A SEMI-EXPLICIT MPC SET-UP FOR CONSTRAINED PIECEWISE AFFINE SYSTEMS

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Abstract

Recently quite some effort has been dedicated towards the use of Model Predictive Control (MPC) for regulating discrete-time piecewise affine systems. One of the obstructions for implementation is the high on-line computational load. In this paper we present an approach to reduce the on-line computations by using an algorithm that solves off-line a controllability problem with respect to an invariant target set. The algorithm generates a *tree* containing *minimal discrete events* controllable paths to the target set. For an initial state (or a measured state), a controllable path to the target set with a minimal number of discrete events is easily obtained and a resulting ordered sequence of state space regions (sets) is pre-computed; each region corresponds to a single sub-model, part of the piecewise affine system. It is then shown how, under certain assumptions, this controllable path can be used to design a computationally more friendly semi-explicit MPC algorithm for constrained piecewise affine systems. Finally, an example is given for illustration purposes.

1 Introduction

Over the recent years, researchers have become increasingly interested in the framework related to the stability and the control problems for piecewise affine systems. This growth in interest is motivated by the fact that many nonlinear systems can be (arbitrarily closely) approximated using piecewise affine systems [12] and by the fact that the interconnection of finite automata and linear systems yields piecewise affine systems as well [13]. Moreover, piecewise affine systems are very useful as they are equivalent to many other relevant classes of hybrid systems [7].

Several control algorithms developed for piecewise affine systems are designed using optimal control or Model Predictive Control (MPC) techniques. The first hybrid MPC algorithm, developed for mixed logical dynamical systems (equivalent to piecewise affine systems under certain mild conditions), has been presented in [1]. Unfortunately, this algorithm has a drawback that consists of a high on-line computational complexity, mainly caused by the mixed integer quadratic programming problem (*NP* hard) that has to be solved on-line, at each discrete-time instant. Then, an explicit piecewise linear control law that reduces the computational burden has been presented in [2]. However, recasting the MPC problem as a multiparametric mixed integer linear programming problem and obtaining an explicit solution off-line did not completely eliminate the computational load, as one still has to perform a search algorithm to determine the relevant control region (and the number of resulting control regions is increasing with the length of the prediction horizon). The present research work in this area deals with finding ways to reduce the number of on-line calculations, such as the reachability based strategy of branch-and-bound nature developed for piecewise affine systems in [3]. An optimal and receding horizon control algorithm for piecewise affine systems with bounded disturbances, based on the robust controllable sets theory [8], has been recently presented in [9]. The optimal control is determined in this case by comparing the solutions of a finite number of multiparametric LP problems. Another MPC algorithm for *continuous* piecewise affine systems, which requires solving on-line a finite number of LP problems, has been presented in [5].

In this paper we present a method that aims at reducing the on-line computational load encountered in MPC algorithms for hybrid systems. This is achieved by formulating a new algorithm (based on the controllable sets theory [6,8]) that solves off-line the controllability problem with respect to an invariant target set for constrained discrete-time piecewise affine systems. The algorithm starts from the target set, which is defined as the root node, calculates the *discrete events* (transitions between affine sub-models) controllable sets with respect to the target set and organizes the resulting state space regions in a *tree-like* structure. Each node of the tree, corresponding to a single sub-model in the piecewise affine system, will be a maximal (discrete-time) controllable set with respect to a *parent* node and a controllable set that is i discrete events away from the target set. For an initial state (or a measured state), a controllable path to the target set with a minimal number of discrete events is easily obtained and a resulting ordered sequence of state space regions (sets) is pre-computed. It is then shown that under some assumptions the *minimal discrete events* controllable path can be used to develop a computationally more friendly semi-explicit (sub-optimal) MPC algorithm for piecewise affine systems.

