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A Semiclassical Treatment of Transition Phenomena by Coherent-State Path Integral

—A Nontrivial Schematic Model—

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A semiclassical method with path integrals in the $SU(2)$ coherent state representation is applied to the investigation of transition phenomena in a schematic model which represents interplay between pairing and quadrupole modes. Numerical evaluation of the semiclassical quantization condition is performed and the result is compared with the exact calculation.

For the last few years, many authors have investigated microscopic descriptions of large-amplitude nuclear collective motion.[1] The time-dependent Hartree-Fock (TDHF) approximation has been considered as a possible candidate for the microscopic theory. The TDHF theory is, however, essentially of classical nature, and subject to the difficulty of finding the unambiguous way to quantization. Recently, path integral approaches to nuclear many-body problems have raised increasing interests[2),3)] mainly because they lead us to an intuitive way of describing the correspondence between classical and quantal concepts. It has been shown in Ref. 3) that path integrals in the coherent state representation may be promising to the investigation of nuclear collective phenomena. In particular, it would be expected that the coherent-state path integral provides us a useful device for the approximate quantization of the TDHF theory. In this short note, we investigate the semiclassical quantization for a nontrivial schematic model by using the semiclassical treatment developed by the present author, together with Kuratsuji.[4)] The obtained result is compared with the exact one.

We will work with the schematic model Hamiltonian suggested by Mottelson[5)]

$$\hat{H}=-g\cdot\hat{P}^{\dagger}\hat{P}-\frac{\chi}{4}\hat{Q}^2,\quad (g,\chi>0) \tag{1}$$

$$\left.\begin{aligned}
&\hat{P}^{\dagger}=\sum_{m>0} c_m^{\dagger}c_{\bar{m}}^{\dagger},\\
&\hat{Q}=\sum_{m}\sigma_m c_m^{\dagger}c_m(=\hat{Q}^{\dagger}),\\
&\sigma_m=\begin{cases}+1 & \text{for } |m|<\Omega/2,\\ -1 & \text{for } |m|>\Omega/2,\quad (\Omega=j+1/2)\end{cases}
\end{aligned}\right\} \tag{2}$$

where the system is composed of n identical particles on a single-j shell (for simplicity, the case of even Ω is considered). The Hamiltonian (1) illustrates an essential nature of the pairing-plus-quadrupole interaction. $\hat{P}^{\dagger}$ and $\hat{P}$ are the pairing operators, and $\hat{Q}$ corresponds to "deformation". These operators are a part of the generators of an $O(4)$ group, and can be expressed in terms of two independent spins (i.e., $O(4)-SU(2)\times SU(2)$):

$$\left.\begin{aligned}
&\hat{P}^{\dagger}=\hat{S}_+^{(1)}+\hat{S}_+^{(2)},\quad \hat{Q}=2(\hat{S}_0^{(1)}-\hat{S}_0^{(2)}),\\
&\left.\begin{matrix}\hat{S}_+^{(1)}\\ \hat{S}_+^{(2)}\end{matrix}\right\}=\sum_{m>0}\frac{1\pm\sigma_m}{2}c_m^{\dagger}c_{\bar{m}}^{\dagger},\\
&\left.\begin{matrix}\hat{S}_0^{(1)}\\ \hat{S}_0^{(2)}\end{matrix}\right\}=\frac{1}{2}\left[\sum_{m}\frac{1\pm\sigma_m}{2}c_m^{\dagger}c_m-\frac{\Omega}{2}\right].
\end{aligned}\right\} \tag{3}$$

Then the Hamiltonian is rewritten as

$$\hat{H}=g(\hat{S}_0^{(1)}+\hat{S}_0^{(2)})$$

$$-\frac{1}{2}g\{\hat{S}_+^{(1)}+\hat{S}_+^{(2)},\ \hat{S}_-^{(1)}+\hat{S}_-^{(2)}\}_+$$

$$-\chi(\hat{S}_0^{(1)}-\hat{S}_0^{(2)})^2\ . \tag{4}$$

Here we find that this model can be regarded as a system of two interacting (quasi-) spins. For simplicity, let the number of particles n be equal to Ω and seniorities in both parts of the shell ($|m|<\Omega/2$ and $|m|>\Omega/2$) be zero, and it is sufficient to consider the case of two spins with the same quantum number $s=\Omega/4$. Thus we can treat the model by using the path integral in the $SU(2)$ coherent state representation.

