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A Shapiro Lemma for Diagrams of Spaces with Applications to Equivariant Topology

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In this paper we introduce a notion of twisted coefficients M for a diagram of spaces \underline{X} , and define the cohomology groups $H^*(\underline{X}, M)$. We prove the invariance of this cohomology under weak homotopy equivalence of diagrams (Theorem 2.3, Corollary 2.6), as well as a Whitehead Theorem for diagrams of spaces (Corollary 3.8). We also study induction and restriction of diagrams of spaces along a change of indexing categories. We prove that the cohomology of the induced diagram is isomorphic to the cohomology of the original diagram with restricted coefficients (Corollary 3.4), and we relate the cohomology of a given diagram to that of its restriction via a spectral sequence (Corollary 3.7). These results are of the form of the Shapiro lemma in group cohomology (see e.g. [Bn, Ec, We]).

These results belong to the homotopy theory of diagrams of spaces, which has recently been studied by various authors; cf. for example [D, DZ, DK1, DK2, H]. However, our motivation for proving a Whitehead theorem and induction-restriction theorems comes from equivariant topology. A space X with an action by a (discrete) group G gives rise to the diagram $X^{(-)}$ of fixed-point spaces X^H , for all subgroups $H \subseteq G$. The Bredon cohomology of X , as defined in [Br], is isomorphic to the cohomology of the diagram $X^{(-)}$. The Whitehead Theorem for diagrams of spaces will be seen to imply an equivariant Whitehead Theorem, while induction and restriction along a group homomorphism or along a suitable subcategory of the orbit category explain, in a unified way, the relation of the cohomology of fixed-point spaces and of orbit spaces to Bredon cohomology. These results on equivariant topology are stated in a first, introductory section. Sections 2 and 3 are

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concerned with cohomology of diagrams, while Section 4 provides the proofs of the results stated in Section 1. We wish to emphasize that some of the results in Section 1 can also be proved in a more ad hoc way. However, our approach of using diagrams of spaces provides a uniform treatment, and opens further possibilities by considering other subcategories of the orbit category. In Section 5 we show, by way of illustration, that our results on induction and restriction also relate to splitting theorems for Bredon cohomology with coefficients in a Mackey functor. The proof of our main technical result, the Invariance Theorem 2.3, has been deferred until the last Section 6. There are two appendices, one providing background for the relation between the homotopy of spaces, of simplicial sets, and of small categories, the other on cohomology of small categories.

Our methods also apply to Bredon homology and homology of diagrams. The details for homology are completely analogous to those for cohomology, and we have refrained from spelling them out.

1. Some results for G -spaces

The techniques used in this paper are most naturally explained in the context of diagrams of spaces; i.e. (contravariant) functors from some small category into a category of spaces. Our motivation, however, comes from the study of spaces with a group action. Indeed, if a group G acts on a space X , then, up to (weak) G -homotopy equivalence, X can be recovered from a diagram of spaces defined over the orbit category $\mathcal{O}(G)$ (the category of all transitive G -sets and all G -maps between them); for details, see e.g. [E1] or §4 below. The purpose of this section is to outline some of the consequences of our results for the special case of G -spaces. Full proofs of these consequences will be given in §4 below.

For a G -space X , Bredon introduced the cohomology of X with coefficients in a functor $M : \mathcal{O}(G)^{\text{op}} \rightarrow \mathbf{Ab}$ with values in the category \mathbf{Ab} of abelian groups (briefly, M is said to be an $\mathcal{O}(G)$ -abelian group); see [B, Br, tD2]. Such a coefficient system M depends on the orbit category $\mathcal{O}(G)$ but not on X , and should be regarded as *constant*, from the point of view of G -spaces. There is also a more general notion of *local* or *twisted* coefficient system M on a G -space X , introduced in [MS] and explicitly described in §4 below. Roughly speaking, such a local coefficient system M is a contravariant abelian group-valued functor defined on a category built up from $\mathcal{O}(G)$ together with the fundamental groupoids of all the fixed-point spaces X^H . (This is unrelated to the “local” coefficient systems occurring in Bredon’s book.) Similar coefficients have been studied by Møller in [Mø]. For such a system M , we introduced in [MS] cohomology groups $H_G^*(X, M)$, which should be viewed as a twisted version of Bredon cohomology. (The definition of these groups is reviewed in §4 below.) Recall that, for G -spaces X and Y , a G -map $f : Y \rightarrow X$ is said to be a weak G -homotopy equivalence if, for each subgroup $H \subseteq G$, the map f induces an ordinary weak homotopy equivalence $Y^H \rightarrow X^H$ between fixed point sets. In case X and Y are (of the G -homotopy type of) G -CW-complexes,