2 Problem formulation and main result

Consider the time-invariant discrete-time piecewise affine system described by equations of the form [12]:

$$x_{k+1} = A_j x_k + B_j u_k + f_j \text{ if } x_k \in \Omega_j, \qquad (1)$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the control input vector, $A_j \in \Re^{nxn}$, $B_j \in \Re^{nxm}$, $f_j \in \Re^n$, $\forall j \in S = \{1, 2, ..., s\}$, $\{\Omega_j \mid j \in S\}$ is a finite set of disjoint-convex polyhedra (not necessarily closed) in $\Re^n$ and $k \geq 0$ denotes the discrete-time instant. In addition, *design constraints* of the form:

$$x_k \in X, \; u_k \in U, \; \forall k \geq 0 \qquad (2)$$

are imposed with respect to the states and the inputs of (1), where X and U are convex and compact sets containing the origin in the interior. In the sequel we will need the concepts of maximal and one-step controllable sets [8] with respect to an affine sub-model j in the piecewise affine system (1)-(2),

$$x_{k+1} = A_j x_k + B_j u_k + f_j \text{ if } x_k \in \Omega_j \cap X =: X_j. \qquad (3)$$

Definition 1 The one-step controllable set for sub-model (3) with respect to a target set X_T is the set of states $K_j^1(X_T)$ in X_j for which there exists an admissible control input such that the target set is reached in one discrete-time step (sampling period), i.e.

$$K_j^1(X_T) = \{x \in X_j \mid \exists u \in U : A_j x + B_j u + f_j \in X_T\}. \qquad (4)$$

Definition 2 The maximal controllable set *without switching* for sub-model (3) with respect to a target set X_T is the set of states $K_j^\infty(X_T)$ in X_j for which there exists a finite sequence of admissible control inputs such that the target set is reached in a finite number of discrete-time steps and the state trajectory does not leave X_j until it enters X_T, i.e.

$$K_j^\infty(X_T) = \{x \in X_j \mid \exists r, \exists \mathbf{u} = (u_0, ..., u_{r-1}) \in U^r : x_0, ..., x_{r-1} \in X_j \text{ and } x_r \in X_T\}, \qquad (5)$$

where $\mathbf{x} = (x_0, ..., x_r)$ is generated by (3) for the input sequence $\mathbf{u}$ and initial state $x_0 = x \in X_j$. These definitions can be straightforwardly extended to the piecewise affine system (1)-(2), i.e. the one-step controllable set is in this case the union of all sets (4) and the maximal controllable set *without switching* is the union of all sets (5) over all $j \in S$.

Note that the algorithms developed for the computation of the controllable sets in [9] require closedness of all regions X_j. Since we chose the Ω_j regions to be disjoint this is not necessarily the case. However, as indicated in Remark 2 of [7], this is numerically not really a problem.

We introduce now the notions of a *controllable path* and a *minimal discrete events* controllable path.

Definition 3 We call $\mathbf{x} = (x_0, ..., x_r)$ with $x_i \in X$ for all $i = 0, ..., r$ a (discrete-time) *controllable path* from x_0 to X_T if $\exists \mathbf{u} = (u_0, ..., u_{r-1}) \in U^r$ such that $(\mathbf{x}, \mathbf{u})$ is a trajectory of system (1)-(2) and $x_r \in X_T$. The corresponding discrete events controllable path is defined by the sequence of affine sub-models in (1) $\mathbf{j} = (j_0, ..., j_r)$ with $j_i \in S$ for all $i = 0, ..., r$ (i.e. $x_i \in X_{j_i}$).

Definition 4 Given a discrete events controllable path $\mathbf{j} = (j_0, ..., j_r)$, then the number of discrete events, denoted by $\#\mathbf{j}$, is defined as the number of elements in the set $\{ i \in \{0, ..., r-2\} \mid j_i \neq j_{i+1} \}$.

Definition 5 A discrete events controllable path $\mathbf{j}$ from x_0 to a target set X_T is a *minimal discrete events* controllable path from x_0 to X_T if it has a minimal number of discrete events, i.e. for all discrete events controllable paths $\tilde{\mathbf{j}}$ from x_0 to X_T, $\#\tilde{\mathbf{j}} \geq \#\mathbf{j}$.

