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A semi-analytical formula for the light variations due to low-frequency g modes in rotating stars

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ABSTRACT

Through the adoption of the so-called 'traditional approximation', a new semi-analytical formula is derived for the light variations produced by low-frequency g modes in uniformly rotating stars. The formula is used to examine the influence of rotation on the variability produced by a stellar model representative of the slowly pulsating B-type class.

It is found that, for all apart from prograde sectoral modes, the Coriolis force acts to trap pulsation within an equatorial waveguide. Towards rapid rotation and/or low pulsation frequency, this waveguide becomes so narrow that only a thin band around the stellar equator makes any appreciable contribution toward flux changes. As a result, unless viewed from near the poles, the variability exhibited by the star becomes very small, possibly explaining why recent photometric observations of rapidly rotating stars have failed to find much evidence for the presence of low-frequency modes.

It is further demonstrated that the ratio between the variability amplitude in pairs of passbands depends, with the introduction of rotation, both on the azimuthal order of a mode, and on the location of the observer in relation to the rotation axis of the star. This means that the standard photometric techniques used to identify modes in non-rotating stars cannot easily be applied to systems where rotation is significant.

Key words: methods: analytical – techniques: photometric – stars: oscillations – stars: rotation.

1 INTRODUCTION

A singular event in the study of stellar non-radial pulsation (NRP) was the derivation, by Dziembowski (1977), of a semi-analytical formula for the light variations produced by a pulsating star. Not only has this formula been used extensively to establish that NRP is responsible for the flux changes seen in many disparate classes of variable star, it has also facilitated the identification of the specific modes responsible for these changes. In particular, the formula underpins the so-called amplitude-phase and multicolour-amplitudes diagnostic techniques, developed by Stamford & Watson (1981) and Heynderickx, Waelkens & Smeyers (1994), respectively, which permit the derivation of the harmonic degree ℓ of pulsation from observations of flux changes in two or more photometric passbands (see, e.g., Cugier, Dziembowski & Pamyatnykh 1994; De Cat 2001).

Historically, these mode identifications have been sought after as an end in themselves. However, with the outstanding success of the *Hipparcos* mission in discovering many new putative non-radial pulsators (see, e.g., Waelkens et al. 1998), and looking forward to current and future space-based missions, which can be expected to find many more (Eyer 2000), the era nears where the technique of asteroseismology can be applied to *all* classes of pulsating star – not just to the Sun, for which the superb quality of observational data permits the establishment of tight constraints on its internal structure (e.g. Gough & Toomre 1991). Toward this end, mode identifications will doubtless play a crucial role.

Since its inception, Dziembowski's (1977) formula has evolved significantly. Stamford & Watson (1981) effected a substantial simplification, by replacing integral terms with analytical equivalents; subsequent work by Watson (1987, 1988) added terms to account for the influence on limb darkening of photospheric temperature and gravity variations. More recently, Heynderickx et al. (1994) have demonstrated that the same formula can be derived from within a Lagrangian (rather than Eulerian) framework for describing stellar-surface perturbations.

Regrettably, Dziembowski's (1977) formula suffers from one notable deficiency: it is applicable only to non-rotating stars. This is less of a problem for stars that are pulsating in p (pressure) modes (e.g. β Cephei and δ Scuti stars; see Gautschy & Saio 1996), the time-scales characteristic of such modes being too short for them to be affected significantly by rotation. However, for those systems for which the NRP is dominated by g (gravity) modes – in particular, the slowly pulsating B (SPB) stars (Waelkens & Rufener 1985; Waelkens 1991) and γ Doradus stars (Balona, Krisciunas & Cousins 1994; Kaye et al. 1999) – the effects of the Coriolis force

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can be dramatic, and the assumption that rotation may be ignored will certainly lead to erroneous results.

The present paper seeks to address this deficiency, by introducing a new semi-analytical formula for the light variations of uniformly rotating stars. The formula is built around the adoption of the so-called 'traditional approximation', and is therefore applicable to low-frequency g modes of slow-to-moderately rotating systems, in which the influence of the Coriolis force may be significant, but where that of the centrifugal force can safely be neglected. The paper is laid out thus: the following section reviews the formula for the light variations of a non-rotating star, with the purpose of introducing the basic concepts and nomenclature used throughout. After a review of the traditional approximation in Section 3, the new formula, applicable to rotating stars, is derived in Section 4. Following a discussion of its implementation in Section 5, this formula is used in Section 6 to examine the effect of rotation on the variability of a model SPB star. Section 7 concludes the paper with a discussion and summary of the principal findings.

2 THE CASE WITHOUT ROTATION

This section reviews the expressions pertinent to the light variations of a non-rotating pulsating star. These expressions will be used extensively during the development, in Section 4, of formulae for the light variations in the rotating case. The exposition has been specifically tailored to highlight the similarities between non-rotating and rotating cases, and the present section also serves to introduce the formalism and nomenclature common to the two.

In a non-rotating star, the equations governing NRP are obtained through a sequence of steps. First, the equations of hydrodynamics and thermodynamics are subjected to small perturbations about the equilibrium state of the star. After discarding terms of quadratic or higher order in the perturbation amplitude, the resulting linearized pulsation equations – which constitute an eigenproblem, with the pulsation frequency σ as the eigenvalue – may be separated in all spherical-polar coordinates. It transpires that the angular dependence of eigenmodes is proportional to the spherical harmonics Y_ℓ^m, where ℓ is the harmonic degree and m the azimuthal order.

A myriad differing formulations exist for the spherical harmonics; throughout the present work, the definition

$$Y_\ell^m(\theta,\phi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_\ell^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi} \tag{1}$$

due to Arfken (1970) is adopted, where P_ℓ^m is an associated Legendre function, and (θ, ϕ) are, respectively, the polar and azimuthal coordinates. The normalization of this equation ensures that the spherical harmonics are orthonormal over the surface of a sphere, namely

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\phi) Y_{\ell'}^{m'*}(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \delta_{\ell,\ell'}\delta_{m,m'}. \tag{2}$$

It should be noted that these expressions are valid for negative m, as well as positive, as long as it is understood that

$$P_\ell^{-m}(\cos\theta) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta) \tag{3}$$

for all $|m| \leqslant \ell$; the latter relation arises directly from Rodrigues' formula for $P_\ell^m(\cos\theta)$ (Arfken 1970).

Under the condition that a single pulsation mode is excited in the star, the Lagrangian perturbations (denoted throughout by the prefix δ) to the stellar radius R, effective temperature T_eff and effective gravity g_eff can be expressed as

$$\frac{\delta R}{R}(\theta,\phi;t) = \mathrm{Re}\left[\Delta_R Y_\ell^m(\theta,\phi)\mathrm{e}^{\mathrm{i}\sigma t}\right], \tag{4}$$

$$\frac{\delta T_\mathrm{eff}}{T_\mathrm{eff}}(\theta,\phi;t) = \mathrm{Re}\left[\Delta_T Y_\ell^m(\theta,\phi)\mathrm{e}^{\mathrm{i}\sigma t}\right] \tag{5}$$

and

$$\frac{\delta g_\mathrm{eff}}{g_\mathrm{eff}}(\theta,\phi;t) = \mathrm{Re}\left[\Delta_g Y_\ell^m(\theta,\phi)\mathrm{e}^{\mathrm{i}\sigma t}\right], \tag{6}$$

respectively. Here, Re[$\cdots$] denotes the real part of a complex quantity, while σ is the eigenfrequency introduced previously, and t the temporal independent variable. The complex perturbation coefficients $\{\Delta_R,\ \Delta_T,\ \Delta_g\}$ appearing in these equations determine the amplitude and phase of the respective perturbations. Specifically, as

$$\int_0^{2\pi}\int_0^{\pi}\left[\frac{\delta R}{R}(\theta,\phi;t)\right]^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{|\Delta_R|^2}{2} \tag{7}$$

via the orthonormality relation (2), it follows that $|\Delta_R|$ is equal to $\sqrt{8\pi}$ times the root-mean square (rms) amplitude of $\delta R/R$ over all solid angles, and likewise for $|\Delta_T|$ and $|\Delta_g|$.

If it is assumed that the radial displacement of fluid elements is constant throughout the photosphere, then – as argued by Cugier & Daszyńska (2001) – the effective gravity perturbation coefficient Δ_g can be related to Δ_R via

$$\Delta_g = -(2+\bar\sigma^2)\Delta_R, \tag{8}$$

where, with M the stellar mass and G the constant of gravitation,

$$\bar\sigma^2 \equiv \frac{\sigma^2 R^3}{GM} \tag{9}$$

defines a dimensionless pulsation frequency $\bar\sigma$, which is typically less than unity for g modes, and greater than unity for p modes. In the expression for Δ_g, the first term in parentheses comes from the r^{-2} dependence of the equilibrium gravitational field, and the second from the radial acceleration of fluid elements.

If the pulsation can be considered adiabatic, then Δ_T may also be related to Δ_R, through the expression (e.g. Buta & Smith 1979)

$$\Delta_T = \nabla_\mathrm{ad}\left[\frac{\ell(\ell+1)}{\bar\sigma^2} - 4 - \bar\sigma^2\right]\Delta_R, \tag{10}$$

where ∇_ad is the adiabatic temperature gradient within the photosphere. In this latter expression, the terms within the brackets account for compression or rarefaction of the atmosphere, caused on the one hand by changes in the stellar surface area spanned by fluid elements (namely the first term and half of the second one), and on the other hand by perturbations to the effective gravity (namely the last term, and the other half of the second term). If the pulsation *cannot* be considered adiabatic, however, calculation of Δ_T requires a solution of non-adiabatic pulsation equations; more will be said regarding this issue in Section 4.

