psi\psi\rangle$ develops, we introduce an external field J coupled to $\bar\psi\psi$ and define

$$\mathcal{L}^J=\mathcal{L}+2gJ\cdot\bar\psi\psi\,, \tag{B·1}$$

where $\mathcal{L}$ is the NJL Lagrangian density given in (2·1). In the same way as in the text, $\mathcal{L}^J$ can be decomposed into the semi-classical and quantum correction parts:

$$\mathcal{L}^J=\mathcal{L}^J_{\text{MFA}}+\mathcal{L}^J_{\text{res}}\,, \tag{B·2a}$$

where

$$\mathcal{L}^J_{\text{MFA}}=\bar{\psi}[i\gamma.\partial-(M-2gJ)]\psi-\mu^2/2\cdot\sigma_0{}^2 \tag{B·2b}$$

and

$$\mathcal{L}^J_{\text{res}}=g[:(\bar{\psi}\psi)^2:+:(\bar{\psi}i\gamma_5\psi)^2:] \tag{B·2c}$$

with $M=G\sigma_0$. Here, if we represent the vacuum of $\mathcal{L}^J_{\text{MFA}}$ by $|\sigma_0\rangle$, which is the state having a condensation $\langle\sigma_0|\bar{\psi}\psi|\sigma_0\rangle$, the self-consistent mean-field σ_0 satisfies the following SCC,

$$-\mu^2/G\cdot\sigma_0=\langle\sigma_0|\bar{\psi}\psi|\sigma_0\rangle, \tag{B·3a}$$

which can be explicitly evaluated as

$$\frac{2N_c}{\Omega}\sum_p\frac{M_J}{\sqrt{M_J{}^2+\boldsymbol{p}^2}}-\frac{M_J}{2g}=J \tag{B·3b}$$

with

$$M_J=M-2gJ\,. \tag{B·3c}$$

(B·3) gives the relation between M and J. The Hamiltonians corresponding to $\mathcal{L}^J$ and $\mathcal{L}$ are given as

$$H^J=\int d^3x[\bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+M_J)\psi+\mu^2/2\cdot\sigma_0{}^2]-g\int d^3x[:(\bar{\psi}\psi)^2:+:(\bar{\psi}i\gamma_5\psi)^2:]\,, \tag{B·4a}$$

$$H=H^J+2g\int d^3xJ\cdot\bar{\psi}\psi\,. \tag{B·4b}$$

Then, the energy density of the state $|\sigma_0\rangle$ with a condensation $\langle\sigma_0|\bar{\psi}\psi|\sigma_0\rangle$ becomes

$$V(\sigma_0)=\frac{1}{\Omega}\langle\sigma_0|H|\sigma_0\rangle, \tag{B·5a}$$

$$=-2N_c/\Omega\cdot\sum_p\sqrt{M_J{}^2+\boldsymbol{p}^2}+\mu^2/2\cdot\sigma_0{}^2-GJ\sigma_0\,, \tag{B·5b}$$

$$=-2N_c/\Omega\cdot\sum_p\sqrt{M_J{}^2+\boldsymbol{p}^2}+M_J{}^2/(4g)-gJ^2\,. \tag{B·5c}$$

In this expression, J is supposed to be written in terms of M or σ_0 by the use of the SCC (B·3b). The differences of $V(\sigma_0)$ from the E.P. $\mathcal{V}(\sigma_0)$ given in the text are i) M is replaced by $M_J=M-2gJ$ and ii) the quadratic term of external field $-gJ^2$ is added. In the approach in this appendix, the optimum value of σ_0 is determined by an energy extremum condition,

$$0=\frac{dV(\sigma_0)}{\delta\sigma_0}=-GJ\,, \tag{B·6}$$

or $J=0$. Inserting this into the SCC (B·3), we obtain the same SCC for σ_0 as that given in the text,

$$\frac{2N_c}{\Omega}\sum_{p}\frac{M}{\sqrt{M^2+\boldsymbol{p}^2}}=\frac{M}{2g}=\frac{\mu^2\sigma_0}{G}. \qquad (\mathrm{B}\cdot7)$$

Appendix C

In this appendix, we derive the energy density of the vacuum of the $O(N)$-ϕ^4 model by using the SCMF theory. Infinite series of higher order corrections in the sense of large N expansion can be easily obtained in our framework.