Definition 6 The *i-discrete-events* controllable set for system (1)-(2) with respect to a target set X_T is the set

$X_T^i = \{x_0 \in X \mid \exists$ a discrete events controllable path $\mathbf{j}$ from x_0 to X_T such that $\#\mathbf{j} = i$ and there does not exist a discrete events controllable path $\tilde{\mathbf{j}}$ from x_0 to X_T with $\#\tilde{\mathbf{j}} < i\}$. (6)

This set is the collection of all the states in X from which a target set X_T can be reached in *i transitions* between the affine sub-models in (1) and not less. It is evident that we have the following result:

Lemma 1 *The zero-discrete-events controllable set for the piecewise affine system* (1)-(2) *with respect to a target set* X_T *is the maximal controllable set without switching for the piecewise affine system* (1)-(2) *with respect to* X_T*, i.e.*

$$X_T^0 = \bigcup_{j \in S} K_j^\infty(X_T). \qquad (7)$$

The controllability problem with respect to discrete events controllable paths for the piecewise affine system (1)-(2) can now be formulated as follows:

Problem 1 *Given a target set X_T and an initial state space region X_0, calculate for every initial state in X_0 a minimal discrete events controllable path (in X) - if it exists - with respect to X_T.*

Problem 1 is formulated such that it requires the calculation of a minimal discrete events controllable path because fewer discrete events yield a smaller storing capacity (this aspect is discussed in Section 3). A similar approach, regarding the reachability problem for piecewise affine systems, but formulated with respect to discrete-time steps rather than discrete events has been presented in [3]. Next, we formulate an algorithm that solves Problem 1 and organizes the resulting state space regions in a *tree-like* structure.

Algorithm 1

0. Normalize all regions Ω_j. The normalized regions will be $X_j = \Omega_j \cap X$ for all $j \in S$.
1. Calculate the maximal controllable sets (5) in all regions X_j, $j \in S$ with respect to X_T, namely $X_{jT}^0 = K_j^\infty(X_T)$, and build the *zero-discrete-events* controllable set (7) for the piecewise affine system (1)-(2) with respect to the target set X_T:
$$X_T^0 = \bigcup_{j \in S} X_{jT}^0 . \quad (8)$$
   - Build a tree with the target set X_T as the root node and with the elements of (8) as *child* nodes, and associate *level* $i = 0$ to this level in the tree, containing all X_{jT}^0 sets.
   - Build the set of remaining *empty* regions for level $i = 0$ (the set of states that cannot be steered to the target set with zero discrete events):
$$X_{empty}^0 = \bigcup_{j \in S} X_{j,empty}^0 \text{ where } X_{j,empty}^0 = X_j \setminus X_{jT}^0 . \quad (9)$$
   - If $X_0 \subseteq X_T^0$ STOP; Else go to step 2.
2. For the *ith* level of the tree perform the following operations:
   - Compute the index sets $\mathrm{Set}(X_T^i) := \{j \in S | X_{jT}^i \neq \phi\}$ and $\mathrm{Set}(X_{empty}^i) := \{j \in S | X_{j,empty}^i \neq \phi\}$;
   - For$_1$ $index_1 \in$ Set(X_T^i) do: Let $X_{index_1 T}^i$ be the new target set;
   - For$_2$ $index_2 \in$ Set(X_{empty}^i) do:
     - Calculate the maximal controllable set (5) in $X_{index_2,empty}^i$ for the affine sub-model ($index_2$) with respect to the target set $X_{index_1 T}^i$:
$$X_{index_2 index_1}^{i+1} := X_{index_2,empty}^i \cap K_{index_2}^\infty (X_{index_1 T}^i) \quad (10)$$
$$X_{index_2,empty}^i := X_{index_2,empty}^i \setminus X_{index_2 index_1}^{i+1} \quad (11)$$

     End$_2$;
     - $\mathrm{Set}(X_{empty}^i) = \{index_2 \in S | X_{index_2,empty}^i \neq \phi\}$;

     End$_1$;
   - Insert the level $i + 1$, X_T^{i+1}, in the controllable sets tree:
$$X_T^{i+1} = \bigcup_{index_1} \bigcup_{index_2} X_{index_2 index_1}^{i+1} . \quad (12)$$
   - Build the empty set for the tree level $i + 1$:
$$X_{empty}^{i+1} = \bigcup_{index_2} X_{index_2,empty}^i . \quad (13)$$
3. Set $i = i + 1$. If $X_0 \subseteq X_T^i$ or if $X_{empty}^i = X_{empty}^{i-1}$, STOP; Else go to step 2.