We starts with the propagator

$$K(\boldsymbol{\zeta}'',\boldsymbol{\zeta}';T)=\langle\boldsymbol{\zeta}''|e^{-i\hat{H}T/\hbar}|\boldsymbol{\zeta}'\rangle. \tag{5}$$

The coherent state $|\boldsymbol{\zeta}\rangle$ is defined as a direct product $|\boldsymbol{\zeta}\rangle=|\zeta_1\rangle\otimes|\zeta_2\rangle$ of the ones for single spins $|\zeta_k\rangle=(1+|\zeta_k|^2)^{-s}\exp(i\zeta_k\hat{S}_+^{(k)})|s,-s\rangle$, ($k=1,2$), and has a property of the completeness relation $\int|\boldsymbol{\zeta}\rangle d\mu(\boldsymbol{\zeta})\langle\boldsymbol{\zeta}|=1$. The use of the completeness relation yields a path integral form of K

$$K=\int D\mu[\boldsymbol{\zeta}]e^{iS/\hbar}, \tag{6}$$

where $D\mu[\boldsymbol{\zeta}]$ is the invariant path measure. S is called the action functional and given by

$$S[\boldsymbol{\zeta}]=\int_0^T dt[i\hbar s\sum_{k=1}^{2}(1+|\zeta_k|^2)^{-1}$$

$$\times(\zeta_k^*\dot{\zeta}_k-\zeta_k\dot{\zeta}_k^*)-\mathcal{H}(\boldsymbol{\zeta})] \tag{7}$$

with

$$\mathcal{H}(\boldsymbol{\zeta})=\langle\boldsymbol{\zeta}|\hat{H}|\boldsymbol{\zeta}\rangle.$$

The dominant contributions to the path integral in (6) come from the "classical paths" which are determined by the extreme condition $\delta S=0$ (stationary phase approximation). Using a solution of the condition $\delta S=0$, we obtain the classical approximation for K

$$K^{\mathrm{cl}}=e^{iS^{\mathrm{cl}}/\hbar}, \tag{8}$$

where S^{cl} is a functional of the classical path.

In order to get the classical trajectories, it is convenient to introduce the angle variables by stereographic projection of $\boldsymbol{\zeta}$ through $\zeta_k=\tan(\theta_k/2)e^{-i\varphi_k}(0\leq\theta_k\leq\pi,0\leq\varphi_k<2\pi;k=1,2)$. In the present model, physical paths are subject to the additional condition $n=\Omega$, i.e.,

$$\langle\boldsymbol{\zeta}|\hat{S}_0^{(1)}+\hat{S}_0^{(2)}|\boldsymbol{\zeta}\rangle\propto\cos\theta_1+\cos\theta_2=0\ . \tag{9}$$

Under the condition (9), the Hamiltonian is reduced to the following form:

$$\mathcal{H}=-4S^2[g\cdot\sin^2\theta\cdot\cos^2\varphi+\chi\cdot\cos^2\theta] \tag{10}$$

with

$$\cos\theta\equiv\cos\theta_1=-\cos\theta_2,\quad \varphi\equiv(\varphi_1-\varphi_2)/2,$$

$$S\equiv s+1/2\ .$$

We note that symmetry between $\hat{S}^{(1)}$ and $\hat{S}^{(2)}$ in $\hat{H}$ causes the degeneracy of states with the same value of $|\cos\theta|$, and reduces the period of φ by half, i.e., π. Since the value of $\mathcal{H}$ remains constant ($\mathcal{H}=E$) along the "classical path", we obtain several trajectories on a unit sphere (S^2) which can be regarded as a "curved" phase space. Figure 1 illustrates some examples of such trajectories, where $\cos\theta$ is shown as a function of φ and arrows indicate directions of the motion. The relation between the force strength parameters is $g<\chi$ in the case of Fig. 1(a); trajectory C_1 is obtained for the energy region $-4\chi S^2<E<-4gS^2$, and C_2 for $-4gS^2<E<0$. Figure 2 shows the case in which $g>\chi$; we get trajectory C_3 for $-4gS^2<E<-4\chi S^2$ and C_2' for $-4\chi S^2<E<0$. Trajectory C_1 corresponds to the "deformation" mode, while C_3 corresponds to the pairing mode. Trajectories C_2 and C_2' lie in the "transitional" region between these modes.