With these definitions, the time-dependent perturbation $\delta\mathcal{F}_x$ to the flux $\mathcal{F}_x$ in the photometric band denoted 'x', arising from the excitation of an individual pulsation mode, is given by

$$\frac{\delta\mathcal{F}_x}{\mathcal{F}_x}(\theta_\mathrm{o},\phi_\mathrm{o};t) = \mathrm{Re}\left[\left\{\Delta_R\mathcal{R}_{\ell;x}^m(\theta_\mathrm{o},\phi_\mathrm{o}) + \Delta_T\mathcal{T}_{\ell;x}^m(\theta_\mathrm{o},\phi_\mathrm{o}) + \Delta_g\mathcal{G}_{\ell;x}^m(\theta_\mathrm{o},\phi_\mathrm{o})\right\}\mathrm{e}^{\mathrm{i}\sigma t}\right], \tag{11}$$

where $(\theta_\mathrm{o}, \phi_\mathrm{o})$ are the angular coordinates of the *observer*, in the spherical-polar reference frame of the star. This expression is based on Watson's (1987, 1988) extension to Dziembowski's (1977) formula for the flux variations, but the nomenclature has been somewhat modified in order to facilitate the developments in Section 4. The three 'differential flux functions' (DFFs),

$$\mathcal{R}^m_{\ell;x}(\theta_\mathrm{o},\phi_\mathrm{o}) \equiv \frac{(2+\ell)(1-\ell)}{\mathcal{I}_{0;x}}\mathcal{I}_{\ell;x} Y^m_\ell(\theta_\mathrm{o},\phi_\mathrm{o}), \tag{12}$$

$$\mathcal{T}^m_{\ell;x}(\theta_\mathrm{o},\phi_\mathrm{o}) \equiv \frac{1}{\mathcal{I}_{0;x}}\frac{\partial \mathcal{I}_{\ell;x}}{\partial \ln T_\mathrm{eff}} Y^m_\ell(\theta_\mathrm{o},\phi_\mathrm{o}) \tag{13}$$

and

$$\mathcal{G}^m_{\ell;x}(\theta_\mathrm{o},\phi_\mathrm{o}) \equiv \frac{1}{\mathcal{I}_{0;x}}\frac{\partial \mathcal{I}_{\ell;x}}{\partial \ln g} Y^m_\ell(\theta_\mathrm{o},\phi_\mathrm{o}), \tag{14}$$

parametrize – as a function of the observer coordinates $(\theta_\mathrm{o}, \phi_\mathrm{o})$ – the individual contributions toward the relative flux variations arising from perturbations to the surface radius, temperature and gravity, respectively. Each DFF depends on the intensity moment

$$\mathcal{I}_{\ell;x} = \int_0^1 \mu P_\ell(\mu)\mathcal{I}_x(\mu)\,\mathrm{d}\mu, \tag{15}$$

characterizing the angular dependence of the photospheric radiation field; here, $\mathcal{I}_x$ is the wavelength-integrated specific intensity of the radiation within passband x, emergent from the equilibrium photosphere at cosinus μ to the local surface normal, while P_ℓ is the Legendre polynomial of degree ℓ. The normalizing coefficient

$$\mathcal{I}_{0;x} = \int_0^1 \mu \mathcal{I}_x(\mu)\,\mathrm{d}\mu, \tag{16}$$

is the value of $\mathcal{I}_{\ell;x}$ for $\ell = 0$ and is proportional to the x-band flux from the quiescent star.

Inspection of the formula (11) reveals that the flux variations depend on the azimuthal order m, and the observer coordinates (θ_o, ϕ_o), only through the same spherical harmonic term Y^m_ℓ appearing in each of the expressions (12)–(14) for the DFFs. Consequently, the ratio between the variation amplitude in two separate passbands is *independent* of both m and θ_o. In fact, this ratio depends only on the harmonic degree ℓ and the relative amplitudes and phases of the three perturbation coefficients $\{\Delta_R, \Delta_T, \Delta_g\}$. This fact underpins the approach used in the amplitude-phase (Stamford & Watson 1981) and multicolour-amplitudes (Heynderickx et al. 1994) techniques for the determination of ℓ.

3 THE TRADITIONAL APPROXIMATION

The traditional approximation, the origins of which lie in the geophysical literature (see Eckart 1960), greatly reduces the difficulty of treating NRP in uniformly rotating stars. It amounts to neglecting the horizontal component of the rotation angular velocity vector $\boldsymbol{\Omega}$, when evaluating the Coriolis-force terms in the pulsation equations. As discussed by Lee & Saio (1997), such an approach is reasonable in layers where the Brunt-Väisälä frequency N greatly exceeds both the rotation angular frequency $\Omega \equiv |\boldsymbol{\Omega}|$ and the corotating-frame pulsation frequency $\omega \equiv \sigma + m\Omega$. These conditions are usually fulfilled for low-frequency g modes trapped in the radiative regions of slowly to moderately rotating stars.

When combined with the adiabatic and Cowling (1941) approximations, where perturbations to the specific entropy and gravitational potential are ignored, and under the further assumption that the centrifugal distortion of the equilibrium star is negligible, the traditional approximation restores the separability of the rotating-star pulsation equations into radial and angular parts. The solutions of the latter depend on the 'spin parameter' $\nu \equiv 2\Omega/\omega$; denoting these angular eigenfunctions by $\tilde{Y}^m_\ell$, the Lagrangian perturbations to R, T_eff and g_eff (cf. equations 4–6) become

$$\frac{\delta R}{R}(\theta,\phi;t;\nu) = \mathrm{Re}\left[\Delta_R \tilde{Y}^m_\ell(\theta,\phi;\nu)\mathrm{e}^{\mathrm{i}\sigma t}\right], \tag{17}$$

$$\frac{\delta T_\mathrm{eff}}{T_\mathrm{eff}}(\theta,\phi;t;\nu) = \mathrm{Re}\left[\Delta_T \tilde{Y}^m_\ell(\theta,\phi;\nu)\mathrm{e}^{\mathrm{i}\sigma t}\right] \tag{18}$$

and

$$\frac{\delta g_\mathrm{eff}}{g_\mathrm{eff}}(\theta,\phi;t;\nu) = \mathrm{Re}\left[\Delta_g \tilde{Y}^m_\ell(\theta,\phi;\nu)\mathrm{e}^{\mathrm{i}\sigma t}\right]. \tag{19}$$

The indices ℓ and m are used here to denote the harmonic degree and azimuthal order of the spherical harmonic Y^m_ℓ to which $\tilde{Y}^m_\ell$ tends in the limit $\nu \to 0$ of no rotation.

There currently exist two approaches to calculating the angular eigenfunctions $\tilde{Y}^m_\ell$. In their seminal treatment, Lee & Saio (1987) addressed the problem using a matrix-algebra formalism; more recently, Bildsten, Ushomirsky & Cutler (1996) have advocated a procedure based around the direct numerical integration of Laplace's tidal equation. Lee & Saio (1997) themselves suggest that this direct approach is superior to the matrix method, but it has long been known in the field of geophysics that the two formalisms are entirely equivalent (e.g. Longuet-Higgins 1968). Here, a useful analogy can be drawn to the complementarity of the wave-mechanical (Schrödinger) and matrix-mechanical (Heisenberg) paradigms in quantum theory.

It transpires that the matrix-based formalism is the more flexible framework for the present purposes. Full details of the technique have been given by Lee & Saio (1987, 1990); the underlying concept is that the angular eigenfunctions $\tilde{Y}^m_\ell$ are expanded as an infinite series of spherical harmonics. For a given $\tilde{Y}^m_\ell$, the appropriate expansion combines only harmonics of the same azimuthal order m and latitudinal parity about the equator (Zahn 1966). Therefore, the eigenfunctions are written as

$$\tilde{Y}^m_{\ell_k}(\theta,\phi;\nu) = \sum_{j=1}^{\infty} \mathbf{B}_{j,k} Y^m_{\ell_j}(\theta,\phi) \tag{20}$$

(e.g. Unno et al. 1989), where

$$\ell_j = \begin{cases} |m| + 2(j-1) & \text{even-parity modes,} \\ |m| + 2(j-1) + 1 & \text{odd-parity modes,} \end{cases} \tag{21}$$

and likewise for ℓ_k. The expansion coefficients $\mathbf{B}_{j,k}$ are the components of a matrix $\mathbf{B}$, for which the column vectors constitute the eigenvectors of the real symmetric coupling matrix $\mathbf{W}$ introduced by Lee & Saio (1987). There is a separate coupling matrix for each value of m, ν and the parity, although the matrix is invariant under a simultaneous change in the sign of both m and ν. The latter property means that it is possible to consider only cases where $\nu \geqslant 0$, leaving the sign of m to determine whether the mode is prograde (negative) or retrograde (positive) in the corotating frame of reference. Note that $\mathbf{W}$ becomes diagonal in the limit of no rotation, with components $\mathbf{W}_{j,j} = \ell_j(\ell_j + 1)$.