Remark 1 If Algorithm 1 has been terminated with $X_0 \subseteq X_T^i$, then a *tree* that contains a minimal discrete events controllable path to the target set from any x_0 in X_0 has been generated. The *l-th* level of the tree consists of the union of state space regions (sets) from which the target set can be reached in *l* discrete events (*l* transitions between different affine sub-models in (1)) and not less. Note that, two consecutive sets in the controllable path cannot belong to the same *original* region X_j (otherwise there does not exist a discrete event), but this is allowed for sets belonging to the same level of the tree. Only a single switching succession has been chosen from the set of possible minimal discrete events paths and this additional freedom might be used for optimising other performance criteria. In this work we are only interested in selecting one minimal discrete events controllable path.

Remark 2 Note that the sets X_{jT}^i are not necessarily convex, which complicates matters for an MPC set-up. Some conditions are known that guarantee convexity (see e.g. [4,8]) but in general these are not satisfied. Actually, every set (10) can be a non-convex or even a non-connected set and then the computational complexity of Algorithm 1 increases (in the "worst" case it leads to the same state space decomposition as in [9]). Still, each *i*-discrete-events controllable set can be decomposed as a union of one-step convex sets [8] that could again be represented as a sub-tree structure. Then, a single switching succession with a minimal number of discrete events can still be chosen and this can be used under suitable assumptions to implement an on-line computationally more friendly MPC algorithm.

3 A semi-explicit MPC algorithm for PWA systems

In this section, a MPC methodology is utilized to develop a controller that *steers* the states of system (1)-(2) along a minimal discrete events controllable path towards the target set. Hence, one could call the resulting MPC algorithm *semi-explicit*, as the minimal discrete events controllable path is calculated off-line, while the computation of the control input is still carried out on-line. The following assumption will be needed in the sequel:

Assumption 1 *The target set X_T contains the origin and is positively invariant if a piecewise linear state feedback $g(x_k)$ is applied. The feedback $g(x_k)$ has the property* *that the state trajectory and the control input of the closed-loop piecewise affine system* (1)-(2) *with* $u_k = g(x_k)$ *converge to the origin, if the initial state* x_0 *satisfies* $x_0 \in X_T$.

Typically, a set X_T satisfying Assumption 1 would be the maximal output admissible set [6] defined for the closed-loop piecewise affine system. Consider now the piecewise affine system (1)-(2), a target set X_T satisfying Assumption 1 and the L levels controllable sets tree, $\bigcup_{l\in\{0,...,L-1\}} X_T^l$, calculated off-line using Algorithm 1. Then, for a known initial state (or a measured state) we find the set $X_{j_0}^l = X_T^l \cap X_{j_0}$ in which x_0 resides. This operation needs to be performed only once, at start-up (or on-line, by a supervisory system, at a slower pace than the sampling frequency of the MPC controller, for the actual measured state). The sets corresponding to the minimal discrete events controllable path are easily obtained from the tree generated by Algorithm 1, i.e.

$$(X_{j_0}^l, X_{j_1}^{l-1}, ..., X_{j_i}^{l-i}, ..., X_{j_l}^0, X_T), \tag{14}$$

where $X_{j_i}^{l-i}$ is shorthand for $X_{j_i j_{i+1}}^{l-i}$ $\forall l \in \{0,...,L-1\}$, $j_0,..., j_l \in S$, $j_i \neq j_{i+1}$ $\forall i \in \{0,...,l-1\}$ and $X_{j_l}^0$ is shorthand for $X_{j_l T}^0$. The control goal is to *drive* the states of the piecewise affine system (1)-(2) to the target set using the sequence of discrete events controllable sets (14) (which are at least unions of convex sets, as indicated in Remark 2). In order to do this, choose for each sub-model j_i in (14) a convex set $\widetilde{X}_{j_{i+1}}^{l-(i+1)}$ in $X_{j_{i+1}}^{l-(i+1)}$ and the corresponding one-step controllable set $K_{j_i}^1(\widetilde{X}_{j_{i+1}}^{l-(i+1)}) \subseteq K_{j_i}^1(X_{j_{i+1}}^{l-(i+1)})$ (always convex [8]) satisfying the following assumption:

Assumption 2 *There exists an equilibrium pair* (x_{ej_i}, u_{ej_i}) *with* x_{ej_i} *in the interior of* $K_{j_i}^1(\widetilde{X}_{j_{i+1}}^{l-(i+1)})$ *and* $u_{ej_i} \in U$ *for each affine sub-model* j_i *in* (14).

Next, consider the optimal control problem given by the QP:

Problem 2

$$\begin{aligned}\min_{\mathbf{u}_k} J_i(\mathbf{u}_k, x_k) = \min_{\mathbf{u}_k}(\left\|x_{k+N} - x_{ej_i}\right\|_{P_{j_i}}^2 + \\ + \sum_{p=0}^{N_{j_i}-1}\left\|x_{k+p} - x_{ej_i}\right\|_{Q_{j_i}}^2 + \left\|u_{k+p} - u_{ej_i}\right\|_{R_{j_i}}^2)\end{aligned} \tag{16}$$

subject to

$$x_{k+1} = A_{j_i} x_k + B_{j_i} u_k + f_{j_i} \tag{17}$$

$$u_{k+p} \in U, \quad p = 0,..., N_{j_i} - 1 \tag{18.1}$$

$$x_{k+p} \in X_{j_i}, \text{ for } p = 0,1,..., N_{j_i}, \tag{18.2}$$

where N_{j_i} is the prediction horizon and the terminal weight matrix P_{j_i} satisfies the discrete-time Lyapunov equation

$$P_{j_i} - A_{j_i}^T P_{j_i} A_{j_i} = Q_{j_i}. \tag{19}$$

Solving Problem 2 for the state measured at each sampling instant and applying to the plant only the first element of the optimal control sequence yields a stabilizing feedback controller for any tuning parameters provided that A_{j_i} is stable and the constraints (18) are feasible. If A_{j_i} has unstable modes, a terminal equality constraint could be added to Problem 2 and the prediction horizon has to be chosen in order to guarantee feasibility under constraints (18) and hence, convergence (see [10] for a possible solution). The control of Problem 2 will asymptotically steer the states of the affine sub-model j_i towards $x_{ej_i} \in K_{j_i}^1(\widetilde{X}_{j_{i+1}}^{l-(i+1)})$. Once the set $K_{j_i}^1(\widetilde{X}_{j_{i+1}}^{l-(i+1)})$ has been reached (convergence to x_{ej_i} is not required), the state can be driven in the next state space region in (14). To guarantee the *transition*, the following one-step optimal control problem has to be solved:

Problem 3

$$\begin{aligned}\min_{u_k} J_i(u_k, x_k) = \min_{u_k}(\left\|x_{k+1} - x_{ej_{i+1}}\right\|_{Q_{j_i}}^2 + \\ + \left\|u_k - u_{ej_{i+1}}\right\|_{R_{j_i}}^2)\end{aligned} \tag{20}$$

subject to

$$x_{k+1} = A_{j_i} x_k + B_{j_i} u_k + f_{j_i} \tag{21}$$

$$u_k \in U, \quad x_{k+1} \in \widetilde{X}_{j_{i+1}}^{l-(i+1)} \tag{22}$$

The solution of Problem 3 will *activate* the transition to the next state space region in (14) and then, the solution of Problem 2 formulated for the next affine sub-model (i.e. sub-model j_{i+1}) will steer the states of system (1)-(2) towards the next one-step controllable set. By repeating this procedure, the target set X_T will ultimately be reached. Moreover, this will be achieved with a minimal number of discrete events. Note that, in principle, x_{ej_i} does not have to be an equilibrium for the affine sub-model j_i, but only a *reference point* since the solution of Problem 2 should only drive the state inside the set $K_{j_i}^1(\widetilde{X}_{j_{i+1}}^{l-(i+1)})$. Still, in this case it is not clear how to formulate a QP complexity MPC algorithm with guaranteed convergence.