such a weak G -homotopy equivalence is automatically a G -homotopy equivalence. As a consequence of our results for diagrams of spaces, we obtain the following theorem. (See [MS] or §4 for the definition of the fundamental groupoid $\Pi_G(X)$ of a G -space X .)

THEOREM 1.1 (Equivariant Whitehead Theorem). *A G -map $f : Y \rightarrow X$ is a weak G -homotopy equivalence iff it induces an equivalence of fundamental groupoids $\Pi_G(Y) \rightarrow \Pi_G(X)$, as well as an isomorphism*

$$H_G^*(X, M) \cong H_G^*(Y, f^*M) \text{ for every local system of coefficients } M \text{ on } X.$$

The “only if” - part of this theorem is proved in [MS]. The key technical result of this paper (Theorem 2.3) is a version of this “only if”-part for diagrams of categories or spaces; it will be applied in our general treatment of induction and restriction in §3 below. There we will consider general constructions on diagrams of spaces \underline{X} and coefficient systems M , and derive an isomorphism having the form of a “Shapiro lemma”,

$$H^*(\text{induced}(\underline{X}), M) \cong H^*(\underline{X}, \text{restricted}(M)), \tag{1}$$

as well as a spectral sequence of the form

$$E_2^{p,q} = H^p(\underline{X}, R^q(\text{induce})(M)) \Rightarrow H^{p+q}(\text{restricted}(\underline{X}), M), \tag{2}$$

where $R^q(\text{induce})$ is the q -th derived functor of the induction functor on coefficients.

To illustrate the meaning of (1) and (2), we will now give their “translations” in various special cases related to the context of spaces with group actions.

The simplest case is where the induction and restriction are along a homomorphism of groups $\varphi : G \rightarrow K$. For such a φ and a G -CW-complex X , the induced K -space will be shown to be K -homotopy equivalent to $K \times_G X$. Any twisted coefficient system M on the K -space $K \times_G X$ restricts in a natural way to a similar such system $\varphi^*(M)$ on the G -space X , essentially by pullback along the natural map $X \rightarrow K \times_G X$. (When M is constant, i.e., $M : \mathcal{O}(K)^{\text{op}} \rightarrow \mathbf{Ab}$, then φ^*M is just the composition of M with the functor $\mathcal{O}(G) \rightarrow \mathcal{O}(K)$ induced by φ .) In this case, the isomorphism (1) takes the following form:

PROPOSITION 1.2. *For any G -CW-complex X and any twisted system of coefficients M on $K \times_G X$, there is a natural isomorphism*

$$H_K^*(K \times_G X, M) \cong H_G^*(X, \varphi^*(M)),$$

For example, if H is a normal subgroup of G , then, for the quotient map $G \rightarrow G/H$, this proposition gives an isomorphism

$$H_{G/H}^*(X/H, M) \cong H_G^*(X, \varphi^*M). \tag{3}$$

When $H = G$ this isomorphism (3) reduces to an isomorphism

$$H^*(X/G, A) \cong H_G^*(X, \text{con}(A)), \tag{4}$$

where $\text{con}(A) : \mathcal{O}(G)^{\text{op}} \rightarrow \underline{\text{Ab}}$ is the constant functor with value the abelian group A . This isomorphism (4) is known, and can be seen as a consequence of the representability of Bredon cohomology with constant coefficients; see [B], [E1].

