

A semiparametric probability distribution estimator of sample maximums

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Abstract

Several approaches of nonparametric inference for extreme values have been studied. This study surveys the semiparametric probability distribution estimation of sample maximums. Moriyama (2021) clarified that the parametric fitting to the generalized extreme value distribution becomes large as the tail becomes light, which means the convergence becomes slow. Moriyama (2021) proposed a nonparametric distribution estimator without the fitting of the distribution and obtained asymptotic properties. The nonparametric estimator was proved to outperform the parametrically fitting estimator for light-tailed data. Moreover, it was demonstrated that the parametric fitting estimator numerically outperformed the nonparametric one in other cases.

Motivated by the study, we construct two types of semiparametric distribution estimators of sample maximums. The proposed distribution estimators are constructed by mixing the two distribution estimators presented in Moriyama (2021). The cross-validation method and the maximum-likelihood method are presented as a way of estimating the optimal mixing ratio. Simulation experiments clarify the numerical properties of the two types of semiparametric distribution estimators.

Keywords: Cross-validation; distribution estimator; sample maximum; semiparametric estimation

1 Introduction

Recently, the semiparametric approach, which is neither parametric nor fully nonparametric, was developed. We consider semiparametric approaches as “parametrically guided” nonparametric approaches in this study. Olkin and Spiegelman (1987) and Soleymani and Lee (2014) proposed a mixed semiparametric density estimator that has the \sqrt{n} consistency under the model assumption. Mazo and Portier (2020) proposed “the fitness coefficient” that maximizes the estimated likelihood and referred the application to Olkin and Spiegelman (1987)’s density estimation. Hjort and Jones (1996) proposed a locally parametric nonparametric density estimator. It is obtained as the minimizer of the local kernel-smoothed likelihood. Localization was also introduced in nonparametric regression. Tibshirani and Hastie (1987) proposed a local likelihood method for polynomial regression estimation. Loader (1996) extended the local likelihood to density estimation. Hjort and Glad (1995) proposed a parametrically guided nonparametric density estimator. The model was extended to the case of censored data by Talamakrouni *et al.* (2016), and the asymptotic properties of the density estimator and the hazard function estimator were derived.

Schuster and Yakowitz (1985) and Olkin and Spiegelman (1987) proposed the following mixed semiparametric density estimator:

$$g(x; p) = pf_{\hat{\theta}}(x) + (1 - p)\hat{f}(x), \quad (1)$$

where $f_{\hat{\theta}}(x)$ and $\hat{f}(x)$ are parametric and nonparametric density estimators, respectively. $0 \leq p \leq 1$ is the mixing parameter. If the underlying distribution F belongs to the parametric class $\{F_{\theta} | \theta \in \Theta \subset \mathbb{R}^d\}$, the mixed density estimator has \sqrt{n} consistency under the model assumption (see Olkin and Spiegelman (1987)). Faraway (1990) investigated numerical properties. While Olkin and Spiegelman (1987) proposed the pseudolikelihood approach for the mixing parameter estimation, Rahman *et al.* (1997) studied the minimized

integrated squared error approach. Soleymani and Lee (2014) proposed the bootstrap method for the mixing parameter estimation and investigated asymptotic properties.

Although we do not usually know the parametric class of distribution F , there are a few cases where we know the class of the approximating distribution. One of them is the case of the distribution estimation of the sample maximum, which is implied by Moriyama (2021). In the independent and identically distributed (i.i.d.) case, it is F^m , and it can be approximated by the generalized extreme value distribution based on the extreme value theory. Moriyama (2021) devised the nonparametric (plug-in) estimator \widehat{F}^m , where \widehat{F} is the nonparametric kernel distribution estimator. Motivated by Olkin and Spiegelman (1987) and Soleymani and Lee (2014), we propose two semiparametric distribution estimators of the distribution of the sample maximum (DSM) given by mixing the approximating generalized extreme value distribution and the kernel type of the nonparametric distribution estimator. The maximum-likelihood approaches presented in Schuster and Yakowitz (1985) and Olkin and Spiegelman (1987) and a cross-validation approach for the mixing parameter selection are introduced in Section 2. Asymptotic properties of the two types of distributions are obtained in Moriyama (2021). In this study, we clarify the numerical properties of the semiparametric distribution estimator of the DSM by conducting numerical studies. Results of the studies are given in Section 3.

