

A SEMI-SYMMETRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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ABSTRACT

We define a linear Connection on a Riemannian manifold which is semi-symmetric but non-metric and study some properties of the curvature tensor and Weyl projective curvature tensor with respect to semi-symmetric connection.

ÖZET

Bir Riemann Manifoldu Üzerinde semi-symmetric non-metric lineer konneksiyon tanımlamak suretiyle eğrilik tensörünün ve Weyl projektif eğrilik tensörünün bazı özellikleri incelendi.

1. SEMI-SYMMETRIC NON-METRIC CONNECTION

Let M be a n -dimensional Riemannian manifold with Riemannian metric g . I define a linear connection $\overset{*}{\nabla}$ on a Riemannian manifold M by

$$\overset{*}{\nabla}_x y = \nabla_x y + w(y)x \quad (1.1)$$

where ∇ is the Riemannian connection on M and w is a 1-form associated with the vector field u on M by

$$w(x) = g(u, x) = \langle u, x \rangle. \quad (1.2)$$

Using (1.1), the torsion tensor $\overset{*}{T}$ of M with respect to connection $\overset{*}{\nabla}$ is given by

$$\begin{aligned} \overset{*}{T}(x, y) &= \overset{*}{\nabla}_x y - \overset{*}{\nabla}_y x - [x, y] \\ &= \overset{*}{\nabla}_x y + w(y)x - \overset{*}{\nabla}_y x - w(x)y - [x, y] \\ &= w(y)x - w(x)y. \end{aligned} \tag{1.3}$$

A linear connection satisfying (1.3) is called a semi-symmetric connection [1].

Further, Using (1.1), we have,

$$\begin{aligned} \overset{*}{\nabla}_x(g(y, z)) &= (\overset{*}{\nabla}_x g)(y, z) + g(\overset{*}{\nabla}_x y, z) + g(y, \overset{*}{\nabla}_x z) \\ &= (\overset{*}{\nabla}_x g)(y, z) + \overset{*}{\nabla}_x(g(y, z)) + w(y)g(x, z) + w(z)g(x, y) \end{aligned}$$

which implies

$$(\overset{*}{\nabla}_x g)(y, z) = -w(y)g(x, z) - w(z)g(x, y) \tag{1.4}$$

for vector fields x, y, z on M .

A linear connection $\overset{*}{\nabla}$ defined by (1.1) satisfies (1.3) and (1.4) and hence we call $\overset{*}{\nabla}$ a semi-symmetric non-metric connection.

Conversely, we will show that a linear connection satisfying (1.3) and (1.4) is defined by (1.1).

Let $\overset{*}{\nabla}$ be a linear connection defined on M by

$$\overset{*}{\nabla}_X y = \nabla_X y + T(x, y) \tag{1.5}$$

where ∇ is the Riemannian connection and T is a tensor of type (1.2) defined on M and $\overset{*}{\nabla}$ satisfies (1.3) and (1.4).

From (1.4) and (1.5), we have

$$\begin{aligned} \overset{*}{\nabla}_X (g(y, z)) &= g(\overset{*}{\nabla}_X y, z) + g(y, \overset{*}{\nabla}_X z) - w(y)g(x, y) - \\ &w(z)g(x, y) \\ &= \nabla_X (g(y, z)) = g(T(x, y), z) + g(T(x, z), y) \\ &\quad - w(y)g(x, z) - w(z)g(x, y). \end{aligned}$$

which implies

$$g(T(x, y), z) + g(T(x, z), y) = w(y)g(x, z) + w(z)g(x, y) \tag{1.6}$$

On the other hand, from (1.5), we have

$$\begin{aligned} \overset{*}{T}(x, y) &= \overset{*}{\nabla}_X y - \overset{*}{\nabla}_y x - [x, y] \\ &= T(x, y) - T(y, x). \end{aligned} \tag{1.7}$$

From (1.2), (1.6) and (1.7), we have

$$\begin{aligned} &g(\overset{*}{T}(x, y), z) + g(\overset{*}{T}(z, x), y) + g(\overset{*}{T}(z, y), x) \\ &= g(T(x, y) - T(y, x), z) + g(T(z, x) - T(x, z), y) \\ &\quad + g(T(z, y) - T(y, z), x) \end{aligned}$$

$$\begin{aligned}
 &= g(T(x, y), z) - g(T(y, x), z) + g(T(z, x), y) + g(T(x, z), y) \\
 &\quad + g(T(z, y), x) - g(T(y, z), x) \\
 &= 2\{g(T(x, y), z) - w(z)g(x, y)\} \\
 &= 2\{g(T(x, y), z) - g(z, u)g(x, y)\} \quad (1.8)
 \end{aligned}$$