For a group homomorphism $\varphi : G \rightarrow K$, there is also an evident restriction functor from K -spaces to G -spaces, and an induction functor $\varphi_* : (\mathcal{O}(G)\text{-abelian groups}) \rightarrow (\mathcal{O}(K)\text{-abelian groups})$, defined as follows. For $M : \mathcal{O}(G)^{\text{op}} \rightarrow \underline{\text{Ab}}$ and any object K/L of $\mathcal{O}(K)$,

$$\begin{aligned} \varphi_*(M)(K/L) &= H_G^0(K/L, M) \\ &\cong \coprod_i M(G/H_i), \end{aligned}$$

where the cosets G/H_i range over a decomposition of the G -set K/L into orbits. This functor φ_* is exact, and the spectral sequence (2) collapses to an isomorphism in this case. More generally, for a K -space Y the functor φ_* can be lifted to twisted coefficients, and we will prove:

PROPOSITION 1.3. *Let $\varphi : G \rightarrow K$ be a group homomorphism, and let Y be a K -space. For any twisted system of coefficients M on Y as a G -space, there is a canonical isomorphism*

$$H_K^*(Y, \varphi_*M) \cong H_G^*(Y, M).$$

Besides induction and restriction along a group homomorphism, there are many other ways of inducing and restricting. For example, for the functor $1 \rightarrow \mathcal{O}(G)$ on the one-point category 1 with as value an orbit G/K , our general induction and restriction operations take the following form: the restriction of a G -space X is the fixed-point set X^K , and for any abelian group A the induced $\mathcal{O}(G)$ -abelian group $\text{ind}(A)$ is given by $\text{ind}(A)(G/H) = \text{Hom}(\mathbb{Z}[\text{Hom}_G(G/K, G/H)], A)$. In this case the spectral sequence (2) collapses to an isomorphism

$$H_G^*(X, \text{ind}(A)) \cong H^*(X^K, A). \tag{5}$$

This shows that the Bredon cohomology groups “contain” the cohomology groups of all the fixed-point sets in X . The same holds for twisted coefficients. Indeed, in §4 we will show that if one replaces the functor $1 \rightarrow \mathcal{O}(G)$ by a suitable inclusion of the fundamental groupoid of X^K into the “equivariant” fundamental groupoid $\Pi_G(X)$ (defined in §4 below), one obtains the following result:

PROPOSITION 1.4. *Let X be a G -space. For any subgroup $K \subseteq G$ and any twisted system of coefficients A on the fixed-point space X^K , there is a natural isomorphism*

$$H_{tw}^*(X^K, A) \cong H_G^*(X, \text{ind}(A)).$$

We will also give an explicit formula for this system $\text{ind}(A)$ in §4 below. (Cf. the proof of Proposition 1.4 there.)

Proposition 1.4 is a key ingredient of the proof of the “if” part of the equivariant Whitehead Theorem (Theorem 1.1). Indeed, from Proposition 1.4 it follows that if a G -map $f : Y \rightarrow X$ induces isomorphisms in (our) twisted Bredon cohomology, then for any subgroup $H \subseteq G$ the map $f^H : Y^H \rightarrow X^H$ between fixed-point sets induces isomorphisms in (ordinary) twisted cohomology. The “if”-part of Theorem 1.1 will then be seen to follow easily from the classical (non-equivariant) Whitehead theorem; cf. the proof of Corollary 3.8 below.

For a final example of induction/restriction in this section, we also write G for the category with one object and elements of G as morphisms, and consider the inclusion $G \hookrightarrow \mathcal{O}(G)$ sending the one object to the orbit $G/1$. A G -space X can be considered as a diagram of spaces indexed by the category G , and the cohomology of this diagram X with coefficients in a G -module A is simply $H_{tw}^*(EG \times_G X, A)$, with twisting arising from the projection $EG \times_G X \rightarrow BG$. We will show in Section 4 that, when the action by G on X is free, the isomorphism (1) takes the following form:

PROPOSITION 1.5. *For any free G -space X and any $\mathcal{O}(G)$ -abelian group M there is a natural isomorphism*

$$H_G^*(X, M) \cong H_{tw}^*(EG \times_G X, M(G/1)).$$

This shows that for free G -spaces, Bredon cohomology reduces to (twisted) Borel cohomology.

2. Twisted cohomology for diagrams of spaces

Let \mathbb{B} be a fixed small category. A diagram of spaces, indexed by \mathbb{B} , is a functor $\underline{X} : \mathbb{B}^{\text{op}} \rightarrow \text{Top}$, where Top is the category of topological spaces. The purpose of this section is to define the cohomology of such a diagram with “local” coefficients, and state the invariance of this cohomology under suitable weak homotopy equivalences between diagrams (Corollary 2.6 below). This Invariance Theorem, and analogous invariance theorems for simplicial sets and for categories, play a central role in this paper. However, the proofs of these results are somewhat technical, and will only be given in Section 6 below.