2 Semiparametric DSM estimators

Throughout the paper, let X_1, X_2, \dots be *i.i.d.* random variables with a continuous distribution function F , which belongs to the maximum domain of attraction. Let us consider the estimation of the probability DSM of the future observations $\{X_{n+1}, \dots, X_{n+m}\}$ from the observations $\{X_1, \dots, X_n\}$. The DSM of F^m is approximated by the generalized extreme value distribution $G_{\gamma m}$, where

$$G_{\gamma m}(x) := G_{\gamma} \left(\frac{x - b_m}{a_m} \right) \quad \text{for } 1 + \gamma \frac{x - b_m}{a_m} > 0,$$

$$G_{\gamma}(x) := \begin{cases} \exp(-(1 + \gamma x)^{-1/\gamma}), & 1 + \gamma x > 0, \quad \gamma \in \mathbb{R} \setminus \{0\} \\ \exp(-\exp(-x)), & x \in \mathbb{R}, \quad \gamma = 0. \end{cases}$$

Dombry and Ferreira (2019) proved that the maximum likelihood estimator $\widehat{\gamma}_k$ based on the Block Maxima method is consistent and asymptotically normal, where k is the block size. Moriyama (2021) researched the convergence rate of the parametrically fitting distribution estimator $G_{\widehat{\gamma}_k}$.

2.1 Maximum-Likelihood DSM estimator

Schuster and Yakowitz (1985) and Olkin and Spiegelman (1987) surveyed the following mixed type of a semiparametric density estimator:

$$g(x; p) = pf_{\widehat{\theta}}(x) + (1 - p)\widehat{f}(x; h), \tag{2}$$

where $f_{\widehat{\theta}}$ is the maximizing likelihood parametric density estimator, and \widehat{f} is the kernel density estimator defined as

$$\widehat{f}(x) := \frac{1}{nh} \sum_{i=1}^n k \left(\frac{x - X_i}{h} \right).$$

A symmetric density is used as the kernel function k , and h is the smoothing parameter. The mixing parameter p in the semiparametric density estimator is the maximizer of the pseudolikelihood g , with a smoothing parameter estimator \widehat{h} . Olkin and Spiegelman (1987) clarified the condition of the convergence of the semiparametric density estimator. Aiming at maximizing the likelihood, we propose the estimator

$$\widetilde{G}(x; p, h) := pG_{\widehat{\gamma}_k}(x) + (1 - p)\widehat{F}^m(x; h),$$

where \widehat{F} is the kernel distribution estimator. The proposed semiparametric distribution estimator of the DSM $\widehat{G}(x; h)$ consists of the two distributions $G_{\widehat{\gamma}_k}$ and \widehat{F}^m . In practice, h is replaced with a data-driven bandwidth \widehat{h} , and p is replaced with

$$\widehat{p} := \operatorname{argmax}_{0 \leq p \leq 1} \left[\sum_{i=1}^n \log \widetilde{g}(x; p, \widehat{h}) \right],$$

where \widetilde{g} is the derivative of \widetilde{G} .

Remark 1 *In Schuster and Gregory (1981), a study on the maximum-likelihood bandwidth estimator in the naive kernel density estimation is presented. In Schuster and Gregory (1981), it is proven that the kernel density estimator with the maximum-likelihood bandwidth is not consistent for distributions with $\gamma \geq 0$.*

The plug-in type of the smoothing estimator proposed in Altman and Léger (1995) is employed in this study. The “kerdiest” package in the R software provides the function “ALbw” for calculating the plug-in type of the estimated value. It is well demonstrated (e.g., by Lejeune and Sarda (1992)) that the kernel function selection does not affect the numerical accuracy significantly.

2.2 Cross-validated DSM estimator

We consider a slight modification to the maximum-likelihood DSM estimator. An alternative estimator is

$$\widehat{G}(x; h) := h(1+h)^{-1}G_{\widehat{\gamma}_k}(x) + (1+h)^{-1}\widehat{F}^m(x; h), \quad (3)$$

where the mixing parameter $h(1+h)^{-1}$, not p , takes a value between 0 and 1. We set $q := h(1+h)^{-1}$ as opposed to the maximum-likelihood estimator, $\widehat{G}(x; h)$, which does not have the secondary hyperparameter (p in (1)). Therefore, the proposed distribution estimator is computationally intensive. Further, the distribution estimator has the following property:

Remark 2 *The bandwidth in the kernel distribution estimation controls the bias-variance tradeoff. It holds that*

$$\widehat{F}(x; h) \rightarrow \begin{cases} F_n(x) & \text{as } h \rightarrow 0 \\ 0.5 & \text{as } h \rightarrow \infty, \end{cases}$$

where F_n is the empirical distribution estimator. $F_n(x)$ is unbiased and has the variance $(n^{-1}F(x)\{1-F(x)\})$. 0.5 has zero variance and is biased. The proposed distribution estimator satisfies

$$\widehat{G}(x; h) \rightarrow \begin{cases} F_n^m(x) & \text{as } h \rightarrow 0 \\ G_{\widehat{\gamma}_k}(x) & \text{as } h \rightarrow \infty, \end{cases}$$

where both $F_n^m(x)$ and $G_{\widehat{\gamma}_k}(x)$ are consistent under certain respective conditions. This means that h is expected to control the MSE tradeoff, which improves the accuracy of the naive nonparametric distribution estimator $\widehat{F}^m(x; h)$.