In combination with a diagonal matrix $\mathbf{D}$, for which the components $\lambda_j \equiv \mathbf{D}_{j,j}$ are the eigenvalues of $\mathbf{W}$, the eigenvector matrix $\mathbf{B}$ may be defined as a solution of the canonical equation

$$\mathbf{WB} = \mathbf{BD}. \tag{22}$$

However, this equation is insufficient to determine $\mathbf{B}$ fully, and two supplementary constraints must be imposed. First, the column vectors of $\mathbf{B}$ – that is, the eigenvectors of $\mathbf{W}$ – are scaled to have a length of unity. Since $\mathbf{W}$ is real and symmetric, this normalization means that $\mathbf{B}$ will be orthogonal,

$$\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}, \tag{23}$$

where the superscript T denotes the transpose and $\mathbf{I}$ is the identity matrix. As a result, the angular eigenfunctions $\tilde{Y}_\ell^m$ will exhibit the same orthonormality property

$$\int_0^{2\pi}\int_0^{\pi} \tilde{Y}_\ell^m(\theta,\phi)\tilde{Y}_{\ell'}^{m'*}(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \delta_{\ell,\ell'}\delta_{m,m'} \tag{24}$$

as the spherical harmonics (cf. equation 2).

Secondly, as ν is taken to zero, $\mathbf{B}$ must transform *continuously* into the identity matrix $\mathbf{I}$. This requires that the columns (eigenvectors) that constitute $\mathbf{B}$ be arranged in a specific order, and furthermore that the overall sign of each column be selected appropriately. The constraint is introduced to ensure that $\tilde{Y}_\ell^m \to Y_\ell^m$ and that $\lambda \to \ell(\ell + 1)$ in the limit $\nu \to 0$ of no rotation (see above). A procedure for choosing the necessary ordering and sign of the eigenvectors will be discussed in Section 5.

4 THE CASE WITH ROTATION

The expressions (17)–(19) for perturbations within the traditional approximation are almost identical to the equivalent non-rotating ones (4)–(6), but angular dependences are described by the $\tilde{Y}_\ell^m$ functions, rather than by the spherical harmonics Y_ℓ^m. It is for this reason that Dziembowski's (1977) formula, and the modified forms introduced by Watson (1987, 1988), cannot be used to model the flux variations of a rotating, pulsating star.

However, these formulae *can* be applied piecewise to the terms appearing in the expansion (20) for the rotationally modified angular eigenfunctions, as these terms have an angular dependence proportional to spherical harmonics. By adding together the contributions arising from each, the observed flux variations due to the mode with angular dependence $\tilde{Y}_{\ell_k}^m$ are readily found as

$$\frac{\delta\mathcal{F}_x}{\mathcal{F}_x}(\theta_\mathrm{o},\phi_\mathrm{o},t;\nu) = \mathrm{Re}\sum_{j=1}^{\infty}\left[\mathbf{B}_{j,k}\left\{\Delta_R\mathcal{R}_{\ell_j;x}^m(\theta_\mathrm{o},\phi_\mathrm{o}) + \Delta_T\mathcal{T}_{\ell_j;x}^m(\theta_\mathrm{o},\phi_\mathrm{o}) + \Delta_g\mathcal{G}_{\ell_j;x}^m(\theta_\mathrm{o},\phi_\mathrm{o})\right\}\mathrm{e}^{\mathrm{i}\sigma t}\right], \tag{25}$$

where all symbols have their usual meanings. It is straightforward to rewrite this equation as

$$\frac{\delta\mathcal{F}_x}{\mathcal{F}_x}(\theta_\mathrm{o},\phi_\mathrm{o},t;\nu) = \mathrm{Re}\left[\left\{\Delta_R\tilde{\mathcal{R}}_{\ell;x}^m(\theta_\mathrm{o},\phi_\mathrm{o};\nu) + \Delta_T\tilde{\mathcal{T}}_{\ell;x}^m(\theta_\mathrm{o},\phi_\mathrm{o};\nu) + \Delta_g\tilde{\mathcal{G}}_{\ell;x}^m(\theta_\mathrm{o},\phi_\mathrm{o};\nu)\right\}\mathrm{e}^{\mathrm{i}\sigma t}\right], \tag{26}$$

which is *identical* to the non-rotating result (11), except for the fact that the three DFFs – $\mathcal{R}_{\ell;x}^m$, $\mathcal{T}_{\ell;x}^m$ and $\mathcal{G}_{\ell;x}^m$ – have been replaced by the rotationally modified equivalents

$$\tilde{\mathcal{R}}_{\ell_k;x}^m(\theta_\mathrm{o},\phi_\mathrm{o};\nu) \equiv \sum_{j=1}^{\infty}\mathbf{B}_{j,k}\mathcal{R}_{\ell_j;x}^m(\theta_\mathrm{o},\phi_\mathrm{o}), \tag{27}$$

$$\tilde{\mathcal{T}}_{\ell_k;x}^m(\theta_\mathrm{o},\phi_\mathrm{o};\nu) \equiv \sum_{j=1}^{\infty}\mathbf{B}_{j,k}\mathcal{T}_{\ell_j;x}^m(\theta_\mathrm{o},\phi_\mathrm{o}) \tag{28}$$

and

$$\tilde{\mathcal{G}}_{\ell_k;x}^m(\theta_\mathrm{o},\phi_\mathrm{o};\nu) \equiv \sum_{j=1}^{\infty}\mathbf{B}_{j,k}\mathcal{G}_{\ell_j;x}^m(\theta_\mathrm{o},\phi_\mathrm{o}), \tag{29}$$

respectively. Using equations (12)–(14), these may be expressed in terms of more elementary functions as

$$\tilde{\mathcal{R}}_{\ell_k;x}^m(\theta_\mathrm{o},\phi_\mathrm{o};\nu) \equiv \sum_{j=1}^{\infty}\mathbf{B}_{j,k}\frac{(2+\ell_j)(1-\ell_j)}{\mathcal{I}_{0;x}}\mathcal{I}_{\ell_j;x}Y_{\ell_j}^m(\theta_\mathrm{o},\phi_\mathrm{o}), \tag{30}$$

$$\tilde{\mathcal{T}}_{\ell_k;x}^m(\theta_\mathrm{o},\phi_\mathrm{o};\nu) \equiv \sum_{j=1}^{\infty}\mathbf{B}_{j,k}\frac{1}{\mathcal{I}_{0;x}}\frac{\partial\mathcal{I}_{\ell_j;x}}{\partial\ln T_\mathrm{eff}}Y_{\ell_j}^m(\theta_\mathrm{o},\phi_\mathrm{o}) \tag{31}$$

and

$$\tilde{\mathcal{G}}_{\ell_k;x}^m(\theta_\mathrm{o},\phi_\mathrm{o};\nu) \equiv \sum_{j=1}^{\infty}\mathbf{B}_{j,k}\frac{1}{\mathcal{I}_{0;x}}\frac{\partial\mathcal{I}_{\ell_j;x}}{\partial\ln g}Y_{\ell_j}^m(\theta_\mathrm{o},\phi_\mathrm{o}). \tag{32}$$

Since $\mathbf{B}$ approaches the identity matrix $\mathbf{I}$ in the limit of no rotation (cf. Section 3), these three DFFs correctly reduce to $\mathcal{R}_{\ell;x}^m$, $\mathcal{T}_{\ell;x}^m$ and $\mathcal{G}_{\ell;x}^m$, respectively, in the same limit.

Combined with the above three expressions, the new formula (26) represents the final result, appropriate to modelling the light variations of low-frequency g modes in rotating stars. Since the adiabatic approximation was employed during its derivation, it would appear that the formula is limited to those ideal cases where the pulsation is adiabatic. Problematically, such cases are very rarely realized in the surface layers of pulsating stars: there, the time-scale for heat exchange between neighbouring fluid elements becomes shorter than the pulsation period, and departures from adiabaticy are large.

To address this difficulty, an approach suggested by Savonije, Papaloizou & Alberts (1995) can be adopted. As before, the traditional and Cowling (1941) approximations are applied to the pulsation equations for a rotating star. However, the restrictive adiabatic approximation is replaced by a partial model for non-adiabatic processes, centred around neglecting the divergence of the (Eulerian) horizontal flux perturbation in the equation governing energy conservation (see, e.g., Unno et al. 1989, their equation 21.5). Without the terms arising from this divergence, the pulsation equations remain separable in all coordinates, even when non-adiabaticy is allowed for.

As Savonije et al. (1995) discuss, this 'non-adiabatic radial-flux' (NARF) stratagem can be expected to yield reasonable results when the pulsation perturbations vary most rapidly in the radial direction. To examine whether this condition is met in the present context, consider the approximate dispersion relation for low-frequency g modes in rotating stars (e.g. Unno et al. 1989),

$$\omega^2 \approx \frac{N^2k_\perp^2 + (2\mathbf{\Omega}\cdot\boldsymbol{k})^2}{k^2}, \tag{33}$$

where $\boldsymbol{k}$ is the local wavevector associated with perturbations, with magnitude k, and components k_r and $k_\perp$ in the radial and horizontal directions, respectively. As a prerequisite for the adoption of the traditional approximation, it was assumed (cf. Section 3) that both ω and Ω are much smaller than the Brunt-Väisälä frequency N. Applying the same assumptions to the above dispersion relation leads to the result $k_\perp \ll k$, implying that $k_r \gg k_\perp$, and confirming that perturbations will indeed vary most rapidly in the radial direction.

