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A SEMILATTICE DECOMPOSITION INTO SEMIGROUPS HAVING AT MOST ONE IDEMPOTENT Mohan S. Putcha and Julian Weissglass

# A SEMILATTICE DECOMPOSITION INTO SEMIGROUPS HAVING AT MOST ONE IDEMPOTENT 

Mohan S. Putcha and Julian Weissglass

A semigroup $\mathbf{S}$ is said to be viable if $a b=b a$ whenever $a b$ and $b a$ are idempotents. The main theorem of this article proves in part that $S$ is a viable semigroup if and only if $S$ is a semi-latice of $\mathscr{S}$-indecomposable semigroups having at most one idempotent.

Furthermore, each semigroup appearing in the decomposition has a group ideal whenever it has an idempotent. Also included as part of the main theorem is the more elementary result that $S$ is viable if and only if every $\mathcal{J}$-class contains at most one idempotent.

Throughout $S$ will denote a semigroup and $E=E(S)$ the set of idemotents of $S$.

Definition. Let $a, b \in S$. We say $a \mid b$ if there exist $x, y \in S$ such that $a x=y a=b$. The set-valued function $\mathfrak{M}$ on $S$ is defined by $\mathfrak{M}(a)=\{e|e \in E, a| e\}$. The relation $\delta$ on $S$ is defined by $a \delta b$ if $\mathfrak{M}(a)=\mathfrak{M}(b)$.

Our first goal is to show that if $S$ is viable then $\delta$ is a congruence on $S$ and $S / \delta$ is the semilattice described above.

Lemma 1. Let $S$ be viable. If $a b=e \in E$, then $b e a=e$.
Proof. $(b e a)^{2}=b e a b e a=b e a$. Hence bea $\in E$. But cleary $a b e=$ $e \in E$. Hence $b e a=a b e=e$.

Lemma 2. Let $S$ be viable. Suppose $a \in S$ and $e \in E$. Then $a \mid e$ if and only if $e \in S^{1} a S^{1}$.

Proof. If $a \mid e$, then $e \in S^{1} a S^{1}$ by definition. Conversely assume $e=s a t$ with $s, t \in S^{1}$. By (1), ates $=e$ and tesa $=e$. Therefore $a \mid e$.

Theorem 3. Let $S$ be viable. Then
(i) $\delta$ is a congruence relation on $S$ containing Green's relation $\mathscr{H}$.
(ii) $S / \delta$ is a semilattice and
(iii) each $\delta$-class contains at most one idempotent and a group ideal whenever it contains an idempotent.

Proof. (i) Clearly $\delta$ is an equivalence relation. We will show that $\delta$ is right compatible. Assume $a \delta b$. If $a c \mid e \in E$, then
$a c x=e$ for some $x \in S$. By (1), cxea $=e$. Hence $a \mid e$. Thus $b \mid e$, so $y b=e$ for some $y \in S$. Therefore $y b c x e a=e$, so $b c \mid e$ by (2). Hence $\mathfrak{M}(a c) \subseteq \mathfrak{M}(b c)$. Similary $\mathfrak{M}(b c) \subseteq \mathfrak{M}(a c)$ and hence $a c o \delta b c$. That $\delta$ is left compatible follows analogously. Consequently, $\delta$ is a congruence. It is immediate that $\mathscr{C} \cong \delta$.
(ii) To show $S / \delta$ is a band, let $a \in S$. If $a^{2} \mid e \in E$ then by (2), $a \mid e$. Hence $\mathfrak{M}\left(a^{2}\right) \subseteq \mathfrak{M}(a)$. Suppose $a \mid e \in E$, say $a x=y a=e, x, y \in$ $S$. Then $y a^{2} x=e$. Again using (2), $a^{2} \mid e$. Thus, $\mathfrak{M}\left(a^{2}\right)=\mathfrak{M}(a)$ and $a \delta a^{2}$. So $S / \delta$ is a band. Now let $a, b \in S$. If $e \in \mathfrak{M}(a b)$, then there exist $x, y \in S$ such that $a b x=y a b=e$. Hence $y a(b a) b x=e$, and by (2), $e \in \mathfrak{M}(b a)$. Therefore $\mathfrak{M}(a b) \subseteq \mathfrak{M}(b a)$. By symmetry, $\mathfrak{M l}(b a) \subseteq$ $\mathfrak{M}(a b)$. Hence $a b \delta b a$ and $S / \delta$ is a semilattice.
(iii) Suppose, $e_{1} \delta e_{2}$ with $e_{1}, e_{2} \in E$. Then $e_{1} \in \mathfrak{M}\left(e_{1}\right)=\mathfrak{M}\left(e_{2}\right)$, so $e_{2} \mid e_{1}$. Similarly $e_{1} \mid e_{2}$. Hence $e_{1} \mathscr{H} e_{2}$ and by [2], Lemma 2.15, $e_{1}=$ $e_{2}$. Thus each $\delta$-class contains at most one idempotent. Now suppose $A$ is a $\delta$-class containing an idempotent $e$. Let $a \in A$. Since $e \in$ $\mathfrak{M}(e)=\mathfrak{M}(a)=\mathfrak{M}\left(a^{2}\right)$, there exists $x \in S$ such that $a^{2} x=e . \quad$ Now $a \delta$ $a^{2}$ implies $a x$ o $a^{2} x$, so $a x$ o $e$ e ò $a$. Hence $a x \in A$ and $a(a x)=e$ implies $e$ is a right zeroid of $A$. Similarly $e$ is a left zeroid and by [2], §2.5, Exercise 6, $A$ has a group ideal.

