A Sequential Locating Game on Graphs

Suzanne Seager

May 28, 2009

Consider the game of locating a marked vertex on a connected graph by repeatedly choosing a vertex of the graph as a probe, and receiving the distance from the probe to the marked vertex. The goal is to minimize the number of probes required. The static version of this game is the well-known problem of finding the metric dimension of the graph. We study the sequential version of this game, and the corresponding sequential location number. Assume throughout that $G = (V, E)$ is a simple connected graph with $n \geq 2$ vertices.

Ou se cache Minou? Where is Minou Hiding?
http://pagesperso-orange.fr/jeux.lulu /html/dicache/cacheM1.htm
is an Internet game intended for young children. Minou the cat is randomly hidden on one cell of a 9×5 grid. The child clicks any cell to see if Minou is there. If so, she wins the game. If not, that cell is labelled with its distance to Minou. The game continues until the child finds Minou. For a child guessing semi-randomly it could take many guesses, but the mathematician will quickly discover that two well-chosen guesses uniquely locate Minou.

Assume throughout that $G = (V, E)$ is a simple connected graph with $n \geq 2$ vertices. A **resolving set** is a set $S = \{v_1, v_2, ..., v_m\} \subseteq V$ such that for all $u, v \in V$, if $d(v_i, u) = d(v_i, v)$ for $i = 1, 2, ..., m$, then $u = v$. The **metric dimension** $dim(G)$ is the minimum number of vertices in a resolving set. A resolving set with $dim(G)$ elements is a **metric basis**. Many people have investigated these concepts, starting with Slater [1975] and Harary & Melter [1976].

Theorem 1 [Slater 1975; Harary, Melter 1976]
Let G be a graph with $n \geq 2$ vertices.
(a) $dim(G) = 1$ if and only if $G = P_n$ is a path.
(b) For cycles, $dim(C_n) = 2$.
(c) For complete graphs, $dim(K_n) = n - 1$.
(d) For complete bipartite graphs,
$dim(K_{r,s}) = r + s - 2$ for $n = r + s \geq 3$.
(e) For grids, $dim(P_r \times P_s) = 2$ for $r, s \geq 2$.

Theorem 2 [Shanmukha, Sooryanarayana, Harinath 2002] *Let W_m be a wheel with $n = m+1$ vertices, $m \geq 3$. Then $dim(W_m) = \lfloor \frac{2m+4}{5} \rfloor$.*

Let T be a tree which is not itself a path. A **leg** at a vertex v is a component of $T-v$ which is a path, and ℓ_v is the number of legs at v. An **exterior major vertex** is a vertex v such that $deg(v) \geq 3$ and $\ell_v > 0$.

Theorem 3 [S,HM] *Let T be a tree which is not a path and let $x_1, x_2, ..., x_m$ be its exterior major vertices. Then $dim(T) = \sum(\ell_i - 1)$, and any set consisting of the leaves at the end of all but one of the legs of each exterior major vertex of T is a metric basis for T.*

Consider the Minou game on a graph G where Minou is hidden at a vertex $M \in V(G)$. The player then has to locate M by choosing a **probe** v_1 from V and receiving the distance $d(v_1, M)$. The player then chooses a second probe from V, and this process continues until the player can uniquely determine the location of M. The player's objective is to locate M with a minimum number of probes.

The **sequential location number of** G, $SL(G)$, is the number of probes required in the worst case by an optimal.

Theorem 4 *$SL(G) \leq dim(G)$ for all G.*
If $dim(G) \leq 2$, then $SL(G) = dim(G)$.

Corollary 5 *$SL(G) = 1$ iff G is a path.*

Corollary 6 *$SL(C_n) = dim(C_n) = 2$.*

Corollary 7 *$SL(P_r \times P_s) = dim(P_r \times P_s) = 2$ for all $r, s \geq 2$.*

Theorem 8 *$SL(K_n) = dim(K_n) = n - 1$.*

Theorem 9 *Let* $1 \leq r \leq s$*, with* $s > 1$*.*
Then $SL(K_{r,s}) = max\{r, s-1\}$*.*

Corollary 10 $SL(K_{1,s}) = dim(K_{1,s}) = s-1$
for $s > 1$*, and* $SL(K_{r,s}) < dim(K_{r,s}) = r+s-2$
for $r = 2, s > r$ *and* $2 < r \leq s$*.*