The Lagrangian density for the $O(N)$-ϕ^4 model is

$$\mathcal{L}=\frac{1}{2}[(\partial_\mu\phi^a)(\partial^\mu\phi^a)-\mu_0{}^2\phi^a\cdot\phi^a]-\frac{\lambda_0}{8N}(\phi^a\cdot\phi^a)^2, \qquad (\mathrm{C}\cdot1)$$

where ϕ^a denotes the N component real scalar field. Although the Lagrangian density has $O(N)$ invariance, there is a possibility that spontaneous symmetry breaking occurs, which was investigated in detail by some authors within the leading order of $1/N$.[3),8)] In the following, we concentrate on the case that the symmetry of the vacuum breaks down to $O(N-1)$ and two types of condensation appear,

$$\langle\phi^a\rangle \quad \text{and} \quad \langle\varphi^a\varphi^b\rangle\equiv F^{ab}, \qquad (\mathrm{C}\cdot2)$$

where φ^a is a shifted field ($\varphi^a=\phi^a-\langle\phi^a\rangle$). If we choose $\langle\phi^a\rangle=\delta^{aN}\varphi_0$ for convenience, F^{ab} becomes

$$F^{ab}=\begin{pmatrix} f_0 & & 0 & \\ & \ddots & & \\ 0 & & f_0 & \\ & & & f_1 \end{pmatrix} \qquad (\mathrm{C}\cdot3)$$

as a result of the $O(N-1)$ invariance of the vacuum. Thus we have only to concern with the three independent condensation (φ_0, f_0, f_1) or (φ_0, $\mathrm{Tr}F$, $\mathrm{Tr}F^2$).

Following the method explained in the text, we decompose the Lagrangian density (C·1) into the semiclassical and quantum correction parts ($\mathcal{L}=\mathcal{L}_{\mathrm{MFA}}+\mathcal{L}_{\mathrm{res}}$) by introducing the c-number constant mean-fields M_{ab},

$$\mathcal{L}_{\mathrm{MFA}}=\frac{1}{2}[(\partial_\mu\varphi^a)(\partial^\mu\varphi^a)-\varphi^a M_{ab}\varphi^b]+C \qquad (\mathrm{C}\cdot4)$$

and

$$\mathcal{L}_{\mathrm{res}}=-\frac{1}{2}\mu_0{}^2\phi^a\cdot\phi^a-\frac{\lambda_0}{8N}(\phi^a\cdot\phi^a)^2+\frac{1}{2}\varphi^a M_{ab}\varphi^b-C \qquad (\mathrm{C}\cdot5)$$

with

$$M_{ab}=\begin{pmatrix} \mu^2 & & 0 & \\ & \ddots & & \\ 0 & & \mu^2 & \\ & & & \mu_N{}^2 \end{pmatrix}, \qquad (\mathrm{C}\cdot6)$$

where C denotes the c-number term which is determined by the requirement that vacuum expectation value of $\mathcal{L}_{\mathrm{res}}$ should vanish. After the expansion of $\mathcal{L}_{\mathrm{res}}$ using Wick's theorem, C and $\mathcal{L}_{\mathrm{res}}$ become

$$C=-\frac{\lambda_0}{8N}[(\mathrm{Tr}F)^2+2\mathrm{Tr}\,F^2]$$
$$+F^{ab}\Big[\frac{1}{2}(M_{ab}-\mu_0{}^2\delta_{ab})-\frac{\lambda_0}{4N}(\langle\phi^c\rangle\langle\phi^c\rangle\delta_{ab}+2\langle\phi^a\rangle\langle\phi^b\rangle)\Big]$$
$$-\frac{\lambda_0}{8N}(\langle\phi^a\rangle\langle\phi^a\rangle)^2-\frac{1}{2}\mu_0{}^2\langle\phi^a\rangle\langle\phi^a\rangle, \tag{C$\cdot$7}$$

and

$$\mathcal{L}_{\mathrm{res}}=P:(\varphi^a\cdot\varphi^a)^2:+Q^{ab}:\varphi^a\varphi^b:+R^a:\varphi^a:+S^a:\varphi^a\varphi^b\varphi^b:, \tag{C$\cdot$8}$$