Remark 3 While Problem 3 is feasible $\forall x_k \in K_{j_i}^1(\widetilde{X}_{j_{i+1}}^{l-(i+1)})$, Problem 2 is feasible $\forall x_k \in \widetilde{X}_{j_i}^{l-i}$ only for some N_{j_i} chosen such that the constraints (18) are feasible. Possible approaches for determining a prediction horizon that ensures feasibility of Problem 2 (for each sub-model in (14)) can be found in [8,10].

Consider now the following algorithm.

MPC Algorithm 1

0. Select the corresponding sequence of sets (14) associated with a minimal discrete events controllable path for $x_0 \in X$ (or for the measured state at a slower pace than the sampling frequency of the MPC controller) from the tree generated by Algorithm 1, store the elements necessary for formulating Problems 2 and 3, select N_{j_i} such that each MPC sub-problem is feasible and set $i = 0$.
1. Measure x_k ($x_k = x_0$ at start-up) and check:
   - If $x_k \in K^1_{j_i}(\widetilde{X}^{l-(i+1)}_{j_{i+1}})$, solve Problem 3, apply u^*_k to system (1), set $i = i + 1$ and go to step 1.
   - Else, solve Problem 2, apply $\mathbf{u}^*_k(1) = u^*_k$ to system (1) and go to step 1.

Remark 4 MPC Algorithm 1 guarantees convergence to the target set with a minimal number of discrete events and with no constraint violation by solving at each discrete-time instant Problem 2 or Problem 3, depending on the state space region in which the measured state resides. Thus, global asymptotic stability is guaranteed for the controlled piecewise affine system (1)-(2) if Assumptions 1 and 2 are satisfied and under feasibility of Problem 2.

Remark 5 Normally, the MPC problem for piecewise affine systems involves solving a kind of MIQP problem. In this respect, one can state that Algorithm 1 removes off-line the *switching* part of the problem (the integer part) by selecting a particular switching sequence. Any MPC algorithm designed using a minimal discrete events controllable path (including MPC Algorithm 1) comes down to solving a single QP problem subject to constraints and to checking a finite number of linear inequalities (Step 1) on-line (at each sampling time instant), i.e. a reduction of the on-line computational burden.

Remark 6 MPC Algorithm 1 achieves a reduced computational effort at the price of storing capacity, which may become a disadvantage for some cases. For every state space region (set) in (14), a finite number of linear inequalities (the ones defining all one-step controllable sets $K^1_{j_i}(\widetilde{X}^{l-(i+1)}_{j_{i+1}})$ and "target" sets $\widetilde{X}^{l-(i+1)}_{j_{i+1}}$), the matrices ($R_{j_i}, Q_{j_i}, P_{j_i}$) and the vectors ($x_{ej_i}, u_{ej_i}$) have to be stored.

4 Example

To illustrate the algorithm presented in Section 2 and the MPC algorithm formulated in Section 3, consider the following piecewise affine system with the partitioning corresponding to the four quadrants of the two dimensional $x_2 - x_1$ plane [11]:

$$x_{k+1} = \begin{cases} A_1 x_k + Bu_k,\ E_1 x_k \ge 0;\ A_1 = \begin{pmatrix} -0.04 & -0.461 \\ -0.139 & 0.341 \end{pmatrix}, E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ A_2 x_k + Bu_k,\ E_2 x_k \ge 0;\ A_2 = \begin{pmatrix} 0.936 & 0.323 \\ 0.788 & -0.049 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ A_3 x_k + Bu_k,\ E_3 x_k \ge 0;\ A_3 = \begin{pmatrix} -0.857 & 0.815 \\ 0.491 & 0.62 \end{pmatrix}, E_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ A_4 x_k + Bu_k,\ E_4 x_k \ge 0;\ A_4 = \begin{pmatrix} -0.022 & 0.644 \\ 0.758 & 0.271 \end{pmatrix}, E_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \quad (23)$$