We will make repeated use of the fact that the categories Cat , Sset and Top , of small categories, simplicial sets and topological spaces, respectively, are all equivalent from a homotopical point of view. Indeed, one can pass freely between these categories, using functorial constructions as displayed in the following diagram:

$$\begin{array}{ccc}
 & \text{Cat} & \\
 f_{\Delta} \nearrow & & \nwarrow \Delta \\
 \text{Sset} & \xleftrightarrow{N} & \text{Top.} \\
 & \xleftarrow{S} &
 \end{array}
 \tag{1}$$

The functors in this diagram are all standard, and their precise definitions will be recalled in Appendix A at the end of this paper. What is important here is that they are mutually inverse up to weak homotopy equivalence (as proved in the appendix).

The construction of the category $\int_{\Delta} Z$ from any simplicial set Z (the upper left functor in (1)) is a special case of the so-called Grothendieck construction, at least if we view sets as “discrete” categories. This construction assembles a diagram of categories $F : \mathbb{B}^{\text{op}} \rightarrow \text{Cat}$ into one large category, denoted

$$\int_{\mathbb{B}} F.
 \tag{2}$$

The objects of this category $\int_{\mathbb{B}} F$ are pairs (B, x) , where B is an object of \mathbb{B} and x is an object of the category $F(B)$. An arrow $(B, x) \rightarrow (B', x')$ between two such objects of $\int_{\mathbb{B}} F$ is a pair (α, u) , where $\alpha : B \rightarrow B'$ is an arrow in \mathbb{B} while $u : x \rightarrow F(\alpha)(x')$ is an arrow in $F(B)$. Composition of such arrows is defined in the evident way.

We will now use this Grothendieck construction to define twisted cohomology of diagrams of spaces.

Recall first that for an arbitrary (small) category \mathbb{C} , and for any contravariant abelian group-valued functor M on \mathbb{C} ,

$$M : \mathbb{C}^{\text{op}} \rightarrow \underline{\text{Ab}},$$

one can define the cohomology groups $H^n(\mathbb{C}, M)$ for any integer $n \geq 0$ (see Appendix B). In general, these cohomology groups are invariant under a weak homotopy equivalence of categories $\mathbb{C}' \rightarrow \mathbb{C}$ only in case the functor M is morphism-inverting (cf. Appendix B (5)). Our main Invariance Theorem 2.3 states that they are also invariant under weaker conditions on M , in the special case where the map $\mathbb{C}' \rightarrow \mathbb{C}$ is obtained by “integrating” (as in the Grothendieck construction) a pointwise weak equivalence.

To express the conditions on the coefficients M , let $F : \mathbb{B}^{\text{op}} \rightarrow \text{Cat}$ be a diagram of categories as above. For any small category \mathbb{C} , the fundamental groupoid $\Pi(\mathbb{C})$ of \mathbb{C} is obtained by formally inverting all the arrows in \mathbb{C} . (This groupoid can also be constructed as the edge-path groupoid of the simplicial set $N\mathbb{C}$; cf. [GZ, pp 10, 39]. It comes equipped with a functor $\mathbb{C} \rightarrow \Pi(\mathbb{C})$. By applying this construction to each of the categories $F(B)$ in our diagram F , one obtains a diagram of groupoids $\Pi(F)$

and a natural transformation of diagrams $F \rightarrow \Pi(F)$. By integration (Grothendieck construction) one next obtains a functor $\int_{\mathbb{B}} F \rightarrow \int_{\mathbb{B}} \Pi(F)$. We will call $\int_{\mathbb{B}} \Pi(F)$ the *fundamental groupoid* of the diagram F , and denote it

$$\Pi_{\mathbb{B}} F. \tag{3}$$

It should be emphasized that this category $\Pi_{\mathbb{B}} F$ is not itself a groupoid, but an integrated diagram of groupoids, or (in Grothendieck’s language) a fibered groupoid over the base category \mathbb{B} (fibré en groupoïde).