Theorem 11 *For a wheel* W_m *with* $n = m+1$*,*
$SL(W_m) = \lfloor \frac{m+1}{3} \rfloor$ *for* $m \geq 5$*,*
$SL(W_3) = 3$*, and* $SL(W_4) = 2$*.*

Corollary 12 $SL(W_m) = dim(W_m)$
for $m = 3, 4, 5, 7, 8, 10, 11, 15$ *and* 16*;*
otherwise $SL(W_m) < dim(W_m)$*.*

Theorem 13 *For any tree T with $n \geq 2$, $SL(T) \geq \Delta(T) - 1$.*

Theorem 14 *For any tree T which is not a path, if there exists a path P in T such that all vertices of degree at least 3 lie on P, then $SL(T) = \Delta - 1$.*

Consider the game where the robber chooses a vertex v_R to hide on. The locator then has to locate the robber by choosing a **probe** vertex v_1 from V, and receiving the distance $d(v_1, v_R)$. The robber may then move to an adjacent vertex or stay put (but may not move to the probe vertex). The locator then chooses a second probe from V, and this process continues until the locator has minimized the number of possible locations for the robber, with a minimum number of probes. At that point the locator sends a cop to each possible location to catch the robber. The minimum number of cops required is the **cop number** $Cop(G)$.

From the cop and robber tradition we are interested in graphs with $Cop(G) = 1$, and from the metric dimension tradition we are interested the **the robber location number** $RL(G)$, the minimum number of probes needed to locate the robber when $Cop(G) = 1$. Thus $RL(G)$ represents the number of probes required in the worst case by an optimal strategy for the locator.

Theorem 15 $Cop(K_n) = n - 2$ *for all* $n \geq 1$.

Theorem 16 *If* $Cop(G) = 1$ *then* $RL(G) = 1$ *if and only if* G *is a path.*

Theorem 17 *$Cop(C_5) = 2$, and $Cop(C_n) = 1$ for all $n \neq 7$. Moreover $RL(C_n) = 3$ for $n = 3, 4, 6, 7, 8, 9, 11$ and $RL(C_n) = 2$ for all $n > 11$.*

Theorem 18 *$Cop(P_r \times P_s) = 2$ for all $r, s > 2$.*

Robber Location for Trees

Let T be a tree with n vertices. A vertex of degree at least 3 is a **major vertex** of T. We define the **frozen root property** for a strategy S on a tree T with root r as the property that if the robber is initially at the root, or if at any point the robber moves to the root, then strategy S will locate the robber at the root with the next probe.

Lemma 19 *Let T be a tree with root r and edge rs. Let T_r with root r and T_s with root s be the two components of $T - rs$. Let S_r and S_s be strategies to locate the robber on T_r and T_s respectively such that both have the frozen root property. Let S_0 be a strategy which either locates the robber on T, determines that the robber is on $T_r - r$, or determines that the robber is on $T_s - s$, and which has the frozen root property. Let S_T be the strategy which consists of applying S_0, then if the robber is on $T_r - r$ applying S_r, and if the robber is on $T_s - s$ applying S_s. Then strategy S_T locates the robber on T and has the frozen root property.*

Theorem 20 *For any tree, T, $Cop(T) = 1$.*

Proof. If T is a path, choose any vertex as root r. If T is not a path, choose a major vertex as r. For every vertex $v \neq r$ of T, let T_v be the subtree of T which is the component of $T - vw$ where vw is the first edge on the path from v to r in T.

We now prove by induction on n that for any tree T with root r there exists a strategy S_T which locates the robber on T and has the frozen root property.

Theorem 21 *For any tree T with $n \geq 3$ vertices, $\Delta - 1 \leq RL(T) < e(T)$.*

Robber Location for Caterpillars

Assume C is a caterpillar with n spine vertices $v_1, v_2, ..., v_n$. Let v_0 be a leaf adjacent to v_1 and v_{n+1} a leaf adjacent to v_n. Let $d_i = deg(v_i)$ for $i = 1, 2, ..., n$. Let Δ be the maximum degree.

Theorem 22 $\Delta - 1 \leq RL(C) \leq \Delta + \lfloor \frac{n-1}{2} \rfloor$.