In going from the adiabatic approximation to the NARF approximation, the angular parts of the separated pulsation equations remain unaltered. Therefore, the new formula (26) and accompanying expressions (30)–(32) for the DFFs require no modification. However, the Δ_T perturbation coefficient is no longer obtained via an adiabatic relation (e.g. equation 10); instead, it must be calculated via solution of the radial parts of the NARF pulsation equations. Under the aegis of the traditional approximation, these equations for a rotating star are identical to their non-rotating NARF counterparts, save for the fact that each $\ell(\ell+1)$ term is replaced by the eigenvalue λ (cf. Section 3) for the mode in question.

5 IMPLEMENTATION

Neither the new formula (26) for the flux variations, nor the accompanying equations (30)–(32) for the DFFs, are mathematically esoteric. However, the numerical implementation of these expressions is far from trivial, involving a number of subtle issues that require careful treatment. The present section delves into these issues, both to document how the calculations that follow in Section 6 were accomplished, and – on epistemic grounds – to serve as a reference for future studies that might adopt the new formula.

When calculating the eigenvector matrix $\mathbf{B}$, it is simpler to solve the inverse eigenproblem

$$\mathbf{W}^{-1}\mathbf{B} = \mathbf{B}\mathbf{D}^{-1} \tag{34}$$

than to deal with equation (22) directly. The reasons for this are twofold. First, expressions for the components of the inverse matrix $\mathbf{W}^{-1}$ are available in closed form (Lee & Saio 1987; Townsend 1997b). Secondly, $\mathbf{W}^{-1}$ is both symmetric and tridiagonal; hence, the efficient *QL* algorithm (e.g. Press et al. 1992) may be used to find eigenvalues and eigenvectors. Since the off-diagonal components of the matrix are all non-zero when $\nu \neq 0$, this algorithm is guaranteed to converge (Parlett 1980).

Although both $\mathbf{W}^{-1}$ and $\mathbf{B}$ are formally infinite in size, numerical calculations require that they be truncated at some finite dimension N. This truncation is equivalent to terminating the spherical harmonic expansion of $\tilde{Y}_\ell^m$ (cf. equation 20) at some finite $\ell_j = \ell_N$, and likewise for the summations in the definitions (30)–(32) of $\tilde{\mathcal{R}}_{\ell;x}^m$, $\tilde{\mathcal{T}}_{\ell;x}^m$ and $\tilde{\mathcal{G}}_{\ell;x}^m$. The absence of the $\ell_j > \ell_N$ spherical harmonics will introduce spurious, high-frequency features in all of these functions; however, choosing N suitably large allows the amplitude of these features to be constrained below any required value. Throughout the subsequent sections, a dimension $N = 200$ was adopted; larger values led to negligible changes in the results, while at the same time increasing the computational costs (here, it is noted that the operation count of the *QL* algorithm scales with N^3).

The eigenvector matrix calculated by the *QL* algorithm is always orthogonal, so the first constraint imposed on $\mathbf{B}$ (cf. Section 3) is automatically satisfied. To fulfil the second constraint, that $\mathbf{B}$ transforms smoothly into $\mathbf{I}$ as ν is taken to zero, the procedure outlined by Townsend (1997a,b) can be used. First, the columns of $\mathbf{B}$ and the diagonal components ${\mathbf{D}^{-1}}_{j,j} \equiv {\lambda_j}^{-1}$ are together arranged so that the latter fall in descending order along the diagonal. Then, the columns of both matrices are permuted cyclically $\Xi(\nu')$ times to the left, where $\Xi(\nu')$ is the number of points between $\nu = \nu'$ and $\nu = 0$ at which the components of $\mathbf{W}^{-1}$ are singular. The locations of these points can readily be found from consideration of the expressions for the components (Lee & Saio 1987; Townsend 1997b).

The final step in the procedure is to select the overall sign of each column of the permuted $\mathbf{B}$ matrix. This is done by requiring that, at the stellar equator ($\theta = 90°$), the angular eigenfunctions satisfy

$$\mathrm{sgn}\left[\tilde{Y}_\ell^m(\theta,\phi;\nu)\right] = \mathrm{sgn}\left[Y_\ell^m(\theta,\phi)\right] \tag{35}$$

for even-parity modes and

$$\mathrm{sgn}\left[\frac{\partial}{\partial\theta}\tilde{Y}_\ell^m(\theta,\phi;\nu)\right] = \mathrm{sgn}\left[\frac{\partial}{\partial\theta}Y_\ell^m(\theta,\phi)\right] \tag{36}$$

for odd-parity modes, where $\mathrm{sgn}\,[a] \equiv a/|a|$ is the sign function. After this last step, it is *guaranteed* that $\mathbf{B} \to \mathbf{I}$ in the limit $\nu \to 0$.

Regarding the evaluation of the integral appearing in the definition (15) of the intensity moments $\mathcal{I}_{\ell;x}$, it should be remarked that the use of numerical quadrature, although certainly the most straightforward approach, can be problematical. The integrand involves the Legendre polynomials P_ℓ which, toward larger values of ℓ, oscillate rapidly between positive and negative values over the integration interval $0 \leqslant \mu \leqslant 1$. Unless this interval is subdivided extremely finely during the quadrature, numerical noise arising from discretization errors will totally swamp the true value of the integral. This difficulty can be addressed by fitting the photospheric intensity data $\mathcal{I}_x$ with an analytical limb-darkening law (assuming, of course, that this has not already been done).

The limb-darkening law adopted herein is that proposed by Claret (2000), which models the angular dependence of $\mathcal{I}_x$ as

$$\mathcal{I}_x(\mu) = \mathcal{I}_x(1)\left[1 - \sum_{r=1}^{4} a_{r;x}\left(1-\mu^{r/2}\right)\right], \tag{37}$$

where the coefficients $a_{r;x}$ are determined from a generalized least-squares fit to the intensity data (e.g. Press et al. 1992). This law reproduces accurately the behaviour of stellar photospheres across the whole Hertzprung–Russell diagram (Claret 2000); it is a superset of the older linear, quadratic (Wade & Rucinski 1985) and square-root (Diaz-Cordoves & Giménez 1992) limb-darkening laws, all of which can be recovered by setting one or more $a_{r;x}$ values to zero.

Substituting the above expression into equation (15), the intensity moments are written as

$$\mathcal{I}_{\ell;x} = \mathcal{I}_x(1)\int_0^1 P_\ell(\mu)\mu\left[1 - \sum_{r=1}^{4} a_{r;x}\left(1-\mu^{r/2}\right)\right]\mathrm{d}\mu. \tag{38}$$

This may be expressed in the more compact form

$$\mathcal{I}_{\ell;x} = \mathcal{I}_x(1)\sum_{r=0}^{4}\hat{a}_{r;x} I_{\ell,1+r/2}, \tag{39}$$

where

$$\hat{a}_{r;x} = \begin{cases} 1 - a_{1;x} - a_{2;x} - a_{3;x} - a_{4;x} & (r=0), \\ a_{r;x} & (r=1,\ldots,4), \end{cases} \tag{40}$$

and, following Dziembowski (1977),

$$I_{\ell,s} = \int_0^1 P_\ell(\mu)\mu^s\,\mathrm{d}\mu. \tag{41}$$

The utility of adopting a limb-darkening law becomes apparent once it is appreciated that the integrals $I_{\ell,s}$ may be evaluated *analytically*. Abrabowitz & Stegun (1964, their equations 22.13.8 and 22.13.9) give expressions for these integrals in terms of Gamma functions, which can readily be converted into the recurrence relation

$$I_{\ell+2,s} = \frac{s-\ell}{s+\ell+3} I_{\ell,s}, \tag{42}$$

with

$$I_{0,s} = \frac{1}{1+s} \qquad (s > -1), \tag{43}$$

$$I_{1,s} = \frac{1}{2+s} \qquad (s > -2). \tag{44}$$

This relation is particular useful in the context of the present work, as it permits efficient calculation of an ascending sequence $\mathcal{I}_{\ell;x}, \mathcal{I}_{\ell+2;x}, \mathcal{I}_{\ell+4;x}, \ldots$ of intensity moments.

Table 1. The effective temperature $T_{\rm eff}$, logarithmic surface gravity log g, mass M, radius R and metallicity Z of the SPB stellar model considered throughout.

$T_{\rm eff}$	log g	M	R	Z
15 300 K	4.07 dex	4.70 $M_\odot$	3.31 $R_\odot$	0.02

6 CALCULATIONS

In this section, the new formula (26) is used to examine the effect of rotation on the light variations of a pulsating star. The calculations are intended not as an exhaustive exploration of parameter space, but to highlight some of the qualitative phenomena arising due to rotation. The focus is placed throughout on a stellar model constructed using the Warsaw–New Jersey stellar evolution code (see, e.g., Dziembowski & Pamiatnykh 1993); the fundamental parameters of this model, given in Table 1, are intended to be broadly representative of the archetypal SPB star 53 Per (see, e.g. De Ridder et al. 1999).

6.1 Angular functions

As a preliminary step in the investigation, it is useful briefly to examine the influence of rotation on the angular eigenfunctions $\tilde{Y}_\ell^m$. Accordingly, Fig. 1 shows the θ dependence of the $\ell = 2$ eigenfunctions, for all permissible values $m = -2, \ldots, 2$ of the azimuthal order, and at three selected values $\nu = 0, 2, 5$ of the spin parameter.