A semigroup is said to be $\mathscr{S}$-indecomposable if it has no proper semilattice decomposition.

Corollary 4. If the viable semigroup $S$ is S-indecomposable then $S / \delta=1$ and is either idempotent-free or has a group ideal and exactly one idempotent.

Lemma 5. Assume $I$ is an idempotent-free ideal of $S$. Then $S$ is viable if and only if the Rees factor semigroup S/I is viable.

Proof. Assume $S$ is viable and that $a b, b a \in E(S / I)$. If $a b \in I$, then $b a=b(a b) a \in I$, so $a b=b a$ in $S / I$. So we may assume $a b$ and $b a$ are not in $I$. But then $a b, b a \in E(S)$. Hence $a b=b a$ in $S$ and so in $S / I$. Therefore $S / I$ is viable. Conversely, let $a b, b a \in E(S)$. Since $S / I$ is viable $a b=b a$ in $S / I$. But $a b, b a \notin \mathrm{I}$ since $I$ is idempotent-free. Hence $a b=b a$ in $S$ and $S$ is viable.

A semigroup $S$ is said to be $E$-inversive if for every $a \in S$ there exists $x \in S$ such that $a x \in E$.

Theorem 6. The following are equivalent.
(i) Every $\mathcal{J}$-class of $S$ contains at most one idempotent
(ii) $S$ is viable.
(iii) $S$ is a smilattice of $\operatorname{Se}$-indecomposable semigroups each of
which contains at most one idempotent and a group ideal whenever it contains an idempotent.
(iv) $S$ is a semilattice of semigroups having at most one idempotent.
(v) $S$ is viable and E-inversive or an ideal extension of an idempotent-free semigroup by a viable E-inversive semigroup.

Proof. $\quad(i) \Rightarrow$ (ii) If $a b$ and $b a$ are idempotents then $a b=a(b a) b \in$ $S^{1} b a S^{1}$. Similarly $b a \in S^{1} a b S^{1}$. Hence $a b \not \mathscr{J} b a$, so $a b=b a$.
(ii) $\Rightarrow$ (iii) By Tamura [3], $S$ is a semilattice of $\mathscr{S}$-indecomposable semigroups. Since subsemigroups of viable semigroups are viable, each component is viable. The result follows from (4).
(iii) $\Rightarrow$ (iv) a fortiori
(iv) $\Rightarrow$ (i) Suppose $e, f \in E$ with $e \mathscr{J} f$. Then $e$ and $f$ are in the same component of the given semilattice decomposition. Hence $e=f$.
(ii) $\Rightarrow$ (v) Let $I=\{a \in S \mid \mathfrak{M}(\alpha)=\varnothing\}$. If $I$ is empty then $S$ is $E$-inversive. Otherwise, $I$ is obviously an idempotent-free $\delta$-class of $S$. Moreover if $a x \mid e$ or $x a \mid e, e \in E$, then by (2), $a \mid e$. Hence, $a \in I$ implies $a x, x a \in I$ so that $I$ is an ideal of $S$. By (5), $S / I$ is viable. Since $S / I$ has a zero, it is $E$-inversive. In fact, every nonzero element of $S / I$ divides a nonzero idempotent of $S / I$.
(v) $\Rightarrow$ (ii) Follows from (5).