Although this model has started with two spins, the condition enables us to adopt a method obtained for the case of a single spin in Ref. 4). Application of the method yields

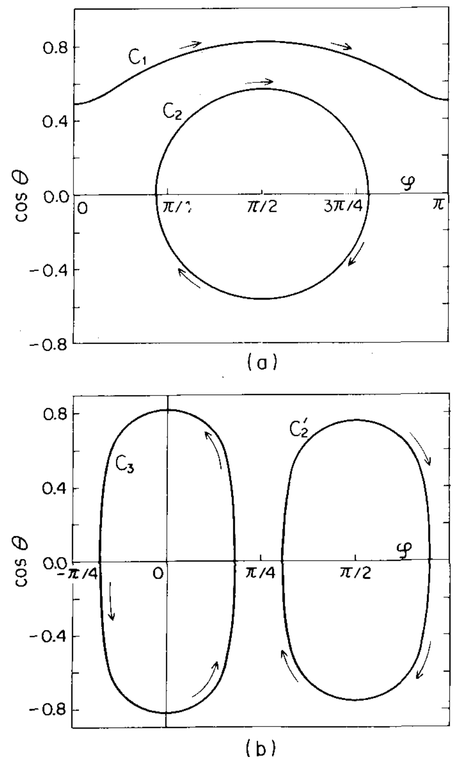

Fig. 1. Examples of the classical trajectories. Values of the force strength are $g=0.5$ and $\chi=0.866$ for the case of (a), and $g=0.866$, $\chi=0.5$ for the case of (b). Energies are $E=-58.62$ and -28.58 for trajectories C_1 and C_2, respectively, in (a), while $E=-62.43$ and -28.58 for C_3 and C_2', respectively, in (b). Ω is set to be 18.

the quantization rule as follows:

$$W(E)=2\left(m+\frac{\nu}{4}\right)\pi, \quad (m=\text{integer}) \qquad (11)$$

where ν is the number of turning points on the trajectory in the course of a period ($\nu=2$ for the present trajectories). $W(E)$ is the action integral defined by

$$W(E)=2S\int_C \cos\theta d\varphi\ . \qquad (12)$$

The integration is to be performed over the interval $0\leq\varphi\leq\pi$ for trajectory C_1 and the closed loops in Fig. 1. for C_2, C_2' and C_3. The action integral (12) is bounded because of compactness of the phase space S^2, so that the number of the "quantum number" m's is finite.

The integration in (12) is numerically performed to obtain the values of E which satisfy the condition (11). The result is shown in Fig. 2 together with the exact one. In Fig. 2, excitation energies are plotted against the force parameters g and χ which are parameterized as $g^2+\chi^2=1$. Dotted lines represent the approximate energies calculated by the present method, which reproduce the main trend of the energy levels given by the exact diagonalization (solid lines). In the $g<\chi$ region, however, there exist discontinuities in the approximate energy levels, which are caused by separatrices that appear

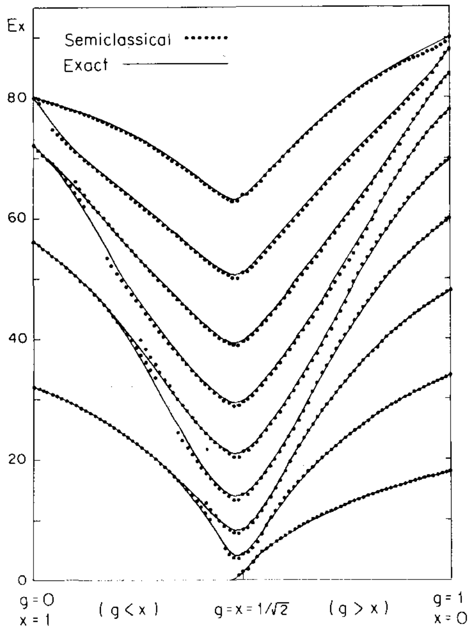

Fig. 2. The excitation energies for $\Omega=18$ protted against g and χ parametrized by $g^2+\chi^2=1$. The dotted lines represent the approximate energies calculated by the present method, and the solid lines the exact ones.

in the course of the transition between trajectories C_1 and C_2. These discontinuities are unavoidable in the semiclassical approximation because the semiclassical quantization cannot be applied to such a critical orbit as a separatrix. Outward discontinuities at $g=\chi$ come from the discontinuity of the ground state energy, while the energy levels of excited states change continuously. The energy is minimum for a trajectory like C_1 in the $g<\chi$ region, for one like C_2 at $g=\chi$, and for one like C_3 in the $g>\chi$ region. So the discontinuity is caused by the separatrix which lies between such trajectories. We note that the change of trajectories in the ground state can be regarded as the phase transition from the "deformed" mode to the pairing mode, and that competition between these modes, which is a collective motion with a large scale, occurs in the excited states arround $g=\chi$. This type of collective motion can be visualized as trajectory C_2 or C_2' in the present method. It should be noted that we have been able to get a new collective mode which manifests itself as interplay of two typical ones and obtain the approximate energy levels which reproduce the exact ones fairly well, though our quantization rule may not be applied to the phase transition point itself. Thus we can conclude that the semiclassical treatment with the coherent-state path integral yields a good approximation to the description of large-amplitude collective phenomena.

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