The figure demonstrates that the effect of increasing ν is to concentrate the eigenfunctions toward the stellar equator. This phenomenon, first noted in the context of stellar pulsation by Lee & Saio (1990), is already familiar in the study of gravity waves within the Earth's atmosphere and ocean (see, e.g. Gill 1982). It occurs because the Coriolis force creates a waveguide centred around the equator, in which pulsation can become trapped. All modes shown in Fig. 1 are subject to confinement within this equatorial waveguide; the retrograde ($m > 0$) modes appear more affected than the prograde ($m < 0$) modes, because the angular eigenfunctions of the former acquire an extra pair of latitudinal nodes when ν exceeds unity (Lee & Saio 1990), which have to be fitted within the waveguide. This extra pair means that, apart from a difference in overall sign, the polar dependence of the $m = 2$ mode approaches that of the $m = 0$ mode as ν increases, as can clearly be seen in the figure.

In the limit of large ν, the boundaries of the equatorial waveguide are situated at $\cos\theta = \pm\nu^{-1}$ (Bildsten et al. 1996) for all but the prograde sectoral ($m = -2$) mode. The latter is unique, in that it remains nodeless in the polar direction at all values of ν; it corresponds to an equatorial Kelvin wave, the angular dependence of which can be approximated by

$$\tilde{Y}_\ell^m(\theta, \phi; \nu) \propto e^{m\nu\cos^2\theta + im\phi} \tag{45}$$

(see Townsend 2003). Clearly, the latitudinal extent of this mode varies as $\nu^{-1/2}$; thus, the degree of equatorial confinement experienced by it will typically be much less than exhibited by others. As will be demonstrated in the following sections, this result has important consequences for the flux variations produced by this mode.

6.2 Differential flux functions

As discussed previously, the differential flux functions (cf. equations 30–32) parametrize the angular dependence of the relative flux variations arising from radius, temperature and gravity perturbations. To demonstrate the general characteristics of these functions, Fig. 2 shows the $\theta_{\rm o}$ (observer colatitude) dependence of the Johnson V-band DFFs appropriate to the model described in Table 1, for the same set of parameters considered in the preceding section. In constructing these functions, the V-band integrated specific intensity $\mathcal{I}_V$ was calculated by interpolating the intensity in a grid of local-thermodynamic equilibrium (LTE) spectra published by Kurucz (1993), and then convolving these data with both the appropriate Johnson-system filter/detector response function given by Bessell (1990), and the transmission functions for two aluminium mirrors and one airmass given by Allen (1973).

The focus is placed first on the $\tilde{\mathcal{T}}_{\ell;V}^m$ (temperature) and $\tilde{\mathcal{G}}_{\ell;V}^m$ (gravity) DFFs. With the introduction of rotation, the overall amplitude of these functions is reduced: as pulsation becomes progressively confined within an equatorial waveguide, less and less of the visible stellar surface is undergoing temperature and gravity perturbations of an amplitude high enough to contribute appreciably toward the respective DFFs. The reduction tends to be rather more pronounced when seen from near the stellar poles ($\theta_{\rm o} \approx 0°$ or $\theta_{\rm o} \approx 180°$), as the contributions from equatorial regions (where the pulsation amplitude remains appreciable) are being viewed edge-on. Generally, the

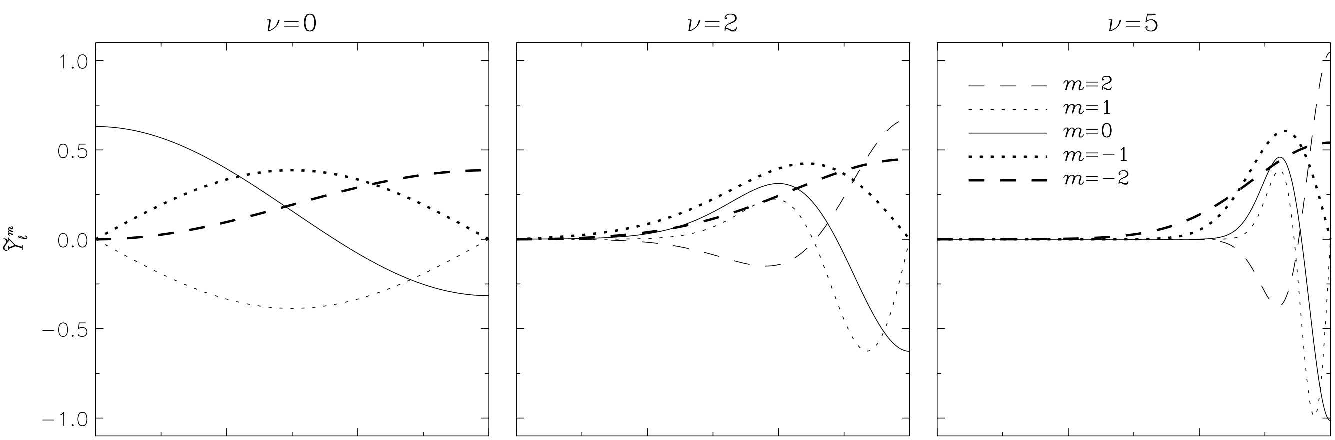

Figure 1. The polar dependence of the rotationally modified angular functions for $\ell = 2$, $m = -2, \ldots, 2$ modes, plotted at azimuth $\phi = 0$ and for three alternative values of the spin parameter ν. The differing line styles are used to indicate the azimuthal order m of each mode.

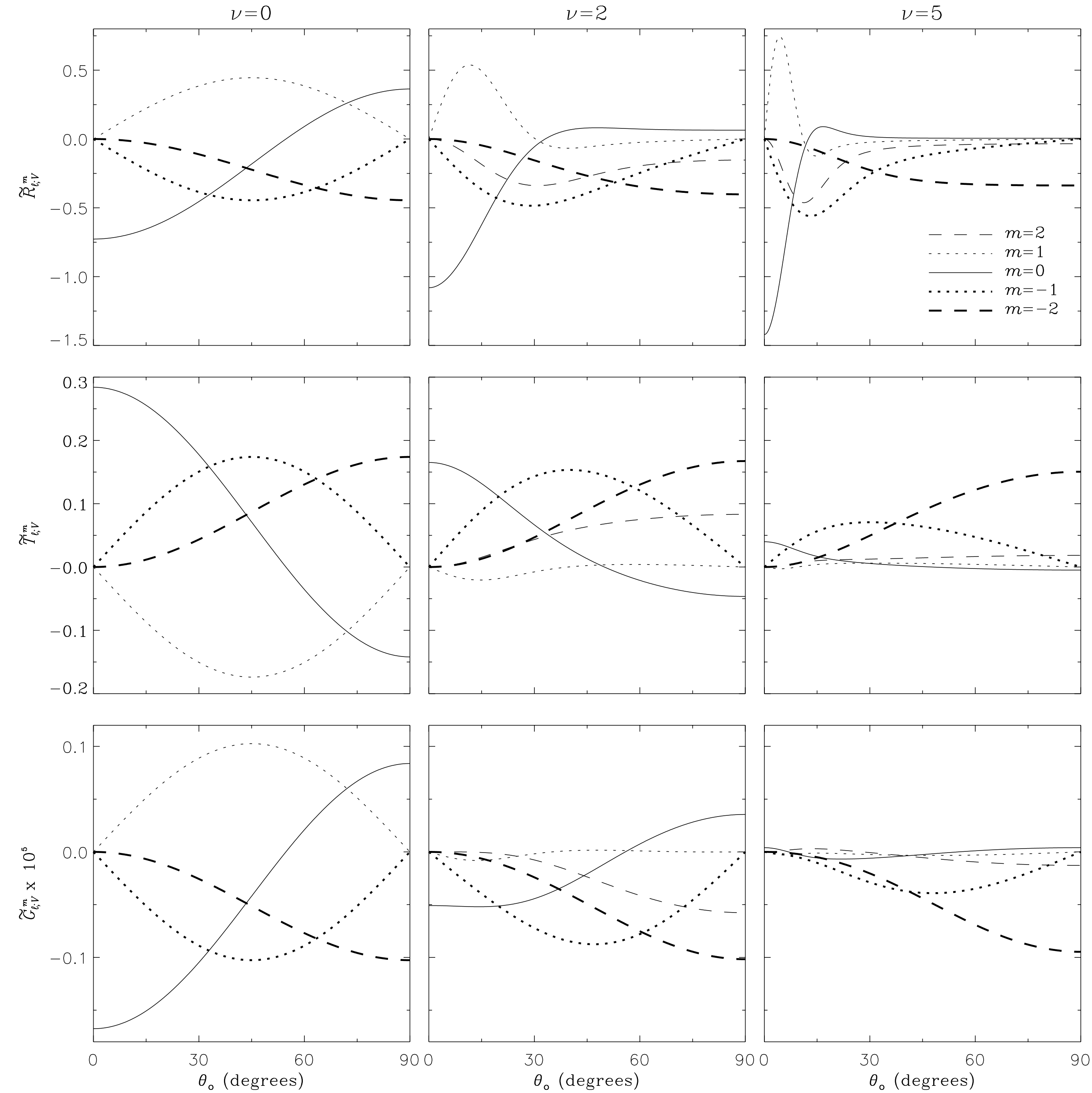

Figure 2. The $\theta_{\rm o}$ dependence of the V-band DFFs for $\ell = 2$, $m = -2, \ldots, 2$ modes of the SPB model (Table 1), plotted at azimuth $\phi = 0$ and for three alternative values of the spin parameter ν. The differing line styles are used to indicate the azimuthal order m of each mode.