We can now define local coefficients:

DEFINITION 2.1. A *local* (or *twisted*) system of coefficients on a diagram $F : \mathbb{B}^{\text{op}} \rightarrow \underline{\text{Cat}}$ of categories is a functor $M : (\int_{\mathbb{B}} F)^{\text{op}} \rightarrow \underline{\text{Ab}}$ which factors, up to natural isomorphism, through the fundamental groupoid of F ,

$$\begin{array}{ccc} (\int_{\mathbb{B}} F)^{\text{op}} & \xrightarrow{M} & \underline{\text{Ab}} \\ \downarrow & \cong & \nearrow \\ (\Pi_{\mathbb{B}} F)^{\text{op}} & = & (\int_{\mathbb{B}} \Pi F)^{\text{op}} \end{array} \tag{4}$$

The *cohomology* of F with respect to such a system of coefficients M is defined to be the cohomology of the category $\int_{\mathbb{B}} F$:

$$H^*(F, M) := H^*\left(\int_{\mathbb{B}} F, M\right). \tag{5}$$

DEFINITION 2.2. A natural transformation $\nu : G \rightarrow F$, between two diagrams of categories F and $G : \mathbb{B}^{\text{op}} \rightarrow \underline{\text{Cat}}$, is said to be a *weak equivalence* if, for any object $B \in \mathbb{B}$, the functor $\nu(B) : G(B) \rightarrow F(B)$ is a weak homotopy equivalence between categories (as defined in Appendix A).

A natural transformation $\nu : G \rightarrow F$ induces, for any local system M on F , an evident local system $\nu^*(F)$ on G .

THEOREM 2.3 (Invariance Theorem). *A weak equivalence $\nu : G \rightarrow F$ between diagrams of categories induces a natural isomorphism*

$$H^*(F, M) \xrightarrow{\cong} H^*(G, \nu^* M)$$

for any local system of coefficients M on F .

REMARK 2.4. From a diagram of categories $F : \mathbb{B}^{\text{op}} \rightarrow \underline{\text{Cat}}$, one obtains a diagram of spaces $B \circ F : \mathbb{B}^{\text{op}} \rightarrow \underline{\text{Top}}$, by pointwise applying the classifying space functor B of (2.1). It is well-known that the classifying space $B(\int_{\mathbb{B}} F)$ of the

Grothendieck construction is a model for the homotopy colimit of this diagram of spaces. (An equivalent statement for simplicial sets is proved in [T].) In particular, a weak equivalence ν as in the theorem induces a weak equivalence of categories $\int_{\mathbb{B}} G \rightarrow \int_{\mathbb{B}} F$. It follows from Appendix B (5) that ν induces isomorphisms in cohomology for any morphism-inverting functor $M : (\int_{\mathbb{B}} F)^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$. We emphasize that the isomorphism in Theorem 2.3 is much more general, since a local coefficient system M on the diagram F need not at all be morphism-inverting on the category $\int_{\mathbb{B}} F$. (For example, any functor $A : \mathbb{B}^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$ yields, by composition, a local system $(\int_{\mathbb{B}} F)^{\text{op}} \rightarrow \mathbb{B}^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$.) On the other hand, Theorem 2.3 need not hold for an arbitrary coefficient system $M : \int_{\mathbb{B}} F^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$. Therefore we reserve the notation $H^*(F, M)$ (as opposed to $H^*(\int_{\mathbb{B}} F, M)$) for *local* coefficients.

Using the functors in diagram (1), and the fact that they are mutually weakly homotopy inverse, Theorem 2.3 immediately gives similar invariance theorems for diagrams of simplicial sets and for diagrams of spaces. We now state these explicitly.

Let \underline{Z} be a diagram of simplicial sets, i.e., a functor $\underline{Z} : \mathbb{B}^{\text{op}} \rightarrow \underline{\mathbf{Sset}}$. Using the functor $\int_{\Delta} : \underline{\mathbf{Sset}} \rightarrow \underline{\mathbf{Cat}}$ of (1), one obtains a diagram of categories $\int_{\Delta} \underline{Z} : \mathbb{B}^{\text{op}} \rightarrow \underline{\mathbf{Cat}}$. We define the fundamental groupoid of \underline{Z} , in terms of the fundamental groupoid of a diagram of categories just considered, as

$$\Pi_{\mathbb{B}} \underline{Z} := \Pi_{\mathbb{B}}(\int_{\Delta} \underline{Z}).$$

A local system of coefficients on \underline{Z} is then (by definition) a local system of coefficients on the diagram of categories $\int_{\Delta} \underline{Z}$, and we will denote the associated cohomology groups by $H^*(\underline{Z}, M)$. (So by definition, $H^*(\underline{Z}, M) = H^*(\int_{\Delta} \underline{Z}, M) = H^*(\int_{\mathbb{B}} \int_{\Delta} \underline{Z}, M)$.)