where

$$P=-\frac{\lambda_0}{8N}, \tag{C$\cdot$9}$$

$$Q^{ab}=\frac{1}{2}(M_{ab}-\mu_0{}^2\delta_{ab})-\frac{\lambda_0}{4N}(\langle\phi^c\rangle\langle\phi^c\rangle\delta_{ab}+2\langle\phi^a\rangle\langle\phi^b\rangle+\mathrm{Tr}\,F\delta_{ab}+2F_{ab}), \tag{C$\cdot$10}$$

$$R^a=-\langle\phi^b\rangle\Big[\mu_0{}^2\delta_{ab}+\frac{\lambda_0}{2N}(\langle\phi^c\rangle\langle\phi^c\rangle\delta_{ab}+\mathrm{Tr}\,F\delta_{ab}+2F_{ab})\Big] \tag{C$\cdot$11}$$

and

$$S^a=-\frac{\lambda_0}{2N}\langle\phi^a\rangle. \tag{C$\cdot$12}$$

The second term in $\mathcal{L}_{\mathrm{res}}$ is the dangerous term which must be already included in $\mathcal{L}_{\mathrm{MFA}}$, and the third term generates the tadpole diagrams. Therefore, the self-consistency condition (SCC) in this model is nothing but the requirement $Q=R=0$, which becomes

$$\mu^2=\mu_0{}^2+\frac{\lambda_0}{2N}(\mathrm{Tr}\,F+2f_0+\varphi_0{}^2), \tag{C$\cdot$13}$$

$$\mu_N{}^2=\mu_0{}^2+\frac{\lambda_0}{2N}(\mathrm{Tr}\,F+2f_1+3\varphi_0{}^2) \tag{C$\cdot$14}$$

and

$$\varphi_0=0 \quad \text{or} \quad \mu_0{}^2+\frac{\lambda_0}{2N}(\mathrm{Tr}\,F+2f_1+\varphi_0{}^2)=0\,. \tag{C$\cdot$15}$$

The SCC (C·13)~(C·15) includes the well-known solutions in the limit of large N; ($\mu^2\simeq 0$, $\mu_N{}^2\simeq(\lambda_0/N)\varphi_0{}^2$), which corresponds to the unstable vacuum first examined by Coleman, Jackiw and Politzer,[8a)] and ($\mu_N{}^2\simeq\mu^2\simeq\mu_0{}^2+(\lambda_0/2)f_0$, $\varphi_0=0$), which corresponds to the stable and symmetric vacuum discussed by Kobayashi and Kugo.[8b)] Detailed discussion on the stability of various vacua is beyond the scope of this short note; we limit ourselves to deriving the vacuum energy density in the case that both condensations occur ($\varphi_0\neq 0$, $\mu^2\neq 0$) and show that the result includes the infinite series of the higher corrections in the $1/N$-expansion. The Hamiltonian of the system under consideration can be easily derived as

$$H=H_{\mathrm{MFA}}+H_{\mathrm{res}} \tag{C$\cdot$16}$$

with

$$H_{\rm MFA}=\frac{1}{\Omega}\sum_{k,a}\left[b_k^{(a)\dagger}b_k^{(a)}+\frac{1}{2}\right]\omega_k^{(a)}-\frac{\lambda_0}{8N}[({\rm Tr}\,F)^2+2{\rm Tr}\,F^2]+\frac{\lambda_0}{8N}\varphi_0^4+\frac{\mu_0^2}{2}\varphi_0^2 \qquad ({\rm C}\cdot 17)$$

and

$$H_{\rm res}=\frac{\lambda_0}{8N}:(\varphi^a\cdot\varphi^a)^2:+\frac{\lambda_0}{2N}\varphi_0:\varphi^N\varphi^a\cdot\varphi^a:, \qquad ({\rm C}\cdot 18)$$

where $\omega_k^{(a)}=\sqrt{\mu^2+\boldsymbol{k}^2}\,(1\le a\le N-1)$, $\omega_k^{(N)}=\sqrt{\mu_N^2+\boldsymbol{k}^2}$ and Ω denotes the volume of the system. To get (C·17), we have used the SCC (C·13) and (C·14). Thus, the energy density difference between the condensed and noncondensed vacua is