DFFs of the prograde sectoral ($m = -2$) mode are affected least by the rotation, as this mode is not as strongly confined as the others (cf. Section 6.1).

The situation is somewhat more complicated in the case of the $\tilde{\mathcal{R}}^m_{\ell;V}$ (radius) DFFs. When the star is viewed from near the equatorial plane ($\theta_{\rm o} \approx 90°$), the effect of increasing ν is to reduce the amplitude of $\tilde{\mathcal{R}}^m_{\ell;V}$, for the same reasons as given above for the $\tilde{\mathcal{T}}^m_{\ell;V}$ and $\tilde{\mathcal{G}}^m_{\ell;V}$ functions. However, as can be seen from Fig. 2, exactly the converse occurs near the polar regions of the star: toward larger values of the spin parameter, the amplitude of $\tilde{\mathcal{R}}^m_{\ell;V}$ becomes enhanced relative to the non-rotating case. The enhancement arises because, when viewed from high latitudes, the equatorially trapped radius perturbations nearly encircle the stellar limb and produce significant modulations in the sky-projected area of the star. Since similar enhancement does not occur to the $\tilde{\mathcal{T}}^m_{\ell;V}$ and $\tilde{\mathcal{G}}^m_{\ell;V}$ functions, the flux variations of an almost pole-on pulsator will become dominated at large ν by contributions from the radius perturbations alone. However, it should be stressed that this effect does not arise for the non-axisymmetric ($m \neq 0$) modes, when the star is *precisely* pole-on: then, the flux variations vanish completely, the same as in the case without rotation. As before, the $m = -2$ mode is the exceptional case; the lesser confinement of this mode produces little polar enhancement in its $\tilde{\mathcal{R}}^m_{\ell;V}$ DFF, as the figure shows.

It is worth calling attention to the fact that the typical amplitude of $\tilde{\mathcal{G}}^m_{\ell;V}$ is approximately five orders of magnitude smaller than those of $\tilde{\mathcal{R}}^m_{\ell;V}$ and $\tilde{\mathcal{T}}^m_{\ell;V}$. This is because, for the stellar parameters given in Table 1, the flux is relatively insensitive to changes in the ionization balance, the latter being governed by the effective gravity. Accordingly, for B-type stars, it is often safe to neglect completely the contribution toward flux variations arising from the $\tilde{\mathcal{G}}^m_{\ell;V}$ term in equation (26).

6.3 Light variations

The new formula (26) indicates that the light variations of a pulsating star can be calculated by summing its three differential flux functions, weighted by the corresponding perturbation coefficients $\{\Delta_R, \Delta_T, \Delta_g\}$. Within the linear theory, the overall scaling of these coefficients is arbitrary; however, their *relative* amplitudes and phases are well defined: for a given Δ_R, Δ_g may be calculated via equation (8) and Δ_T can be found through solution of the pulsation equations under the combined auspices of the traditional, Cowling (1941) and NARF approximations (cf. Section 4).

In the context of the stellar model introduced in Table 1, solution of the approximate equations was accomplished using a modified form of the NARK pulsation code (Townsend 2002). As a result of the appearance of the eigenvalue λ in these equations, the solutions exhibit an implicit dependence on the rotation angular frequency Ω and azimuthal order m. However, it was decided during the NARK calculations to suppress these dependences by setting $\nu = 0$, on the grounds that it is the influence of rotation on the DFFs, rather than on the perturbation coefficients, which is of principal interest in the present paper.

Of the solutions found to the approximated pulsation equations, one was singled out for particular scrutiny, as the lowest-frequency $\ell = 2$ mode unstable due to the iron-group κ mechanism (see, e.g. Dziembowski, Moskalik & Pamyatnykh 1993). This mode, identified within the canonical classification scheme as g_{28}, has a period of 1.40 d, corresponding to a dimensionless pulsation frequency $\bar{\sigma} = 0.229$; Table 2 lists the amplitudes and phases of its three perturbation coefficients. By combining these coefficients with the DFFs presented previously (cf. Fig. 2), the new formula (26) was used to calculate the semi-amplitude $\mathcal{A}_V$ of the V-band variations generated by the $l = 2$ g_{28} modes of azimuthal orders $m = -2, \ldots, 2$.

Fig. 3 presents these data, plotted as a function of θ_o at the three values of ν considered previously. Also shown in the figure are the corresponding $\mathcal{A}_V$ data in situations where solely the radius perturbations or temperature perturbations of the g_{28} modes were allowed to contribute toward the variability. The inclusion of these 'R-only' and 'T-only' cases aims to furnish an indication of what proportion of the variability of the modes arises from radius and temperature perturbations. As discussed in the preceding section, gravity perturbations make a negligible contribution toward the flux changes; thus, a putative 'g-only' case was not considered. Note that, on account of the $\sim 21°$ phase difference between radius and temperature perturbations of the g_{28} modes (cf. Table 2), their $\mathcal{A}_V$ data cannot be calculated simply by summing (or differencing) the $\mathcal{A}_V$ data of the R-only and T-only cases.

Table 2. The amplitudes and phases of the $\{\Delta_R, \Delta_T, \Delta_g\}$ perturbation coefficients for the g_{28} mode of the model star introduced in Table 1.

	Δ_R	Δ_T	Δ_g
Amplitude	8.26×10^{-3}	1.04×10^{-1}	1.70×10^{-2}
Phase	0°.0	−21°.3	−180°.0

Focusing first on the non-rotating ($\nu = 0$) panels, it is clear from a comparison of the R-only and T-only cases that the variability of the g_{28} modes is mostly due to temperature perturbations. This follows from the fact that, as attested by the coefficients in Table 2, the temperature perturbations of the modes are over an order of magnitude stronger than the corresponding radius perturbations – a property characteristic of all low-frequency modes, for which the predominantly horizontal fluid motions produce significant compressive heating and expansive cooling of the stellar surface, but little change in the surface's radius.

Moving now to the $\nu = 2$ and 5 panels, it is evident that the principal effect of rotation is to suppress the variability generated by the g_{28} modes. This reflects the decline in the amplitude of the DFFs contributing toward $\mathcal{A}_V$, due to the equatorial confinement of pulsation (cf. Section 6.2). Near the poles, the suppression of variability is somewhat ameliorated by the fact that that there the $\tilde{\mathcal{R}}^m_{\ell;V}$ DFF is enhanced by the rotation. In fact, from comparing the R-only and T-only cases in the $\nu = 5$ panels, it can be seen that the polar-viewed variability of the g_{28} modes arises predominantly from radius perturbations, even though these perturbations are much weaker than the corresponding temperature perturbations.

As in the preceding sections, the exception to the behaviour described is the prograde sectoral ($m = -2$) mode. The variability produced by this mode is far less affected by rotation than the others, due once again to its reduced equatorial confinement. Since no polar enhancement arises in the $\tilde{\mathcal{R}}^m_{\ell;V}$ DFF of the mode, its light variations are always dominated by temperature perturbations, with only a small component coming from radius perturbations (compare the $m = -2$ data of the R-only and T-only cases).

To illustrate further the effects of rotation, Fig. 4 plots the rms angle-average of the semi-amplitude,

$$\langle \mathcal{A}_V \rangle = \left[(4\pi)^{-1} \int_0^{2\pi} \int_0^{\pi} \mathcal{A}_V^2 \sin\theta_o \, d\theta_o \, d\phi_o \right]^{1/2} \tag{46}$$

as a function of spin parameter, for the same g_{28} modes considered previously, and with the same inclusion of the R-only and T-only cases. This quantity, a function of ν alone, characterizes the mean level of variability exhibited by a large sample of stars which, although otherwise identical, possess randomly oriented rotation axes.

The $\langle \mathcal{A}_V \rangle$ data plotted in Fig. 4 highlight quite dramatically the rotation-promoted suppression of the photometric variability of the star. For $\nu \gtrsim 1$, the rms semi-amplitude of all but the usual prograde sectoral mode declines rapidly with increasing spin parameter. The rate of decline depends on the azimuthal order, being most rapid for the $m = 1$ mode. However, below the $\nu \approx 1$ cut-off – the point at which the equatorial waveguide forms (see Townsend 2003) – the rms semi-amplitude of *all* modes is approximately (or completely, in the limit $\nu \to 0$) independent of m.

From inspection of the R-only panel, it can be seen that the polar enhancement of the $\tilde{\mathcal{R}}^m_{\ell;V}$ DFF is insufficient to reverse the decline in $\langle \mathcal{A}_V \rangle$. This is because the polar regions account for only a small fraction of the total solid angle subtended by the star, and therefore cannot counteract the reduction in variability occurring at the mid- and equatorial latitudes. However, a comparison with the T-only panel reveals that the polar enhancement of $\tilde{\mathcal{R}}^m_{\ell;V}$ does have the effect of slowing the rate of decline in $\langle \mathcal{A}_V \rangle$.