Hence we obtain

$$T(x, y) = \frac{1}{2} \{ \overset{*}{T}(x, y) + \overset{*}{T}(x, y) + \overset{*}{T}(y, x) \} + g(x, y)u \quad (1.9)$$

where the tensor $\overset{*}{T}$ of type (1.2) is defined on M by

$$g(\overset{*}{T}(z, x), y) = g(\overset{*}{T}(x, y), z) \quad (1.10)$$

From (1.3) and (1.10), we have

$$\begin{aligned}
 g(\overset{*}{T}'(x, y), z) &= g(w(x)z - w(z)x, y) \\
 &= w(x)g(z, y) - w(z)g(x, y) \quad (1.11)
 \end{aligned}$$

which implies

$$\overset{*}{T}(x, y) = w(x)y - g(x, y)u. \quad (1.12)$$

Hence from (1.3), (1.9) and (1.12), we have

$$\begin{aligned}
 T(x, y) &= \frac{1}{2} \{ w(y)x - w(x)y + w(x)y - g(x, y)u + w(y)x \\
 &\quad - g(y, x)u \} + g(x, y)u \\
 &= w(y)x. \quad (1.13)
 \end{aligned}$$

on M. Hence from (1.5) and (1.13), we obtain

$$\overset{*}{\nabla}_x y = \nabla_x y + w(y)x.$$

Further, for a 1-form U on M, we have

$$\begin{aligned} \overset{*}{\nabla}_X(U(y)) &= (\overset{*}{\nabla}_X U)y + U(\overset{*}{\nabla}_X y) \\ &= (\overset{*}{\nabla}_X U)y + U(\overset{*}{\nabla}_X y) + w(y)U(x) \end{aligned}$$

$$= (\overset{*}{\nabla}_X U)y + \overset{*}{\nabla}_X(U(y)) - (\overset{*}{\nabla}_X U)y + w(y)U(x)$$

which implies

$$(\overset{*}{\nabla}_X U)y = (\overset{*}{\nabla}_X U)y - w(y)U(x) \tag{1.14}$$

for vector fields x, y on M .

Convariant differentiation of the torsion tensor $\overset{*}{T}$ is given by

$$\begin{aligned} (\overset{*}{\nabla}_X \overset{*}{T})(y, z) &= \overset{*}{\nabla}_X(\overset{*}{T}(y, z)) - \overset{*}{T}(\overset{*}{\nabla}_X y, z) - \overset{*}{T}(y, \overset{*}{\nabla}_X z) \\ &= ((\overset{*}{\nabla}_X w)z)y - ((\overset{*}{\nabla}_X U)y)z. \end{aligned} \tag{1.15}$$

Further we define

$$\overset{*}{T}(x, y, z) = g(T(x, y), z) \tag{1.16}$$

From (1.3) and (1.16), we have

$$\overset{*}{T}(x, y, z) + \overset{*}{T}(y, z, x) + \overset{*}{T}(z, x, y) = 0. \tag{1.17}$$

This identity is true for any semi-symmetric connection on M .

2. CURVATURE TENSOR OF M WITH RESPECT TO SEMI-SYMMETRIC CONNECTION

Analogous to the definition of curvature tensor of a Riemannian manifold M with respect to the Riemannian connection ∇ , we define the curvature tensor of M with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$ by

$$R(x, y, z) = \overset{*}{\nabla}_x \overset{*}{\nabla}_y z - \overset{*}{\nabla}_y \overset{*}{\nabla}_x z - \overset{*}{\nabla}_{[x, y]} z. \tag{2.1}$$

From (1.5) and (2.1), we have

$$\begin{aligned} R(x, y)z &= \overset{*}{\nabla}_x (\overset{*}{\nabla}_y z + w(z)y) - \overset{*}{\nabla}_y (\overset{*}{\nabla}_x z + w(z)x) \\ &\quad - \overset{*}{\nabla}_{[x, y]} z - w(z)[x, y] \\ &= \overset{*}{\nabla}_x \overset{*}{\nabla}_y z + w(\overset{*}{\nabla}_x z + w(z)x)y \\ &\quad - \overset{*}{\nabla}_y \overset{*}{\nabla}_x z - w(\overset{*}{\nabla}_y z + w(z)y)x \\ &\quad - \overset{*}{\nabla}_{[x, y]} z - w(z)[x, y] \end{aligned}$$

$$= R(x, y)z + s(x, z)y - s(y, z)x - s(y, z)x \tag{2.2}$$

where

$$R(x, y)z = \overset{*}{\nabla}_x \overset{*}{\nabla}_y z - \overset{*}{\nabla}_y \overset{*}{\nabla}_x z - \overset{*}{\nabla}_{[x, y]} z.$$

is the curvature tensor of a Riemannian manifold M with respect to the Riemannian Connection ∇ and S is a tensor of type (0,2) defined on M by

$$\begin{aligned} S(x, y) &= (\overset{*}{\nabla}_x w)y - w(x)w(y) \\ &= (\overset{*}{\nabla}_x w)y. \end{aligned} \tag{2.3}$$