The proposed estimator also relates to the studies on kernel density estimation with *large* bandwidth (Jones (1991) and Jones (1993)). Jones (1993) refers to the semiparametric density estimation using the conventional kernel estimator. We refer to Fryer (1976) as the case study on Gaussian density estimation.

Remark 3 *It is noted from Moriyama (2021) that there are cases where both the parametric estimator $G_{\widehat{\gamma}_k}(x)$ and $NE \widehat{F}(x; h)$ do not converge. In fact, the proposed semiparametric estimator does not converge in the cases.*

The mixing and smoothing parameter aims at minimizing the mean integrated squared error

$$\int \{\widehat{G}(x; h) - F^m(x)\}^2 dx,$$

, and sets

$$\widehat{h} := \operatorname{argmin}_{h>0} \left[\int \widehat{G}^2(x; h) dx - \frac{2}{n} \sum_{i=1}^n \widehat{G}^{(-i)}(X_i; h) \right],$$

where $\widehat{G}^{(-i)}$ is the distribution estimator without X_i . The “leave-one-out” cross-validation approach in the naive kernel distribution estimation was studied by Sarda (1993), and its theoretical superiority is provided in Sarda (1993).

3 Simulation study

By simulating the following mean integrated squared error (*MISE*)

$$L_m^{-1} \int_{Q_m(0.1)}^{Q_m(0.9)} (\check{G}(x) - F^m(x))^2 dx,$$

, we surveyed the numerical accuracy of the two semiparametric estimators in finite-sample cases where $L_m := Q_m(0.9) - Q_m(0.1)$, and $Q_m(q)$ denotes the q th quantile of the SMD. \check{G} is the maximum-likelihood estimator \tilde{G} or the cross-validation estimator \widehat{G} . The convergence rates of the parametrically fitting estimator $G_{\widehat{\gamma}_k}$ and the nonparametric estimator \widehat{F}^m are given in Moriyama (2021). Table 1 in this manuscript summarizes the faster rates of the two.

We simulated *MISE* values 100 times between the 10th quantile and 90th quantile of *smd*. *MISE* and *sd* in Tables 2 and 4 denote the mean *MISE* values and their standard deviation, respectively. The sample sizes were $(n =)2^8$, and the forecast periods were $m = n^{1/4}$, $m = n^{1/2}$, and $m = n^{3/4}$. The underlying distributions F were the Pareto distributions, T distributions, Burr distributions, Frchet distributions, Weibull distributions, reversed Burr distributions, and R. von Mises distributions (see e.g., De Haan and Ferreira (2006))—which are given by

$$1 - F(x) = e^{-x - \sin x} \quad x > 0,$$

which is known not to belong to the maximum domain of attraction. The kernels k are Epanechnikov for the reversed Burr distributions and Gaussian for the other distributions. The mixing parameter p of \tilde{G} in Tables 2–3 were estimated by the maximum-likelihood approach presented in Section 2.1. h of \widehat{G} in Tables 4–5 were estimated by the cross-validation approach introduced in Section 2.2. The mean and standard deviation of the estimated “mixing ratio” values p and q are summarized in Tables 3 and 5, respectively.

Moriyama (2021) demonstrated that the parametric distribution estimator is both theoretically and numerically better than the nonparametric distribution estimator in heavy tailed cases. Tables 3 and 5 show that the mixing ratio parameter values estimated by either approach become close to 1 (i.e., the semiparametric distribution estimator becomes parametric) to a varying degree as the tail becomes heavy. The numerical property coincides with what we expected in advance. With R. von Mises, where the parametric distribution estimator is inconsistent, the estimated values of p and q are almost 0.

From the definition of the maximum-likelihood estimator, it usually returns the values between the parametric and nonparametric estimates, as numerically confirmed by comparing Table 2 with Tables 2–5 in Moriyama (2021). However, Table 3 shows that the estimated ratio values p are larger than 0.5 in most of the cases. Moreover, p are almost 1 at the middle period $m(= \sqrt{n})$ for most of the distributions and even larger than those at $m = n^{3/4}$. These results are not preferable as seen from comparing Table 2 with the *MISE* values given in Moriyama (2021). Moreover, the estimates of p are almost 1 for the reversed Burr distributions and tend to 1 for the Weill distributions as at least one of m and n becomes large. This happens even though the numerical superiority of the nonparametric estimator in such cases was demonstrated in Moriyama (2021).