$$\mathcal{V}(\varphi_0,f_0,f_1)=\frac{1}{\Omega}[\langle\varphi_0,\,f_0,\,f_1|H|\varphi_0,\,f_0,\,f_1\rangle-\langle 0,\,0,\,0|H|0,\,0,\,0\rangle] \qquad ({\rm C}\cdot 19{\rm a})$$

$$=\frac{N-1}{2\Omega}\sum_k(\sqrt{\mu^2+\boldsymbol{k}^2}-\sqrt{\mu_0^2+\boldsymbol{k}^2})+\frac{1}{2\Omega}\sum_k(\sqrt{\mu_N^2+\boldsymbol{k}^2}-\sqrt{\mu^2+\boldsymbol{k}^2})$$

$$-\frac{\lambda_0}{8N}[({\rm Tr}\,F)^2+2\,{\rm Tr}\,F^2]+\frac{\lambda_0}{8N}\varphi_0^4+\frac{\mu_0^2}{2}\varphi_0^2\,, \qquad ({\rm C}\cdot 19{\rm b})$$

where $(\mu^2,\,\mu_N^2)$ are related to $(\varphi_0,\,f_0,\,f_1)$ through the SCC (C·13), (C·14), (C·15), ${\rm Tr}\,F=(N-1)f_0+f_1$ and ${\rm Tr}\,F^2=(N-1)f_0^2+f_1^2$. Note that the first line in (C·19b) arises from the zero point energy of the bose particles; this situation is obviously different from the case of fermion (see Eq. (3·7a) in the text). In the covariant notation, $\mathcal{V}(\varphi_0,\,f_0,\,f_1)$ reads

$$\mathcal{V}(\varphi_0,\,f_0,\,f_1)=-\frac{i}{2}{\rm Tr}\int\frac{d^4p}{(2\pi)^4}\ln\left(\frac{M-p^2}{\mu_0^2-p^2}\right)-\frac{\lambda_0}{8N}[({\rm Tr}\,F)^2+2{\rm Tr}\,F^2]+\frac{\lambda_0}{8N}\varphi_0^4+\frac{\mu_0^2}{2}\varphi_0^2\,, \qquad ({\rm C}\cdot 20)$$

where M is the diagonal matrix defined in (C·6).

Here we mention briefly the relation between (C·20) and the effective potential of the $O(N)$-ϕ^4 model. (C·20) denotes the energy density of the true vacuum and $(\varphi_0,\,f_0,\,f_1)$ in the expression must be the solution of the SCC (C·13), (C·14) and (C·15). However, if we regard $(\varphi_0,\,f_0,\,f_1)$ in (C·20) as free-parameters, (C·20) is nothing but the effective potential which does not have the meaning of the energy density of the vacuum except for the stationary point in the case that there exists a condensation of a composite operator.[3] One can easily prove above facts, if one formulates the SCMF theory including the external source, which is the straightforward generalization of the method explained above (in the case of NJL model, see Appendix B).

Let us show that $\mathcal{V}(\varphi_0,\,f_0,\,f_1)$ includes higher order corrections in the sense of large N expansion. To simplify the argument, we restrict ourselves to the case of symmetric vacuum ($\varphi_0=0$, $f_0=f_1$) and rewrite $\mathcal{V}$ as a function of $\mu^2\equiv\chi$;

$$\mathcal{V}(\chi)=-\frac{i}{2}N\int\frac{d^4p}{(2\pi)^4}\ln\left(\frac{\chi-p^2}{\mu_0^2-p^2}\right)-\frac{1}{(1+2/N)}\frac{N}{2\lambda_0}(\chi-\mu_0^2)^2\,. \qquad ({\rm C}\cdot 21)$$

Comparing (C·21) to the leading order results of large N expansion,[8] one can see that $(1+2/N)^{-1}$- factor in (C·21) and χ (C·13) obviously include the higher order corrections in the sense of large N expansion.

More about the vacuum structure, SCMF theory including the external source, and the excited states of $O(N)$-ϕ^4 model will be reported elesewhere.

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