A natural transformation $\varphi : \underline{Z} \rightarrow \underline{W}$ between two diagrams of simplicial sets is said to be a *weak equivalence* if, for any object $B \in \mathbb{B}$, the map $\varphi(B) : \underline{Z}(B) \rightarrow \underline{W}(B)$ is a weak equivalence of simplicial sets. As explained in Appendix A, this is equivalent to the condition that $\int_{\Delta} \varphi : \int_{\Delta} \underline{Z} \rightarrow \int_{\Delta} \underline{W}$ is a weak equivalence between diagrams of categories. Thus from Theorem 2.3 we obtain the following corollary.

COROLLARY 2.5. *A weak equivalence $\varphi : \underline{Z} \rightarrow \underline{W}$ between diagrams of simplicial sets induces an isomorphism*

$$H^*(\underline{W}, M) \rightarrow H^*(\underline{Z}, \varphi^* M)$$

for any local coefficient system M on \underline{W} .

In a similar fashion, one can use the functors in (1) to derive from Theorem 2.3 an invariance theorem for diagrams of spaces. Explicitly, from a diagram of spaces $\underline{X} : \mathbb{B}^{\text{op}} \rightarrow \underline{\mathbf{Top}}$, one obtains a diagram $\Delta \underline{X} : \mathbb{B}^{\text{op}} \rightarrow \underline{\mathbf{Cat}}$ by composing \underline{X} with

the functor $\Delta : \mathbf{Top} \rightarrow \mathbf{Cat}$ in (1). A local system of coefficients on \underline{X} is then, by definition, such a system on this diagram of categories $\Delta \underline{X}$, as defined in 2.1 (see also 2.7 below). For such a local system M , we define the cohomology groups $H^*(\underline{X}, M)$ as $H^*(\underline{X}, M) = H^*(\Delta \underline{X}, M)$. Now call a natural transformation $f : \underline{X} \rightarrow \underline{Y}$ between two diagrams of spaces a weak equivalence if for each object $B \in \mathbb{B}$, the map $f(B) : \underline{X}(B) \rightarrow \underline{Y}(B)$ is a weak homotopy equivalence of topological spaces in the usual sense. By Appendix A, this is equivalent to the condition that $\Delta f : \Delta \underline{X} \rightarrow \Delta \underline{Y}$ is a weak equivalence of diagrams of categories. Thus from 2.3 one obtains:

COROLLARY 2.6. *A weak equivalence $f : \underline{X} \rightarrow \underline{Y}$ between diagrams of spaces induces an isomorphism*

$$H^*(\underline{Y}, M) \cong H^*(\underline{X}, f^*M)$$

for any local coefficient system M on \underline{Y} .

REMARK 2.7. For a diagram of simplicial sets \underline{Z} , we defined local coefficients on \underline{Z} using the category $f \underline{Z}$. But note that, for any simplicial set S and its fundamental groupoid $\Pi(S)$, there is a natural equivalence of groupoids $\Pi(fS) \rightarrow \Pi(S)$. (One way to see this is to use the weak homotopy equivalence $\rho : N(fS) \rightarrow S$ of Appendix A). Consequently, for a diagram \underline{Z} as above, there is a natural equivalence of diagrams of groupoids $\Pi(f \Delta \underline{Z}) \rightarrow \Pi \underline{Z}$. Thus local coefficient on \underline{Z} can equivalently be described as abelian group valued contravariant functors on $\int_{\mathbb{B}} (\Pi \underline{Z})$, rather than on the (equivalent but larger) category $\int_{\mathbb{B}} \Pi(f \Delta \underline{Z})$.