Table 1: The polynomial convergence rates of MSE of the estimator and the lengths

Pareto			$m = n^{1/4}$	$m = n^{1/2}$	$m = n^{3/4}$
ℓ	α	β			
1/2	1/2	1	-3/4	-1/2	-1/4
1	1	1	-3/4	-1/2	-1/4
3	3	1	-3/4	-1/2	-1/4
10	10	1	-3/4	-1/2	-1/4
T dist.			$m = n^{1/4}$	$m = n^{1/2}$	$m = n^{3/4}$
ℓ	α	β			
1/2	1/2	2	-3/4	-1/2	-1/4
1	1	2	-3/4	-1/2	-1/4
3	3	2	-3/4	-1/2	-1/4
10	10	2	-3/4	-1/2	-1/4
Burr			$m = n^{1/4}$	$m = n^{1/2}$	$m = n^{3/4}$
c, ℓ	α	β			
1/2, 1/2	1/4	1/2	-3/4	-1/2	-1/4
1, 1/2	1/2	1	-3/4	-1/2	-1/4
3, 1/2	3/2	3	-3/4	-1/2	-1/4
1/2, 1	1/2	1/2	-3/4	-1/2	-1/4
1, 1	1	1	-3/4	-1/2	-1/4
3, 1	3	3	-3/4	-1/2	-1/4
1/2, 3	3/2	1/2	-3/4	-1/2	-1/4
1, 3	3	1	-3/4	-1/2	-1/4
3, 3	9	3	-3/4	-1/2	-1/4
Frechet			$m = n^{1/4}$	$m = n^{1/2}$	$m = n^{3/4}$
γ	α	β			
5	1/5	1/5	-3/4	-1/2	-1/4
2	1/2	1/2	-3/4	-1/2	-1/4
1	1	1	-3/4	-1/2	-1/4
1/2	2	1	-3/4	-1/2	-1/4
1/4	4	1	-3/4	-1/2	-1/4
Weibull			$m = n^{1/4}$	$m = n^{1/2}$	$m = n^{3/4}$
κ	γ	ρ			
1/2	0	0	-3/4	-1/2	-1/4
1	0	0	-3/4	-1/2	-1/4
3	0	0	-3/4	-1/2	-1/4
10	0	0	-3/4	-1/2	-1/4
rev. Burr			$m = n^{1/4}$	$m = n^{1/2}$	$m = n^{3/4}$
c, ℓ	μ	σ			
-1/2, -1/3	-6	-2	-3/4	-1/2	-1/4
-1, -1/3	-3	-1	-3/4	-1/2	-1/4
-3, -1/3	-1	-1/3	-3/4	-	-
-1/2, -1	-2	-2	-3/4	-	-
-1, -1	-1	-1	-3/4	-	-
-3, -1	-1/3	-1/3	-	-	-
-1/2, -2	-1	-2	-3/4	-	-
-1, -2	-1/2	-1	-	-	-
-3, -2	-1/6	-1/3	-	-	-
R. von Mises			$m = n^{1/4}$	$m = n^{1/2}$	$m = n^{3/4}$
-	-	-	-3/4	-1/2	-1/4

Table 5 shows that the estimated ratio values q are comparatively moderate and tend to be what we expected except for some of the Weibull cases. However, as $q \downarrow 0$, the semiparametric estimator does not necessarily get close to the nonparametric one with the previously estimated bandwidth. Actually, the semiparametric estimator is much worse than that of the nonparametric estimator in the Weibull case with $\kappa = 10$ and much better in the reversed Burr case with $c = -1/2$.

Comparing Tables 3 and 5, we can summarize the results as follows. The maximum-likelihood approach tends to return a parametric result. The cross-validation approach changes according to the heavy-tailness of data but does not coincide with the nonparametric estimator. The maximum-likelihood approach is better for cases where (i) small $\gamma > 0$ with small m and (ii) $\gamma = 0$, and the cross-validation approach is better for cases where (iii) large $\gamma > 0$ with small m , (iv) $\gamma < 0$, and (v) F does not belong to any maximum domain of attraction.

Table 2: Scaled MISE values ($\times 100$) and sd values ($\times 100$) with maximum-likelihood bandwidth