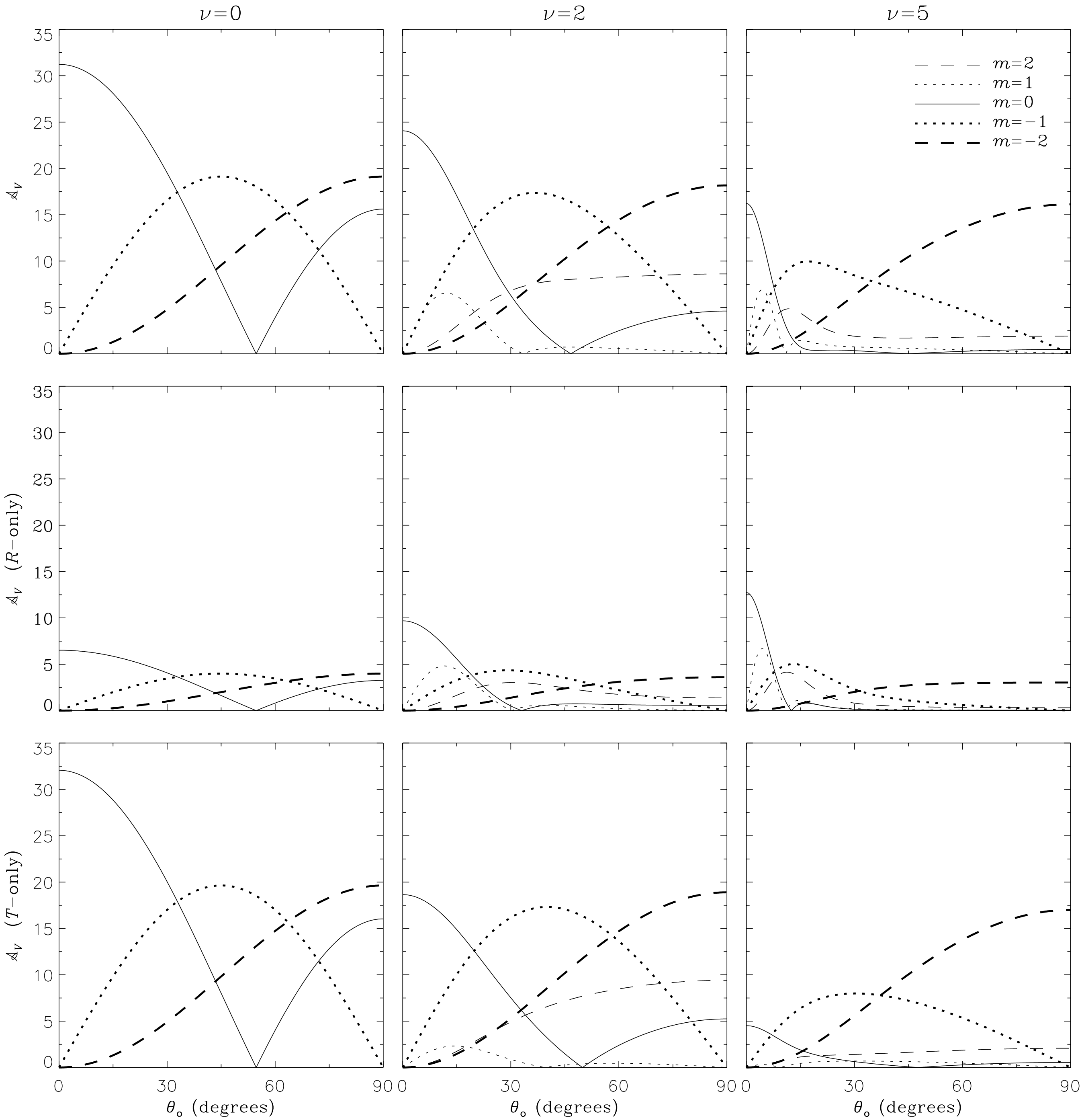

Figure 3. The θ_o dependence of the V-band flux semi-amplitude (in millimagnitudes) for the $\ell = 2$, $m = -2, \ldots, 2$ g_{28} modes of the SPB model (Table 1), plotted at three alternative values of the spin parameter ν; also shown are the R-only and T-only cases discussed in the text. The differing line styles are used to indicate the azimuthal order *m* of each mode.

The above findings are interesting, in light of the recent photometric surveys of open clusters by Balona (1994), Balona & Koen (1994, 1995) and Balona & Laney (1996). These authors failed to find much evidence for low-frequency NRP, but did note that the stars observed appeared to be, systematically, rapid rotators (V sin $i \geqslant 100$ km s^{-1}); this led them to infer that rapid rotation may actively suppress the excitation of g modes. However, the behaviour exhibited in Fig. 4 suggests an alternative hypothesis: that g modes *are* excited, but that equatorial confinement by the Coriolis force may push their photometric signatures below typical observational detection thresholds.

The recent stability calculations of Lee (2001) partly support this hypothesis, by finding significant numbers of unstable g modes for a $M = 4$ M$_\odot$ stellar model, even in the inertial ($\nu > 1$) regime. However, for the hypothesis to remain standing, two further conditions must be met. First, that the dominant-amplitude unstable modes are *not* prograde sectoral, because the latter would probably remain detectable even in the limit of rapid rotation. Secondly, that

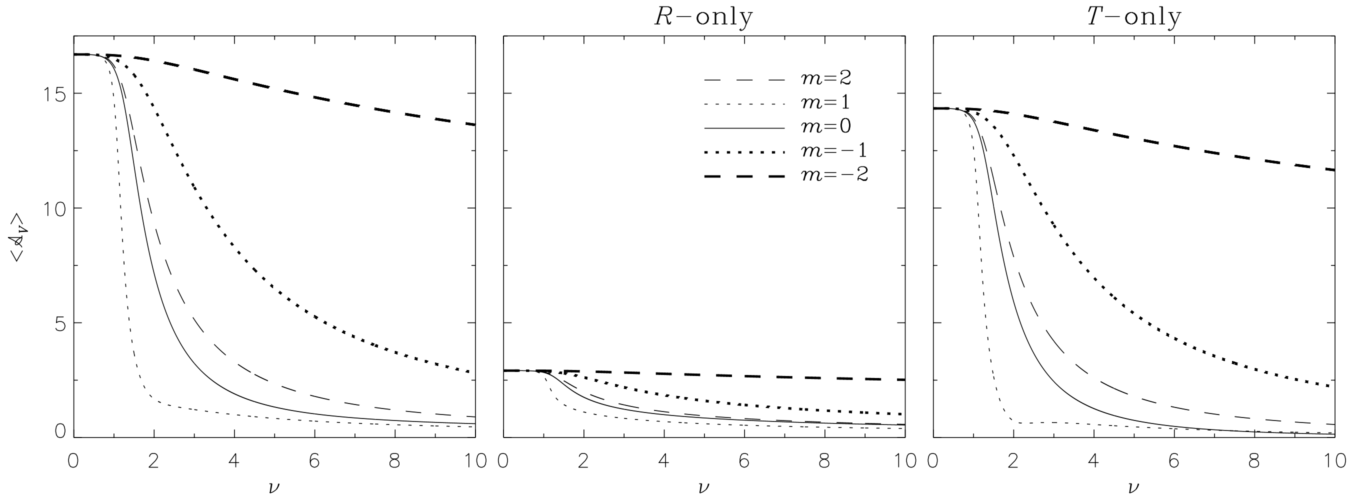

Figure 4. The rms angle-averaged semi-amplitude (in millimagnitudes) of the V-band light variations for the $\ell = 2$, $m = -2, \ldots, 2$ g_{28} modes of the SPB model (Table 1), plotted as a function of the spin parameter ν; also shown are the R-only and T-only cases discussed in the text. The differing line styles are used to indicate the azimuthal order m of each mode.

the frequencies of these dominant modes are sufficiently low to yield corresponding spin-parameter values in excess of unity; here, it is noted that the condition $\nu > 1$, applied to the g_{28} mode considered above, corresponds to the requirement that $\Omega > 0.21\ \Omega_{\rm crit}$, where $\Omega_{\rm crit}$ is the critical rotation angular frequency at which centrifugally assisted equatorial mass loss will commence.

Whether or not these two conditions are met is difficult to establish. Currently, there exists no theory capable of predicting the steady-state saturation amplitude of those NRP modes known, from linear analyses, to be unstable. Therefore, determining which modes are dominant is not as yet possible. Nevertheless, the hypothesis advanced above remains a viable explanation for the apparent paucity of low-frequency NRP in the clusters studied by Balona (1994) and others.

6.4 Colour variations

Toward the end of securing NRP mode identifications, photometric observations come into their own when the variability in differing passbands is compared. For the reasons discussed in Section 2, the ratio

$$\mathcal{A}_{[x,x']} \equiv \frac{\mathcal{A}_x}{\mathcal{A}_{x'}} \tag{47}$$

between the semi-amplitudes of the flux variations in two differing passbands, here denoted by x and x', will be dependent solely on the harmonic degree ℓ and the relative amplitudes and phases of the perturbation coefficients $\{\Delta_R, \Delta_T, \Delta_g\}$. Therefore, measurement of such ratios, characterizing the fluctuations in the colour of a star, can be used to ascertain the harmonic degree of the mode(s) responsible for observed variability (see, e.g., Heynderickx et al. 1994; De Cat 2001).