A similar remark applies to a diagram of spaces \underline{X} , to the effect that local coefficients on \underline{X} are essentially abelian group-valued functors on $\int_{\mathbb{B}} \Pi \underline{X}$, where $\Pi \underline{X}$ is the diagram of groupoids on \mathbb{B} given by $\Pi \underline{X}(B) = \Pi(\underline{X}(B)) =$ the fundamental groupoid of $\underline{X}(B)$.

REMARK 2.8. Consider the case where the index category \mathbb{B} is the one-object category, so that a diagram \underline{X} of spaces on \mathbb{B} is just a single space. A local system on \underline{X} is then a twisted system of coefficients in the usual sense. The weak homotopy equivalence $B \Delta \underline{X} \rightarrow \underline{X}$ of Appendix A, together with the fact that $H^*(\mathbb{C}, A) = H_{tw}^*(B\mathbb{C}, A)$ for any category \mathbb{C} and any local system A on \mathbb{C} (Appendix B (4)), show that our notion of cohomology with local coefficients $H^*(\underline{X}, A)$ agrees with the usual one in the case where \mathbb{B} is the one-point category. Corollary 2.6 reduces in this special case to the familiar invariance of twisted cohomology under weak homotopy equivalence of spaces.

3. Induction and restriction for diagrams

In this section we will describe the general operations of induction and restriction for diagrams of categories and of spaces. The main results of this section, Theorems

3.2 and 3.5 (together with their corollaries 3.4, 3.7), explain how these operations relate to twisted cohomology. We will also derive a general Whitehead Theorem for diagrams of spaces (Corollary 3.8). The Invariance Theorem 2.3, together with the analogous statements 2.5 and 2.6, will be seen to play an essential role.

Let $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ be a functor between small categories. Recall that a system of abelian coefficients on \mathbb{C} , i.e., a functor $M : \mathbb{C}^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$, induces a similar system φ^*M on \mathbb{D} , simply by composition (so $\varphi^*(M)(D) = M(\varphi D)$, for any object $D \in \mathbb{D}$). In the other direction, from a functor $B : \mathbb{D}^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$ one can construct a functor $\varphi_*(N) : \mathbb{C}^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$, by defining, for each object $C \in \mathbb{C}$,

$$\varphi_*(N)(C) = \lim_{\leftarrow \varphi/C} \omega_C^*(N). \tag{1}$$

Here, for a fixed object C , φ/C is the ‘‘comma-category’’ with as objects the pairs $(D, \alpha : \varphi(D) \rightarrow C)$ and as arrows $(D, \alpha) \rightarrow (D', \alpha')$ those arrows $\beta : D \rightarrow D'$ in \mathbb{D} for which $\alpha' \circ \varphi(\beta) = \alpha$ in \mathbb{C} . Furthermore, $\omega_C : \varphi/C \rightarrow \mathbb{D}$ is the ‘‘forgetful’’ functor $(D, \alpha) \mapsto D$.

There is a similar dual comma-category C/φ , with as objects the pairs $(D, \alpha : C \rightarrow \varphi(D))$, and a similar forgetful functor which we denote again by $\omega_C : C/\varphi \rightarrow \mathbb{D}$. It is this latter comma category which we use to define similar operations for diagrams of categories and of spaces. Specifically, the functor $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ yields for a diagram of categories $F : \mathbb{C}^{\text{op}} \rightarrow \underline{\mathbf{Cat}}$, indexed by \mathbb{C} , an evident diagram $\varphi^*(F)$ indexed by \mathbb{D} , by composition with φ (so $\varphi^*(F)(D) = F(\varphi D)$). In the other direction, we define, from a diagram $G : \mathbb{D}^{\text{op}} \rightarrow \underline{\mathbf{Cat}}$, a new diagram $\varphi_!(G) : \mathbb{C}^{\text{op}} \rightarrow \underline{\mathbf{Cat}}$, by setting, for each object $C \in \mathbb{C}$,

$$\varphi_!(G)(C) = \int_{C/\varphi} \omega_C^*(G). \tag{2}$$

Recall here that the integral sign refers to the Grothendieck construction, described in Section 2. For a morphism $\alpha : C' \rightarrow C$ in \mathbb{C} , there is an evident functor $\alpha^* : C/\varphi \rightarrow C'/\varphi$ defined by composition, for which $\omega_{C'} \circ \alpha^* = \omega_C$. Thus such an α induces a functor $\varphi_!(G)(\alpha) : \varphi_!(G)(C) \rightarrow \varphi_!(G)(C')$, showing that $\varphi_!(G)$ is indeed a contravariant functor on \mathbb{C} .

