A SEMISHRINKING BASIS WHICH IS NOT SHRINKING

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A Schauder basis (x_i, f_i) for a Banach space X is

(1) *shrinking* provided (f_i) is a basis for X^*; and

(2) *semishrinking* provided $0<\inf_n\|x_n\| \leqq \sup_n\|x_n\| < +\infty$ and (x_n) is weakly convergent to 0.

A. Pełczynski and W. Szlenk [2], answering a question raised by I. Singer, constructed a Banach space with a semishrinking basis which was not shrinking. Their construction, while elegant, is very complicated. Our purpose here is to show that the space (d) constructed by Davis and Dean [1, p. 214] has such a basis.

The space (d) consists of those real sequences $a=(a_i)$ for which

$$\|a\| = \sup_{p\in\mathcal{P}} \sum_{i=1}^{\infty} \frac{|a_{p(i)}|}{i} < +\infty$$

where $\mathcal{P}$ denotes the collection of all 1-1 maps from the positive integer ω into ω. Davis and Dean [1, p. 214] have shown that the unit vectors (e_n) form a semishrinking basis for (d) (although their terminology is different). We show that (e_n) is not shrinking.

Let $h_n = \sum_{i=1}^{n}(1/i)$ and define $\mathfrak{U}_n \in (d)$ by $\mathfrak{U}_n = h_n^{-1}\sum_{i=1}^{n} e_i$. Clearly $\|\mathfrak{U}_n\|=1$ for $n=1, 2, 3, \cdots$. Define a linear functional F on (d) by $F(x)=\sum_{i=1}^{\infty}(x_i/i)$ where $x=(x_i)$. By the definition of the norm on (d) it follows that $F\in(d)^*$. Moreover, $F(\mathfrak{U}_n)=1$ for each n and so $(\mathfrak{U}_n)$ does not converge weakly to 0. However if (f_i) denotes the coefficient functionals associated with (e_n) then $\lim_{n\to\infty} f_i(\mathfrak{U}_n)=\lim_{n\to\infty} h_n^{-1}=0$. Thus by a theorem of Wilansky [4] (see also [3, property (E)]) (e_n) is not shrinking.

Bibliography

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