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A SEMIDIRECT PRODUCT DECOMPOSITION FOR CERTAIN HOPF ALGEBRAS OVER AN ALGEBRAICALLY CLOSED FIELD

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ABSTRACT. Let H be a finite dimensional Hopf algebra over an algebraically closed field. We show that if H is commutative and the coradical H_0 is a sub Hopf algebra, then the canonical inclusion $H_0 \to H$ has a Hopf algebra retract; or equivalently, if H is cocommutative and the Jacobson radical $J(H)$ is a Hopf ideal, then the canonical projection $H \to H/J(H)$ has a Hopf algebra section.

For a Hopf algebra H we denote the coradical (i.e. the sum of the simple subcoalgebras of H) by H_0, and the Jacobson radical by $J(H)$. If π: $H \to K$ is a surjective (resp. injective) Hopf algebra map we say it splits if there exists a Hopf algebra map τ: $K \to H$ with $\pi \circ \tau = I_K$ (resp. $\tau \circ \pi = I_H$). The purpose of this paper is to prove that if H is a finite dimensional Hopf algebra over an algebraically closed field we have the following:

(A) If H is commutative and H_0 is a sub Hopf algebra, then the canonical inclusion $H_0 \to H$ splits as a map of Hopf algebras; or equivalently,

(B) If H is cocommutative and $J(H)$ is a Hopf ideal, then the canonical projection $H \to H/J(H)$ splits as a map of Hopf algebras.

If follows from the results of [3] that the existence of a Hopf algebra splitting in (A) or (B) induces a semidirect product decomposition of the Hopf algebra H, and that such splittings are necessarily unique. For the standard facts about Hopf algebras see [1] or [7]; for splittings and exact sequences see [3].

It is easy to see that (A) and (B) are equivalent, for by finite dimensionality we have $J(H^*) = (H_0)^\perp$ and so $H_0 \cong (H^*/J(H^*))^*$. Thus a splitting in one case induces a splitting in the other by transposing. We shall verify (B). We begin by establishing a special case of (B) which is valid over any field. If G is a group, let $k[G]$ denote the group algebra of G over k.

PROPOSITION 1. *Let $H = k[G]$ where G is a finite group and k is any field. If $J(H)$ is a Hopf ideal of H then the canonical projection π: $H \to H/J(H)$ splits as a map of Hopf algebras.*

PROOF. If the characteristic of k is zero (or is relatively prime to the order

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of G) then $J(H) = (0)$ by Maschke's theorem so the result is obvious. So we may assume that the characteristic of k is $p > 0$, and that p divides the order of G.

Since π is a Hopf algebra map, it is easy to verify (see 3.6(a) of [3]) that $N = \ker(\pi|_G)$ is a normal subgroup of G and $H/J(H) \cong k[G/N]$, i.e. $J(H)$ is equal to the ideal in H generated by the augmentation ideal of $k[N]$.

Now $k[G/N]$ is semisimple, so p does not divide the order of G/N. Thus N must contain all elements of G having order a power of p. But if $g \in N$ we have $e - g \in \ker(\pi) = J(H)$ (where e is the identity of G). Thus $e - g$ is nilpotent, so $0 = (e - g)^{p^\alpha} = e - g^{p^\alpha}$ for some positive integer α, i.e. g has order a power of p. It follows that N is a normal p-Sylow subgroup of G.

Now the order of N is a power of p by the above, and is thus relatively prime to the order of G/N. By Schur's theorem (10.5 of [2]) there is a group homomorphism i: $G/N \to G$ which splits the restriction of π to G, and this group homomorphism induces the desired Hopf algebra splitting.

LEMMA 1. *Let $K \to H \to L$ be an exact sequence of finite dimensional Hopf algebras. Then H is semisimple as an algebra if and only if K and L are semisimple.*

PROOF. Since everything is finite dimensional it is immediate that the given sequence is exact if and only if the induced sequence $L^* \to H^* \to K^*$ is exact. Now the lemma follows from the corresponding theorem with "semisimple" replaced by "cosemisimple" (see 2.20 of [5], or [4]).

In [9] M. Takeuchi proved that a commutative or cocommutative Hopf algebra H is faithfully flat over any sub Hopf algebra K.

LEMMA 2. *Let H be a cocommutative Hopf algebra over a field k and K a sub Hopf algebra. Then $J(H) \cap K \subseteq J(K)$.*

PROOF. If $\mathfrak{m}$ is a maximal left ideal of K by faithful flatness we have $\mathfrak{m}H \cap K = \mathfrak{m}$. The lemma then follows from the fact that the Jacobson radical is the intersection of the maximal left ideals.