A relation between the curvature tensors of M with respect to the semi-symmetric non-metric Connection $\overset{*}{\nabla}$ and the Riemannian Connection ∇ is given by (2.2).

Using (2.3), we have

$$\begin{aligned} S(x, y) - S(y, x) &= (\overset{*}{\nabla}_x w)y - (\overset{*}{\nabla}_y w)x \\ &= xw(y) - yw(x) - w[x, y] \\ &= dw(x, y) \end{aligned} \tag{2.4}$$

Thus a tensor S is symmetric iff the 1-form w is closed. Let

$$R(x, y, z, w) = g(R(x, y)z, w)$$

and

$$\overset{*}{R}(x, y; z, w) = g(\overset{*}{R}(x, y)z, w) \tag{2.5}$$

for vector fields x,y,z,w on M. From (2.2) and (2.5), we have

$$\begin{aligned} \overset{*}{R}(x, y, z, w) &= R(x, y, z, w) + S(x, z)g(y, w) \\ &\quad - S(y, z)g(x, w) \end{aligned} \tag{2.6}$$

From (2.1) and (2.5) we have

$$\overset{*}{R}(x, y, z, w) + \overset{*}{R}(x, y, z, w) = 0. \tag{2.7}$$

Using (2.2),(2.6) and first Bianchi identity with respect to the Riemannian Connection, we have

$$\begin{aligned} \overset{*}{R}(x, y)z + \overset{*}{R}(y, z)x + \overset{*}{R}(z, x)y \\ = \{S(z, y) - S(y, z)\}x + \{S(x, z) - S(z, x)\}y \\ + \{S(y, x) - S(x, y)\}z \end{aligned} \tag{2.8}$$

and hence

$$\overset{*}{R}(x, y, z, w) + \overset{*}{R}(y, z, x, w) + \overset{*}{R}(z, x, y, w)$$

$$= \{S(z, y) - S(y, z)\}g(x, w) + \{S(x, z) - S(z, x)\}g(y, w) + \{S(y, x) - S(x, y)\}g(z, w)$$

We call (2.8) as the first Bianchi identify with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$.

In particular, if the 1-form w is closed, then (2.8) reduces to

$$\overset{*}{R}(x, y)z + \overset{*}{R}(y, z)x + \overset{*}{R}(z, x)y = 0. \tag{2.9}$$

Using (2.6), we have

$$\overset{*}{R}(x, y, z, w) + \overset{*}{R}(x, y, w, z) = S(x, z)g(y, w) + S(x, w)g(y, z) - S(y, z)g(x, w) - S(y, w)g(x, z). \tag{2.10}$$

and

$$\overset{*}{R}(x, y, z, w) - \overset{*}{R}(z, w, x, y) = \{S(x, z) - S(z, x)\}g(y, w) + S(w, x)g(y, z) - S(y, z)g(x, w). \tag{2.11}$$

Using (1.1), (2.7) and the second Bianchi identify for the Riemannion Connection, we obtain the second Bianchi identify associated with semi-symmetric non-metric connection which is given by

$$\begin{aligned} & (\overset{*}{\nabla}_x \overset{*}{R})(y, z) + (\overset{*}{\nabla}_y \overset{*}{R})(z, x) + (\overset{*}{\nabla}_z \overset{*}{R})(x, y) \\ &= -\overset{*}{R}(\overset{*}{T}(x, y), z) - \overset{*}{R}(\overset{*}{T}(y, z), x) - \overset{*}{R}(\overset{*}{T}(z, x), y) \\ &= 2\{w(x)\overset{*}{R}(y, z) + w(y)\overset{*}{R}(z, x) + w(z)\overset{*}{R}(x, y)\} \end{aligned} \tag{2.12}$$

Analogous to the definition of Ricci tensor of a Riemannion manifold M with respect to the Reimannion connection ∇ , we define Ricci tensor of M with respect to semi-symmetric non-metric Connection $\overset{*}{\nabla}$ by.

$${}^*R_{ic}(y, z) = \sum_{i \Rightarrow}^n {}^*R(E_i, y, z, E_i) \tag{2.13}$$

where $E_i, (1 \leq i \leq n)$ are orthonormal vector fields on M . From (2.5) and (2.13), we have

$${}^*R_{ic}(y, z) = \overset{*}{R}_{ic}(y, z) - (n \rightarrow)S(y, z) \tag{2.14}$$

where

$$\overset{*}{R}_{ic}(y, z) = \sum_{i=1}^n R(E_i, y, z, E_i)$$

is Ricci tensor of M with respect to the Riemannian Connection.