	$n = 2^8$						$n = 2^{12}$					
	MISE	sd	MISE	sd	MISE	sd	MISE	sd	MISE	sd	MISE	sd
Pareto	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
1/2	0.561	1.218	5.202	0.864	5.602	5.202	0.264	0.923	8.241	0.067	4.231	2.619
1	0.477	1.057	5.237	0.950	5.890	5.507	0.280	0.922	8.242	0.065	4.187	2.399
3	0.545	1.042	5.251	0.837	5.660	5.225	0.254	0.922	8.247	0.066	3.852	1.982
10	0.523	1.158	5.289	0.864	5.139	4.938	0.264	0.917	8.243	0.068	3.815	2.687
T	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
1/2	5.007	4.113	1.585	2.266	11.11	12.85	11.10	3.556	2.930	0.236	5.768	9.306
1	1.211	1.756	3.768	1.196	5.107	4.531	1.212	2.701	6.736	0.535	5.348	3.061
3	0.187	0.210	10.97	1.745	4.652	2.860	0.025	0.028	15.12	0.780	13.89	1.319
10	0.179	0.195	16.58	1.381	10.18	3.553	0.025	0.026	21.03	0.546	18.49	0.851
Burr	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
1/2, 1/2	3.959	5.092	0.867	2.233	16.94	20.20	5.940	9.827	1.710	1.595	9.573	17.40
1, 1/2	2.664	3.732	1.961	1.434	10.33	12.86	5.221	5.528	3.591	0.026	5.473	8.325
3, 1/2	0.296	0.568	7.649	1.035	3.867	3.090	0.038	0.203	11.98	0.101	6.701	2.728
1/2, 1	2.058	4.022	1.655	1.264	10.62	11.94	1.596	4.330	3.214	0.288	5.480	9.143
1, 1	0.458	1.019	5.297	0.856	5.213	4.808	0.267	0.961	8.242	0.069	4.280	2.631
3, 1	0.183	0.190	11.73	1.244	3.629	2.003	0.026	0.029	16.55	0.244	13.66	2.146
1/2, 3	0.216	0.432	5.385	0.975	4.524	3.195	0.026	0.028	8.956	0.319	5.324	2.524
1, 3	0.188	0.220	10.79	0.864	3.849	2.313	0.024	0.028	15.46	0.171	12.71	2.558
3, 3	0.188	0.212	16.19	1.181	7.255	4.031	0.023	0.026	20.83	0.373	17.92	0.784
Frechet	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
5	3.134	9.050	1.682	6.276	18.91	22.86	11.23	16.72	3.067	9.259	11.85	20.78
2	2.655	3.862	1.790	1.000	9.683	11.46	3.749	5.117	3.458	0.096	5.425	8.183
1	0.542	1.046	5.173	0.972	5.648	5.089	0.294	0.964	8.488	0.066	4.229	2.539
1/2	0.212	0.275	9.273	1.095	3.456	2.328	0.026	0.034	13.93	0.147	8.740	3.370
1/4	0.193	0.216	12.66	1.139	3.971	2.523	0.026	0.027	17.82	0.218	14.39	2.984
Weibull	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
1/2	0.215	0.231	4.190	2.607	3.865	2.710	0.042	0.041	14.56	0.660	6.857	2.023
1	0.213	0.235	5.285	3.414	4.789	3.658	0.027	0.032	20.99	0.180	18.91	0.439
3	0.177	0.202	1.219	2.145	7.869	3.608	0.025	0.028	24.46	0.339	21.48	0.594
10	0.180	0.213	0.562	0.622	5.142	3.890	0.025	0.027	24.77	0.462	22.38	0.644
rev.Burr	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
-1/2, -1/3	20.19	13.57	71.68	1.205	71.82	1.937	3.985	4.626	4.673	0.062	72.38	0.252
-1, -1/3	2.191	3.060	47.33	4.069	54.37	2.536	1.282	0.612	3.507	0.028	64.58	0.264
-3, -1/3	0.289	0.463	29.42	2.346	36.56	2.529	49.35	6.064	0.054	0.000	0.003	0.000
-1/2, -1	13.32	8.343	65.08	1.179	63.98	2.815	2.644	3.941	55.82	7.040	57.36	0.259
-1, -1	1.626	2.140	37.27	2.741	39.91	2.840	4.779	1.265	42.88	3.956	43.90	0.446
-3, -1	0.221	0.247	22.45	1.841	21.26	4.158	12.82	2.780	3.936	0.496	0.562	0.016
-1/2, -2	11.86	10.71	55.21	1.070	49.94	2.504	0.094	0.321	40.78	0.677	43.95	0.159
-1, -2	0.651	1.018	28.99	2.462	27.50	2.837	0.058	0.303	29.18	0.453	31.57	0.605
-3, -2	0.235	0.286	19.47	1.837	16.58	4.455	0.039	0.248	2.564	0.077	0.927	0.034
R.vonMises	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
	0.239	0.221	16.76	1.206	6.003	3.257	0.234	0.302	29.39	0.282	18.46	0.448