Remark. Observe that the semilattice decomposition of (iii) in general will not be isomorphic to $S / \delta$ since in fact $S$ may be idempotent free. Also, $\mathscr{F}$ may be replaced $\mathscr{O}$ in the theorem.

Lemma 7. $S$ is an ideal extension of a group by a nil semigroup if and only if $S$ is a subdirect product of a group and a nil semigroup.

Proof. Suppose $S$ is an ideal extension of a group $G$ by a nil semigroup $N$. Let $e$ be the identity of $G$. It is easy to see that $e$ is central in $S$. It is well known that $S$ is a subdirect product of subdirectly irreducible semigroups $S_{\alpha}(\alpha \in \Omega)$. Let $\sigma_{\alpha}: S \rightarrow S_{\alpha}$ be the natural map. Let $e_{\alpha}=e \sigma_{\alpha}$. Then $e_{\alpha}$ is a central idempotent in $S_{\alpha}$ and hence is zero or 1 (cf. [1]). If $e_{\alpha}=0$, then $\sigma_{\alpha}(G)=0$ and hence $S_{\alpha}=\sigma_{\alpha}(S)$ is a nil semigroup. If $e_{\alpha}=1$, then all of $S_{\alpha}$ is contained in $\sigma_{\alpha}(G)$ and hence $S_{\alpha}$ is a group. Consequently each $S_{\alpha}$ is a nil semigroup or a group. Let $\Omega_{1}=\left\{\alpha \mid \alpha \in \Omega, S_{\alpha}\right.$ is nil $\}$ and let $\Omega_{2}=$ $\left\{\alpha \mid \alpha \in \Omega, S_{\alpha}\right.$ is a group $\}$. Let $\psi_{i}=\prod_{\alpha \in \Omega_{i}} \sigma_{\alpha}: S \rightarrow \prod_{\alpha \in \Omega_{i}} S_{\alpha}$ be defined for $i=1,2$. One can check that $S$ is a subdirect product of $S \psi_{1}$ and $S \psi_{2}$ with $S \psi_{1}$ a nil semigroup and $S \psi_{2}$ a group.

Conversely, suppose $S$ is a subdirect of a group $G$ and a nil
semigroup $N$. Consider $S$ embedded in $G \times N$. Let $e$ be the identity of $G$. There exists $a \in N$ such that $(e, a) \in S$. There exists a positive integer $k$ such that $a^{k}=0$. Hence $(e, 0)=\left(e, a^{k}\right)=(e, a)^{k} \in S$. If $g \in$ $G$, there exists $b \in N$ such that $(g, b) \in S$. Thus $(g, 0)=(e, 0)(g, b) \in$ $S$. Hence $G \times\{0\} \subseteq S$ and $G \times\{0\}$ is an ideal of $S$. Let $(g, a) \in S$. Since $a \in N$, there exists a positive integer $k$ such that $\mathrm{a}^{k}=0$. Hence $(g, a)^{k}=\left(g^{k}, a^{k}\right)=\left(g^{k}, 0\right) \in G \times\{0\}$. Therefore $S$ is an ideal extension of the group $G \times\{0\}$ by a nil semigroup.

Corollary 8. The following are equivalent.
(i) $S$ is viable and a power of each element lies in a subgroup.
(ii) $S$ is a semilattice of semigroups which are ideal extensions of groups by nil semigroups.
(iii) $S$ is a semilattice of semigroups each of which is a subdirect product of a nil semigroup.
Moreover the decompositions in (ii) and (iii) are the same and coincide with the $\delta$-decomposition as specified in Theorem 3.

A semigroup $S$ is separative if $x^{2}=x y=y^{2}(x, p \in S)$ implies $x=y$.
Corollary 9. The following are equivalent.
(i) $S$ is viable, separative and a power of each element of $S$ lies in a subgroup.
(ii) $S$ is a semilattice of groups.

Proof. (i) $\Rightarrow$ (ii) By (8), it suffices to show that if $T$ is separative and an ideal extension of a group $G$ by a nil semigroup, then $T=G$. Let $e$ be the identity of $G$. Then $e$ is central in $T$. If $T \neq$ $G$, then there exists $a \in T, a \notin G$ with $a^{2} \in G$. Then $a^{2}=(a e)^{2}=a(a e)$. Thus $a=a e \in G$, a contradiction. Hence $T=G$.
(ii) $\Rightarrow$ (i) Obvious.

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