A relation between Ricci tensor with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$ and the Riemannian Connection ∇ is given by (2.14).

Further, if $\overset{*}{R}_{ic}(x, y) = 0$ on M , then (2.14).

implies is symmetric. From (2.4) and (2.14), we have

$$\overset{*}{R}_{ic}(x, y) - \overset{*}{R}_{ic}(y, x) = -(n \rightarrow)dw(x, y).$$

Hence, Ricci tensor with respect to semi-symmetric non-metric Connection $\overset{*}{\nabla}$ is symmetric iff the I-form w is closed and hence if S is symmetric.

Using (1.15) and (2.3), we have.

$$(\overset{*}{\nabla}_x \overset{*}{T})(y, x) = S(x, z)y - S(x, y)z. \tag{2.15}$$

In particular, if either the I-form w is closed or Ricci tensor with respect to semi-symmetric non-metric connection $\overset{*}{\nabla}$ vanishes then from (2.15), we have

$$(\overset{*}{\nabla}_x \overset{*}{T})(y, z) + (\overset{*}{\nabla}_y \overset{*}{T})(z, x) + (\overset{*}{\nabla}_z \overset{*}{T})(x, y) = 0. \tag{2.16}$$

Analogous to the definition of the scalar curvature of a Riemannian manifold M with respect to the Riemannian Connection, we define the scalar curvature of M with respect to semi-symmetric non-metric Connection by

$$r^* = \sum_{i=1}^n R_{ic}^*(E_i, E_i) \tag{2.17}$$

From (2.14) and (2.17), we obtain a relation between the scalar curvature of M with respect to the Riemannian Connection and the semi-symmetric non-metric connection which is given by

$$r^* = r - (n-1)\text{trace}S \tag{2.18}$$

where

$$r = \sum_{i=1}^n R_{ic}(E_i, E_i)$$

Is the scalar curvature of M with respect to the Riemannian Connection and s is a tensor of type (1,1) defined on M by

$$s(x, y) = g(Sx, y).$$

3. PROJECTIVE CURVATURE TENSOR OF A RIEMANNIAN MANIFOLD

Weyl projective curvature tensor of a Riemannian Manifold M with respect to the Riemannian Connection is given by

$$P(x, y)z = R(x, y)z - \frac{1}{n-1} \{ \overset{*}{R}_{ic}^*(y, z)x - \overset{*}{R}_{ic}^*(x, z)y \} \tag{3.1}$$

(3.1)

$$P(x, y)z = R(x, y)z - \frac{1}{n-1} \{ \overset{*}{R}_{ic} (y, z)x - \overset{*}{R}_{ic} (x, z)y \} \tag{3.1}$$

Analogous to this definition, we define projective curvature tensor of M with respect to semi-symmetric non-metric connection by

$$\overset{*}{P}(x, y)z = \overset{*}{R}(x, y)z - \frac{1}{n-1} \{ \overset{*}{R}_{ic} (y, z)x - \overset{*}{R}_{ic} (x, z)y \} \tag{3.2}$$

From (2.2), (2.14), (3.1) and (3.2), we have

$$\overset{*}{P}(x, y)z = P(x, y)z \tag{3.3}$$

on M.

Theorem 3.1. If M is a Riemmanian manifold admitting semi-symmetric non-metric connection, then the weyl projective curvature tensor with respect to semi-symmetric non-metric connection is equal to the Weyl projective curvature tensor with respect to Riemannian Connection.

From (3.3), we have, the projective curvature tensor with respect to semi-symmetric non-metric connection satisfies the following algebraic properties

$$\overset{*}{P}(x, y)z + \overset{*}{P}(y, x)z = 0$$

and

$$\overset{*}{P}(x, y)z + \overset{*}{P}(y, z)x + \overset{*}{P}(z, x)y = 0 \tag{3.4}$$

for vector fields x,y,z on M.