Table 3: Estimated maximum-likelihood mixing parameter p and sd values

	$n = 2^8$						$n = 2^{12}$					
	MISE	sd	MISE	sd	MISE	sd	MISE	sd	MISE	sd	MISE	sd
Pareto	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
1/2	0.626	0.335	0.949	0.026	0.706	0.023	0.666	0.300	0.994	0.003	0.923	0.061
1	0.644	0.335	0.950	0.025	0.708	0.017	0.667	0.306	0.994	0.004	0.923	0.060
3	0.629	0.338	0.950	0.025	0.709	0.016	0.649	0.311	0.994	0.004	0.925	0.061
10	0.645	0.334	0.950	0.024	0.709	0.014	0.667	0.308	0.994	0.004	0.930	0.062
T	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
1/2	0.671	0.092	0.935	0.009	0.749	0.03	0.593	0.055	0.983	0.002	0.997	0.018
1	0.247	0.256	0.931	0.014	0.708	0.022	0.190	0.215	0.985	0.002	1.000	0.006
3	0.055	0.172	0.974	0.020	0.697	0.062	0.000	0.008	0.997	0.001	1.000	0.000
10	0.176	0.341	0.998	0.006	0.849	0.087	0.004	0.045	1.000	0.000	1.000	0.000
Burr	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
1/2, 1/2	0.831	0.171	0.939	0.019	0.999	0.012	0.984	0.077	0.998	0.004	0.970	0.052
1, 1/2	0.794	0.167	0.934	0.017	0.795	0.102	0.759	0.144	0.998	0.002	0.937	0.063
3, 1/2	0.242	0.352	0.942	0.018	0.693	0.022	0.177	0.330	0.989	0.003	0.999	0.011
1/2, 1	0.871	0.174	0.935	0.026	0.841	0.122	0.941	0.130	0.999	0.001	0.936	0.062
1, 1	0.619	0.342	0.949	0.026	0.706	0.017	0.662	0.298	0.994	0.004	0.927	0.061
3, 1	0.238	0.403	0.982	0.015	0.675	0.034	0.077	0.236	0.997	0.001	1.000	0.000
1/2, 3	0.890	0.250	0.932	0.041	0.696	0.033	1.000	0.001	0.993	0.003	0.930	0.060
1, 3	0.562	0.457	0.956	0.010	0.670	0.038	0.936	0.197	0.993	0.001	1.000	0.000
3, 3	0.308	0.447	0.999	0.004	0.755	0.101	0.066	0.236	1.000	0.000	1.000	0.000
Frechet	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
5	0.924	0.145	0.963	0.026	1.000	0.000	0.995	0.042	0.999	0.002	0.998	0.013
2	0.802	0.169	0.929	0.017	0.801	0.102	0.824	0.161	0.992	0.003	0.931	0.063
1	0.568	0.326	0.931	0.022	0.707	0.017	0.563	0.309	0.992	0.004	0.943	0.062
1/2	0.324	0.431	0.951	0.018	0.689	0.023	0.437	0.483	0.991	0.002	1.000	0.004
1/4	0.357	0.454	0.978	0.014	0.677	0.030	0.457	0.489	0.998	0.001	1.000	0.000
Weibull	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
1/2	0.000	0.000	0.544	0.249	0.689	0.124	1.000	0.007	0.989	0.001	1.000	0.000
1	0.000	0.000	0.508	0.209	0.663	0.13	0.989	0.075	1.000	0.001	1.000	0.000
3	0.000	0.290	0.111	0.178	0.664	0.133	0.101	0.290	1.000	0.000	1.000	0.000
10	0.000	0.000	0.043	0.000	0.458	0.297	0.001	0.014	1.000	0.000	1.000	0.000
rev.Burr	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
-1/2, -1/3	0.980	0.078	0.998	0.007	0.997	0.012	0.509	0.068	0.964	0.003	1.000	0.001
-1, -1/3	1.000	0.000	1.000	0.000	1.000	0.000	0.429	0.053	0.964	0.003	1.000	0.001
-3, -1/3	1.000	0.000	1.000	0.000	1.000	0.000	0.412	0.050	0.964	0.003	1.000	0.002
-1/2, -1	0.981	0.075	0.998	0.007	0.995	0.019	0.325	0.127	0.999	0.007	1.000	0.000
-1, -1	1.000	0.009	1.000	0.000	1.000	0.000	0.319	0.150	0.999	0.006	1.000	0.000
-3, -1	1.000	0.000	1.000	0.000	1.000	0.000	0.305	0.172	0.998	0.010	1.000	0.000
-1/2, -2	0.979	0.085	0.998	0.008	0.995	0.018	0.398	0.411	1.000	0.000	1.000	0.000
-1, -2	1.000	0.003	1.000	0.000	1.000	0.000	0.000	0.001	1.000	0.000	1.000	0.000
-3, -2	1.000	0.000	1.000	0.000	1.000	0.000	0.001	0.028	1.000	0.000	1.000	0.000
R.vonMises	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
	0.260	0.403	0.947	0.008	0.680	0.029	0.334	0.452	0.976	0.002	1.000	0.000

Table 4: Scaled MISE values ($\times 100$) and sd values ($\times 100$) with cross-validated bandwidth