Unfortunately, the foregoing discussion applies only to non-rotating stars. This is demonstrated in Fig. 5, where $\mathcal{A}_{[V,U]}$ – the ratio between the Johnson V- and U-band semi-amplitudes – is plotted as a function of ν, for the modes considered in the preceding sections, and at three alternative values of $\theta_{\rm o}$. The semi-amplitude data for the U band were calculated in an identical manner to the V-band data (cf. Section 6.3), save for the fact that the Bessell (1990) filter/detector response function appropriate to the U band was used to obtain the integrated specific intensity.

The figure indicates quite clearly that rotation causes the amplitude ratio to acquire a dependence on both m and $\theta_{\rm o}$. For non-zero ν, the DFFs associated with a given passband do not share the same angular (i.e. $\theta_{\rm o}$ and m) dependence as their counterparts in another passband; accordingly, when equation (47) is used to calculate $\mathcal{A}_{[V,U]}$, this dependence will not cancel between the numerator and the denominator. An inevitable consequence of this result is that photometric amplitude ratios cannot easily be used as a diagnostic for the harmonic degree of a mode; indeed, the extremely complicated behaviour of the $\mathcal{A}_{[V,U]}$ data plotted in Fig. 5 raises the question of whether these ratios can offer *any* useful information concerning the NRP of a rotating star.

In fact, the data in the figure do indicate one important heuristic: that, in the limit of rapid rotation and/or low pulsation frequency, the amplitude ratio is approximately independent of propagation direction (as determined by the sign of m). This is most evident in the sin $\theta_{\rm o} = 0.9$ panels of the figure: with increasing ν, the $m = \pm1$ curves approach one another quite closely. The $m = \pm2$ curves exhibit similar behaviour, albeit to a lesser extent. The reasons underlying such behaviour are difficult to fathom; as Fig. 1 shows, the polar dependence of the same $|m|$ modes is different for prograde and retrograde propagation directions. One therefore would not anticipate that their amplitude ratios coincide in the limit of large ν.

7 DISCUSSION AND SUMMARY

In the preceding sections, a new semi-analytical formula was presented for the flux variations originating from low-frequency g modes in uniformly rotating stars. Although the formula (equations 26 and 30–32) was developed under the auspices of the adiabatic approximation, it was argued that it can also be applied to non-adiabatic pulsation, as for low-frequency modes perturbations will vary most rapidly in the radial direction, and the NARF approximation (cf. Section 4) may therefore be employed.

Of the findings of the present work, the most significant is that, for all modes apart from prograde sectoral, the confinement of pulsation within an equatorial waveguide suppresses the photometric variability exhibited by a star. In the limit of rapid rotation and/or small pulsation frequency, the variability will remain at detectable levels only when viewed from very near the stellar poles. This

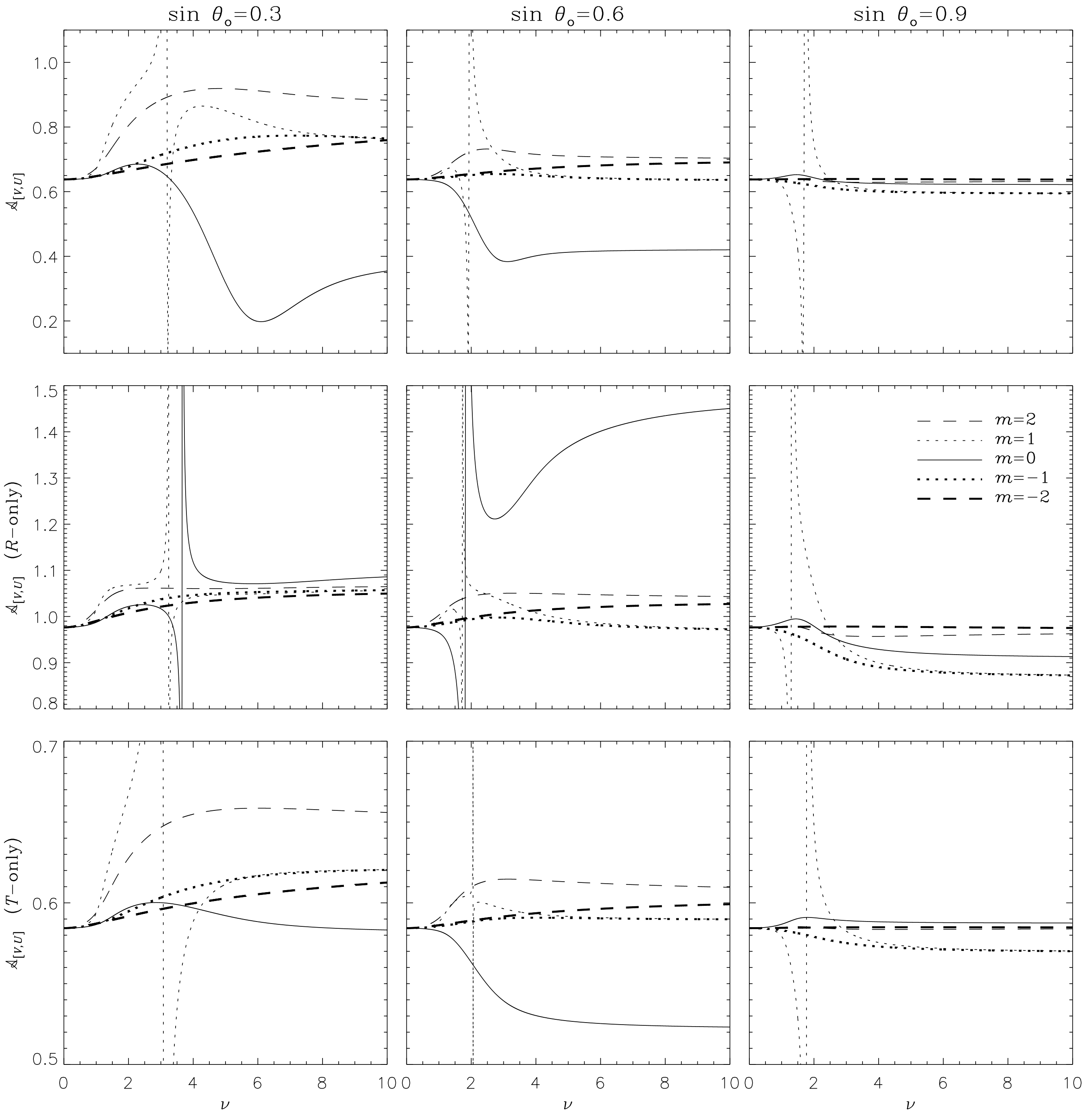

Figure 5. The ratio $\mathcal{A}_{[V,U]}$ between the V- and U-band semi-amplitudes, for the $\ell = 2$, $m = -2\ldots2$ g_{28} modes of the SPB model (Table 1), plotted as a function of the spin parameter ν at three alternative values of θ_o; also shown are the R-only and T-only cases discussed in Section 6.3. The differing line styles are used to indicate the azimuthal order m of each mode.

may explain why Balona (1994), and subsequent authors, found so few long-period modes in their surveys of rapidly rotating cluster members.

A secondary finding is that rotation completely disrupts the independence, from the azimuthal order and observer colatitude, of the amplitude ratio between variability in differing passbands. This result, anticipated by a number of authors (e.g. Balona & Dziembowski 1999), means that it will be very difficult to ascertain, from multicolour photometry alone, the identity of modes excited in a given rotating star. Likewise, as Daszyńska-Daszkiewicz et al. (2002) have pointed out in a recent paper, great care must be used when applying standard (i.e. non-rotating) mode-identification procedures to rotating stars, lest these procedures lead to incorrect parameters.

The Daszyńska-Daszkiewicz et al. (2002) study sought to address issues similar to those discussed in the present work, and a comparison between the two is therefore appropriate. These authors considered the influence on photometric variability of rotation-induced coupling phenomena, whereby modes of the same azimuthal order, but differing harmonic degree, interact with one another via the action of both Coriolis and centrifugal forces. Such interactions occur only when the modes are undergoing an avoided crossing, and thereby exhibit similar pulsation frequencies; by restricting their analysis to couplings between a small number of modes (up to three), the authors were able to incorporate a full treatment of non-adiabatic processes in their calculations.

With regard to present work, the traditional approximation neglects, a priori, the possibility of such mode couplings (see, e.g. Lee & Saio 1989). However, it is this very neglect which permits the separation of the pulsation equations, allowing investigations to be extended into the inertial regime ($\nu > 1$). This regime remains inaccessible to the Daszyńska-Daszkiewicz et al. (2002) study, as a large number of spherical harmonics (in the present work, 200) are required to express accurately the angular dependence of a given mode.

Accordingly, it can be concluded that the present study is complementary to, rather than in competition with, that of Daszyńska-Daszkiewicz et al. (2002); while the latter recommends itself on its consideration of rotation-induced mode coupling, and on its full incorporation of non-adiabatic effects, the former provides valuable insights into photometric variations in the inertial rapid-rotation/low-frequency limit, with a restricted (NARF) non-adiabatic treatment. Clearly, what is required of future investigations is the unification of these strengths in a single NRP model, which treats both non-adiabaticy and rotation without recourse to approximations. From such a model can be devised the tools necessary to secure mode identifications for those classes of variable star strongly influenced by rotation, at last opening them up to asteroseismological scrutiny.

ACKNOWLEDGMENTS

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