In particular, Let M be a Riemmanian Manifold Satisfying

$$\overset{*}{R}(x, y)z = 0 \tag{3.5}$$

which implies

$$\overset{*}{R}_{ic}(y, z) = 0 \tag{3.6}$$

on M. From (3.2), (3.3), (3.5) and (3.6), we have

$$P(x, y)z = 0$$

on M.

Necessary and sufficient condition for a manifold with a symmetric linear connection to be projectively flat is that the projective curvature tensor with respect to it vanishes identically on a manifold [2].

From (2.3), (2,14) and (3.6), we have

$$(\nabla_x w)y = \frac{1}{n-1}R_{ic}(x, y) + w(x)w(y) \tag{3.7}$$

Using (3.7), we have

$$\begin{aligned} -w(R(x, y)z) &= (\nabla_x \nabla_y w - \nabla_y \nabla_x w - \nabla_{[x, y]} w)z \\ &= \frac{1}{n-1}((\nabla_x R_{ic})(y, z) - (\nabla_y R_{ic})(x, z) \\ &\quad + w(y)R_{ic}(x, z) - w(x)R_{ic}(y, z)) \end{aligned} \tag{3.8}$$

Further, from (3.1), we have

$$w(R(x, y, z)) = \frac{1}{n-1}(R_{ic}(x, z)w(x) - R_{ic}(x, z)w(y)) \tag{3.9}$$

From (3.8) and (3.9), we have

$$(\nabla_x R_{ic})(y, z) - (\nabla_y R_{ic})(x, z) = 0.$$

Theorem 3.2. If M is a Riemmannian manifold with vanishing curvature tensor with respect to semi-symmetric non-metric connection, then M is projektively flat and

$$(\nabla_x R_{ic})(y, z) = (\nabla_y R_{ic})(x, z)$$

on M.

It is well Known that a Riemannian manifold is of constant curvature if it is projectively flat and a Riemmanian manifold of constant is Conformally flat [4].

Theorem 3.3. If M is a Riemannian manifold with vanishing curvature tensor, with respect to semi-symmetric non-metric connection, then M is a space of constant curvature and hence is conformally flat.

From (2.2), (2,15), (3,5) and (3.6), we have

$$\begin{aligned} R(x, y)z &= S(z, y)x - S(z, x)y \\ &= (\overset{*}{\nabla}_x \overset{*}{T})(x, y). \end{aligned} \tag{3.10}$$

Theorem 3.4. If M is a Riemannian manifold with vanishing curvature tensor with respect to semi-symmetric non-metric connection, then M is flat iff

$$(\overset{*}{\nabla}_x \overset{*}{T})(y, z) = 0$$

on M.

A Riemannian manifold M is a group manifold [5] with respect to semi-symmetric non-metric connection if

$$\overset{*}{R}(x, y)z = 0$$

and

$$(\overset{*}{\nabla}_x \overset{*}{T})(y, z) = 0 \tag{3.11}$$

on M.

Theorem 3.5. If a Riemannian manifold M is a group manifold with respect to semi-symmetric non-metric connection, then M is flat and consequently M is projectively flat and conformally flat.

In particular if either the 1-form w is closed or Ricci tensor with respect to semi-symmetric non-metric connection vanishes, then from (2.2) and (2.15), we have

$$\overset{*}{R}(x, y)z = R(x, y)z + \left(\overset{*}{\nabla}_Z \overset{*}{T}\right)(y, x). \tag{3.12}$$

Theorem 3.6. Let M be a Riemannian manifold with semi-symmetric non-metric connection. If either the 1-form w is closed or Ricci tensor with respect to semi-symmetric non-metric connection vanishes, then

$$R(x, y)z = \overset{*}{R}(x, y)z + \left(\overset{*}{\nabla}_Z \overset{*}{T}\right)(x, y)$$

on M .

In particular, if M is a Riemannian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connection, then from (3.2), (3.3) and (3.12), we have.

$$P(x, y)z = \overset{*}{R}(x, y)z = R(x, y)z + \left(\overset{*}{\nabla}_Z \overset{*}{T}\right)(y, x) \tag{3.13}$$

on M .

Theorem 3.7. If a Riemannian manifold M with vanishing Ricci tensor with respect to semi-symmetric non-metric connection is projectively flat, then the curvature tensor with respect to semi-symmetric non-metric connection vanishes.

Theorem 3.8. If M is a Riemannian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connection, then M is projectively flat iff the curvature tensor with respect to semi-symmetric non-metric connection vanishes.

Since a flat manifold is projectively flat, from (3.13), we have

$$\overset{*}{R}(x, y)z = 0 \text{ and } \left(\overset{*}{\nabla}_Z \overset{*}{T}\right)(x, y)$$

on M .

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