	$n = 2^8$						$n = 2^{12}$					
	MISE	sd	MISE	sd	MISE	sd	MISE	sd	MISE	sd	MISE	sd
Pareto	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
1/2	0.394	2.368	4.878	7.132	12.14	12.62	0.209	1.163	0.432	1.688	10.16	11.37
1	0.190	0.235	1.157	2.474	8.326	6.521	0.040	0.038	0.258	0.872	5.714	5.531
3	2.231	0.867	3.049	2.114	5.239	5.067	2.801	0.329	2.214	0.903	2.477	2.833
10	10.12	1.025	17.40	2.211	20.45	4.951	14.69	0.398	19.22	1.068	19.07	3.167
T	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
1/2	0.369	2.099	4.310	6.934	12.06	12.03	0.073	0.510	0.590	2.566	9.706	10.43
1	0.498	0.457	0.825	1.450	8.069	7.188	0.202	0.095	0.206	0.753	5.318	5.367
3	1.157	0.675	2.120	1.770	5.128	5.776	1.608	0.271	1.719	0.818	2.330	2.875
10	1.316	0.624	3.673	2.210	7.508	5.505	2.262	0.309	4.820	1.085	6.338	3.712
Burr	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
1/2, 1/2	0.939	4.842	4.290	9.951	15.60	20.89	6.088	9.703	0.395	0.868	11.70	16.74
1, 1/2	0.299	1.159	4.248	6.673	12.78	12.88	0.108	0.851	0.499	2.015	8.484	7.957
3, 1/2	0.417	0.415	0.795	1.200	6.451	5.758	0.239	0.113	0.170	0.195	2.688	3.712
1/2, 1	0.346	1.363	4.511	6.836	12.14	12.48	0.160	0.893	0.275	0.948	9.518	9.255
1, 1	0.203	0.260	1.113	2.402	8.406	6.763	0.038	0.032	0.259	0.700	6.776	6.096
3, 1	1.863	0.805	2.853	2.030	5.576	5.788	2.440	0.287	2.019	0.786	2.010	2.138
1/2, 3	0.849	0.541	0.911	1.173	6.406	5.743	0.519	0.155	0.240	0.234	2.818	3.051
1, 3	2.303	0.889	3.157	2.077	5.883	6.082	2.784	0.337	2.276	0.801	2.464	2.994
3, 3	5.014	0.939	11.18	2.499	15.35	5.577	8.409	0.379	13.89	1.076	14.68	3.651
Frechet	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
5	1.642	7.574	3.429	10.08	18.01	23.57	11.910	17.77	1.764	8.596	15.05	22.53
2	0.313	1.578	4.335	7.244	14.03	11.71	0.200	1.124	0.366	2.037	10.29	9.848
1	0.196	0.241	1.090	2.430	8.856	6.399	0.039	0.039	0.181	0.207	5.968	5.952
1/2	0.377	0.389	0.876	1.097	5.612	4.603	0.256	0.118	0.240	0.281	2.359	2.715
1/4	0.594	0.500	1.239	1.430	4.802	4.700	0.671	0.198	0.721	0.567	2.088	2.553
Weibull	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
1/2	0.226	0.273	0.905	1.083	5.251	4.917	0.125	0.087	0.272	0.342	2.116	2.424
1	1.120	0.665	2.500	1.911	5.678	5.125	1.640	0.281	2.762	0.955	4.138	3.572
3	4.175	0.853	11.07	2.249	16.69	4.342	7.659	0.356	15.90	1.001	20.51	2.656
10	9.047	0.789	21.01	1.426	26.84	2.005	15.84	0.270	26.74	0.463	30.67	0.900
rev.Burr	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
-1/2, -1/3	2.395	0.884	1.740	0.206	2.445	0.108	0.330	0.040	2.387	0.043	2.359	0.088
-1, -1/3	0.776	0.432	4.441	0.835	7.939	0.427	1.683	0.142	8.031	0.190	8.728	0.076
-3, -1/3	1.880	0.556	7.976	1.530	13.39	1.639	4.204	0.328	13.40	0.560	17.34	0.335
-1/2, -1	2.201	1.754	1.583	0.470	4.831	0.216	1.424	0.175	4.857	0.082	5.097	0.097
-1, -1	0.959	0.495	5.791	1.455	12.58	1.751	2.100	0.328	12.57	0.439	17.31	0.338
-3, -1	1.792	0.548	8.166	2.036	15.30	3.241	4.308	0.293	14.68	0.787	21.55	1.131
-1/2, -2	2.654	2.976	1.081	1.029	4.326	2.205	1.058	1.362	3.512	0.639	13.72	0.670
-1, -2	0.509	0.468	3.001	1.637	9.618	4.202	0.849	0.154	8.142	0.953	17.90	2.019
-3, -2	1.206	0.457	6.128	2.059	13.55	4.397	2.974	0.291	12.43	1.133	20.10	2.771
R.vonMises	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^5$		$m = 2^9$	
	0.384	0.198	2.628	1.548	4.865	4.348	0.696	0.109	0.838	0.524	2.185	2.124