In addition, system (23) is subject to *design constraints* of the form:

$$x_k \in X = [-5, 5] \times [-5, 5], \quad u_k \in U = [-1, 1], \quad \forall k \ge 0 \cdot \quad (24)$$

The target set X_T defined for system (23)-(24) and the corresponding *normalized* state space regions X_1, X_2, X_3, X_4 (obtained accordingly with Step 0 of Algorithm 1) are depicted in Fig. 2. Consider now Problem 1 formulated for system (23)-(24). For simplicity, Algorithm 1 will be used to calculate only the controllable path with respect to the initial state (utilized in the simulations) $x_0 = [-4 \;\; 4]^T$. The tree generated by Algorithm 1, which contains the minimal discrete events controllable paths to the target set X_T, is given in Fig. 1. Once the tree has been obtained, the sequence of sets (14) corresponding to a minimal discrete events controllable path is easily attained, i.e. $X^2_{j_0} = X^2_{42}$, $X^1_{j_1} = X^1_{21}$, $X^0_{j_2} = X^0_{1T}$ and X_T. The controllable sets employed in Algorithm 1 have been calculated using the Matlab Invariant Set Toolbox [8].

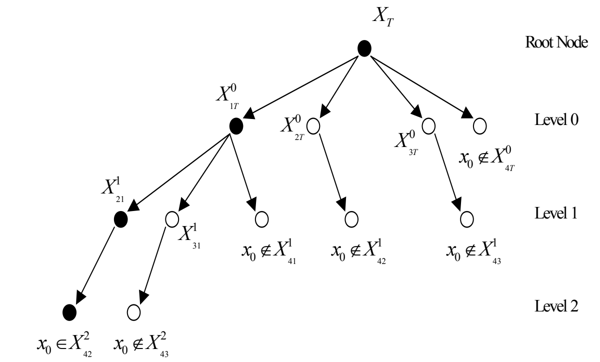

Fig. 1 The discrete events controllable sets tree for system (23)-(24) and x_0.

For system (23)-(24) and the chosen initial state, the number of discrete events needed to reach the target set coincides with the number of discrete-time steps. The target set is reached by implementing the MPC Algorithm 1 for the one-step controllable sets plotted in Fig. 2 and with the tuning parameters $N = 1$, $Q = I_2$ and $R = 0.2$ at each sampling instant and then, global asymptotic stability is achieved using the stabilizing state feedback $g(x_k)$ of [11]. The simulation results are plotted in Fig. 3.

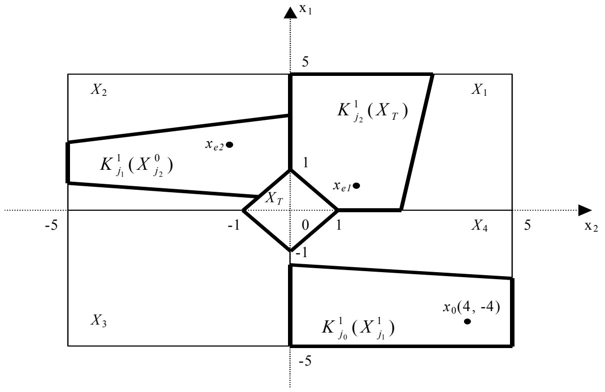

Fig. 2 The graphical illustration of the one-step sets.

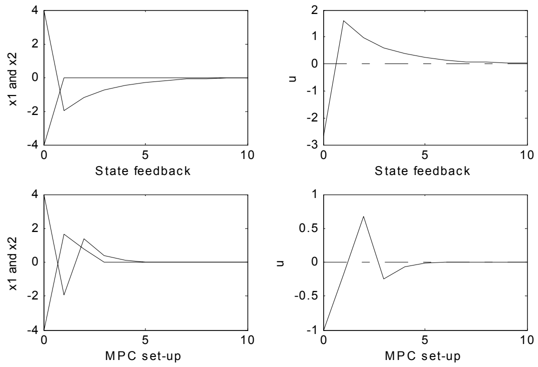

Fig. 3 State histories and control input.