In [6] J. B. Sullivan proved that if H is a cocommutative Hopf algebra over an algebraically closed field and H_0 is spanned by grouplike elements then the inclusion $H_0 \hookrightarrow H$ splits as a map of Hopf algebras. The following proposition is an easy consequence of Sullivan's theorem.

PROPOSITION 2. *Let H be a finite dimensional, irreducible, cocommutative Hopf algebra over an algebraically closed field k. If $J(H)$ is a Hopf ideal of H then the canonical projection $H \to H/J(H)$ splits as a map of Hopf algebras.*

PROOF. We may assume that the characteristic of k is $p \neq 0$ because in characteristic 0 finite dimensionality implies $H = k$ (see 13.0.1 of [8]). Now H^* is local since H is irreducible, and $(H^*)_0 \cong (H/J(H))^*$ is a sub Hopf algebra. We have sep $((H^*)_0) \subseteq$ sep $(H^*) = k$ by 3.2 of [7] since H^* is local. So $(H^*)_0$ is cocommutative by Theorem 4.1 of [7] and hence must be spanned by its grouplike elements. But then $(H^*)_0 \to H^*$ splits by Sullivan's theorem and so $H \to H/J(H)$ splits by duality.

We are now ready to prove our main result.

THEOREM. *If H is a cocommutative, finite dimensional Hopf algebra over an algebraically closed field k and $J(H)$ is a Hopf ideal, then there exists a Hopf algebra map which splits the canonical projection $\pi\colon H \to H/J(H)$.*

PROOF. The proof follows by pasting together the special cases in Propositions 1 and 2 by means of the structure theorem for cocommutative Hopf algebras. We recall (8.15 of [8], or see [4]) that this says $H \cong H^1 \,\sharp\, k[G]$ (Hopf algebra isomorphism) where H^1 is the irreducible component containing 1, and G is a finite group. Note that we may assume the characteristic of k is $p > 0$ since otherwise $H^1 = k$ and, as in Proposition 1, $J(H) = (0)$.

Let $L = H/J(H)$ and $\pi\colon H \to L$ be the canonical map. Now $L \cong L^1 \,\sharp\, k[G/N]$ where $N = \ker(\pi|_G)$ and $L^1 = \pi(H^1)$ is the irreducible component of L containing 1. Moreover L is semisimple so L^1 and $k[G/N]$ are semisimple by Lemma 2 and 3.6(c) of [3].

If we let $\pi_1 = \pi|_{H^1}$ and $\pi_2 = \pi|_{k[G]}$, then $\pi = \pi_1 \,\sharp\, \pi_2$, and we have

$$\ker(\pi_1) = H^1 \cap J(H) \subseteq J(H^1) \subseteq \ker(\pi_1),$$

the first containment following from Lemma 2 and the second from the fact that $L^1 = \pi_1(H^1)$ is semisimple. A similar argument shows

$$\ker(\pi_2) = k[G] \cap J(H) \subseteq J(k[G]) \subseteq \ker(\pi_2),$$

and so we have (Hopf ideals!) $\ker(\pi_1) = J(H^1)$ and $\ker(\pi_2) = J(k[G])$. Thus from Propositions 1 and 2 we have Hopf algebra maps τ_1 and τ_2 splitting π_1 and π_2 respectively. We have the following commutative diagram:

$$\begin{array}{ccccc}
H^1 & \xhookrightarrow{\;i\;} & H^1\#k[G] & \underset{\longrightarrow\!\!\!\!\rightarrow}{\overset{j}{\longleftarrow\!\!\!\supset}} & k[G] \\
{\scriptstyle \pi_1}\downarrow\uparrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \pi} & & {\scriptstyle \tau_2}\uparrow\downarrow{\scriptstyle \pi_2} \\
L^1 & \hookrightarrow & L^1\#k[G/N] & \rightleftarrows & k[G/N]
\end{array}$$

where the horizontal maps are the canonical ones, the rows are exact (3.6(c) of [3]), $\pi_1 \circ \tau_1 = I_{L^1}$, $\pi_2 \circ \tau_2 = I_{k[G/N]}$, $H \cong H^1 \,\sharp\, k[G]$, and $L \cong L^1 \,\sharp\, k[G/N]$.

Thus we have Hopf algebra maps $i \circ \tau_1\colon L^1 \to H$ and $j \circ \tau_2\colon k[G/N] \to H$ and it is clear from the diagram that $i \circ \tau_1$ is a morphism of $k[G/N]$-algebras. So by the universal property of the smash product (1.8 of [1]) there is a map $\tau\colon L^1 \,\sharp\, k[G/N] \to H$, $\tau = (i \circ \tau_1) \,\sharp\, (j \circ \tau_2)$. This map is clearly a Hopf algebra map (see §2 of [3]) and splits π, so we are done.

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