Table 5: Estimated cross-validated mixing parameter q and sd values

	$n = 2^8$						$n = 2^{12}$					
	MISE	sd	MISE	sd	MISE	sd	MISE	sd	MISE	sd	MISE	sd
Pareto	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
1/2	0.620	0.465	0.725	0.429	0.964	0.149	0.647	0.468	0.491	0.486	0.906	0.270
1	0.826	0.223	0.624	0.456	0.891	0.291	0.930	0.029	0.580	0.483	0.788	0.390
3	0.598	0.009	0.667	0.013	0.676	0.194	0.638	0.004	0.732	0.005	0.788	0.165
10	0.310	0.006	0.291	0.007	0.284	0.011	0.301	0.002	0.282	0.004	0.302	0.005
T	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
1/2	0.884	0.100	0.690	0.445	0.958	0.170	0.914	0.148	0.541	0.487	0.918	0.252
1	0.827	0.014	0.749	0.338	0.891	0.267	0.866	0.005	0.709	0.422	0.750	0.415
3	0.814	0.010	0.726	0.025	0.708	0.201	0.720	0.002	0.764	0.005	0.785	0.201
10	0.675	0.010	0.633	0.012	0.592	0.079	0.655	0.005	0.601	0.004	0.591	0.010
Burr	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
-1/2, -1/3	0.664	0.020	0.799	0.151	0.813	0.309	0.760	0.006	0.877	0.200	0.737	0.422
-1, -1/3	0.599	0.009	0.667	0.013	0.684	0.183	0.638	0.004	0.732	0.005	0.807	0.115
-3, -1/3	0.449	0.008	0.400	0.009	0.371	0.015	0.427	0.005	0.370	0.002	0.371	0.007
-1/2, -1	0.500	0.475	0.684	0.446	0.966	0.149	0.598	0.482	0.582	0.478	0.884	0.298
-1, -1	0.813	0.245	0.615	0.459	0.901	0.277	0.931	0.003	0.529	0.491	0.828	0.354
-3, -1	0.637	0.009	0.683	0.012	0.691	0.185	0.661	0.003	0.740	0.005	0.800	0.140
-1/2, -2	0.639	0.458	0.512	0.483	0.985	0.068	0.592	0.478	0.355	0.436	0.751	0.419
-1, -2	0.589	0.473	0.707	0.438	0.965	0.147	0.696	0.451	0.587	0.483	0.979	0.098
-3, -2	0.812	0.012	0.779	0.295	0.826	0.317	0.858	0.004	0.851	0.282	0.673	0.455
Frechet	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
5	0.800	0.386	0.424	0.477	0.990	0.005	0.808	0.374	0.316	0.411	0.744	0.422
2	0.626	0.442	0.682	0.45	0.970	0.130	0.648	0.461	0.531	0.485	0.915	0.250
1	0.822	0.234	0.646	0.449	0.908	0.261	0.926	0.071	0.576	0.484	0.772	0.397
1/2	0.831	0.028	0.800	0.271	0.821	0.306	0.868	0.004	0.879	0.218	0.668	0.447
1/4	0.782	0.008	0.811	0.081	0.770	0.252	0.803	0.005	0.851	0.003	0.796	0.281
Weibull	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
1/2	0.812	0.170	0.776	0.323	0.807	0.323	0.890	0.001	0.875	0.234	0.726	0.407
1	0.694	0.008	0.702	0.010	0.675	0.126	0.702	0.004	0.697	0.004	0.694	0.032
3	0.459	0.009	0.371	0.010	0.297	0.014	0.418	0.004	0.298	0.004	0.232	0.006
10	0.268	0.008	0.165	0.006	0.105	0.006	0.211	0.003	0.105	0.005	0.069	0.002
rev.Burr	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
-1/2, -1/3	0.104	0.039	0.017	0.005	0.010	0.000	0.054	0.014	0.010	0.002	0.010	0.000
-1, -1/3	0.336	0.041	0.137	0.016	0.030	0.007	0.229	0.011	0.030	0.002	0.010	0.000
-3, -1/3	0.479	0.021	0.280	0.018	0.123	0.014	0.373	0.007	0.124	0.005	0.029	0.003
-1/2, -1	0.220	0.099	0.083	0.014	0.011	0.004	0.185	0.010	0.012	0.004	0.010	0.000
-1, -1	0.504	0.061	0.313	0.025	0.133	0.018	0.412	0.019	0.133	0.004	0.030	0.002
-3, -1	0.538	0.018	0.373	0.018	0.232	0.019	0.452	0.006	0.231	0.006	0.106	0.006
-1/2, -2	0.419	0.468	0.434	0.168	0.289	0.055	0.191	0.227	0.283	0.018	0.056	0.007
-1, -2	0.685	0.169	0.563	0.032	0.387	0.036	0.644	0.016	0.384	0.009	0.197	0.008
-3, -2	0.610	0.018	0.459	0.019	0.332	0.022	0.533	0.006	0.329	0.006	0.209	0.008
R.vonMises	$m = 2^2$		$m = 2^4$		$m = 2^6$		$m = 2^3$		$m = 2^6$		$m = 2^9$	
	0.003	0.055	0.003	0.055	0.004	0.063	0.000	0.000	0.000	0.000	0.000	0.000

4 Conclusion

This study proposes two semiparametric approaches for the estimation of the sample maximum distribution. The proposed estimator includes the mixing ratio parameter, and numerical properties of the two different approaches for determining the parameter are studied. The simulation studies demonstrate that both semiparametric estimators can be parametric or nonparametric. As the tail becomes lighter, the mixing ratio parameter tends to zero. This means the semiparametric estimator becomes the nonparametric estimator, which is superior to the parametrically fitting estimator. It is demonstrated that the proposed semiparametric estimators are not always the best but are reasonable for DSM estimation.

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