The proposed MPC set-up yields performances comparable with the ones obtained with the control law of [11] and, in addition, the input constraint (24) is satisfied, which is not the case for the state feedback.

5 Conclusions

An approach for reducing the on-line computational load encountered in MPC algorithms for piecewise affine systems has been presented. The method is based on an algorithm that solves off-line the controllability problem with respect to an invariant target set. The algorithm calculates minimal discrete events controllable paths to the target set and organizes the resulting state space regions (sets) in a *tree-like* structure. For an initial state (or a measured state), a controllable path to the target set with a minimal number of discrete events is easily obtained and a resulting ordered sequence of state space regions is pre-computed; each region corresponds to a single sub-model of the piecewise affine system. Then, it has been shown that under certain assumptions the minimal discrete events controllable path can be used to develop a semi-explicit (sub-optimal) computationally more friendly MPC algorithm for piecewise affine systems.

The algorithm that solves the minimal discrete events controllability problem and the MPC algorithm have been illustrated on an example. The simulations show the satisfactory performance of the proposed control scheme.

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References

[1] A. Bemporad, M. Morari. "Control of systems integrating logic, dynamics, and constraints", *Automatica*, **35**, pp. 407-427, (1999).

[2] A. Bemporad, F. Borrelli, M. Morari. "Optimal controllers for hybrid systems: Stability and piecewise linear explicit form", In *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, pp. 1810-1815, (2000).

[3] A. Bemporad, L. Giovanardi, F.D. Torrisi. "Performance driven reachability analysis for optimal scheduling and control of hybrid systems", In *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, pp. 969-974, (2000).

[4] F. Blanchini. "Set invariance in control", *Automatica*, **35**, pp. 1747-1767, (1999).

[5] B. De Schutter, T.J.J. van den Boom and G.J. Benschop. "MPC for continuous piecewise-affine systems", In *Proceedings of the 15th IFAC World Congress*, Barcelona, Spain, (2002).

[6] E.M. Gilbert, K.T. Tan. "Linear systems with state and control constraints: The theory and application of maximal output admissible sets", *IEEE Transactions on Automatic Control*, **36**(9), pp. 1008-1020, (1991).

[7] W.P.M.H. Heemels, B. De Schutter, A. Bemporad, "Equivalence of hybrid dynamical models", *Automatica*, **37**, pp. 1085-1091, (2001).

[8] E.C. Kerrigan. *Robust Constraint Satisfaction: Invariant Sets and Predictive Control*, PhD thesis, University of Cambridge, UK, (2000). Thesis and Matlab Invariant Set Toolbox downloadable from http://www-control.eng.cam.ac.uk/eck21/.

[9] E.C. Kerrigan, D.Q. Mayne. "Optimal control of constrained, piecewise affine systems with bounded disturbances", In *Proceedings of the 41st Conference on Decision and Control*, Las Vegas, (2002).

[10] E.S. Meadows, K.R. Muske, J.B. Rawlings. "Implementable Model Predictive Control in the State Space", In the *Proceedings of the American Control Conference*, Seattle, pp. 3699-3703, (1995).

[11] D. Mignone, G. Ferrari-Trecate, M. Morari. "Stability and stabilization of piecewise affine and hybrid systems: An LMI approach", In *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, pp. 504-509, (2000).

[12] E.D. Sontag. "Nonlinear regulation: the piecewise linear approach", *IEEE Transactions on Automatic Control*, **26**(2), pp. 346-357, (1981).

[13] E.D. Sontag. "Interconnected automata and linear systems: A theoretical framework in discrete-time", In *Hybrid systems III: Verification and Control*, (R. Alur, T.A. Henzinger, and E.D. Sontag, Ed. NY: Springer), pp. 436-448, (1996).