For diagrams of spaces we define similar functors φ^* and $\varphi_!$, except that we use homotopy colimits instead of the Grothendieck construction. Thus, for a diagram of spaces $\underline{X} : \mathbb{C}^{\text{op}} \rightarrow \underline{\mathbf{Top}}$, we denote by $\varphi^*(\underline{X})$ the diagram $\mathbb{D}^{\text{op}} \rightarrow \underline{\mathbf{Top}}$ obtained by composition with φ . And for a diagram of spaces $\underline{Y} : \mathbb{D}^{\text{op}} \rightarrow \underline{\mathbf{Top}}$, we construct a diagram $\varphi_!(\underline{Y}) : \mathbb{C}^{\text{op}} \rightarrow \underline{\mathbf{Top}}$, by defining

$$\varphi_!(\underline{Y})(C) = \text{hocolim}_{C/\varphi} \omega_C^*(\underline{Y}). \tag{3}$$

REMARK 3.1. These operations φ^* and $\varphi_!$ respect the passage between categories and spaces in diagram (2.1). More explicitly, for any diagrams of

categories $F : \mathbb{C}^{\text{op}} \rightarrow \underline{\text{Cat}}$ and $G : \mathbb{D}^{\text{op}} \rightarrow \underline{\text{Cat}}$, and for any diagrams of spaces $\underline{X} : \mathbb{C}^{\text{op}} \rightarrow \underline{\text{Top}}$ and $\underline{Y} : \mathbb{D}^{\text{op}} \rightarrow \underline{\text{Top}}$, one has the identities

$$\varphi^* \Delta(\underline{X}) = \Delta \varphi^*(\underline{X}), \quad B\varphi^*(F) = \varphi^*B(F), \tag{4}$$

and natural weak homotopy equivalences of diagrams

$$\varphi_!(BG) \simeq B\varphi_!(G), \quad \varphi_!(\Delta \underline{Y}) \simeq \Delta \varphi_!(\underline{Y}). \tag{5}$$

For details, we refer to the end of Appendix A.

Consider again a functor $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ and a diagram of categories G on \mathbb{D} . Then φ lifts to a functor

$$\tilde{\varphi} : \int_{\mathbb{D}} G \rightarrow \int_{\mathbb{C}} \varphi_!(G),$$

defined on an object $(D, x) \in \int_{\mathbb{D}} G$ by $\tilde{\varphi}(D, x) = (\varphi(D), (D, id, x))$; the definition of $\tilde{\varphi}$ is extended to arrows in the evident way. Thus if $M : (\int_{\mathbb{C}} \varphi_!(G))^{\text{op}} \rightarrow \underline{\text{Ab}}$ is any system of abelian coefficients on the category $\int_{\mathbb{C}} \varphi_!(G)$, we obtain by restriction along $\tilde{\varphi}$ a system $\tilde{\varphi}^*(M)$ on $\int_{\mathbb{D}} G$. However, it is easy to see that if M is a local system on the diagram $\varphi_!(G)$, then $\tilde{\varphi}^*(M)$ is a local system on G .

THEOREM 3.2. *Let $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ be a functor between small categories. For any diagram of categories G on \mathbb{D} and any local system of coefficients M on $\varphi_!(G)$, there is a natural isomorphism*

$$H^*(G, \tilde{\varphi}^*M) \cong H^*(\varphi_!(G), M).$$

REMARK 3.3. Recall that we use the notation $H^*(\varphi_!G, M)$ only for local systems of coefficients. However, Theorem 3.2 actually holds for any coefficient system on $\int_{\mathbb{C}} \varphi_!(G)$, and we will prove the theorem in this generality. Of course, the Invariance Theorem 2.3 allows us to deduce a similar result for spaces (Corollary 3.4 below) only for local coefficients.

Proof. Consider the functor $\epsilon : \int_{\mathbb{C}} \varphi_!(G) \rightarrow \int_{\mathbb{D}} G$, defined on objects by

$$\epsilon(C, (D, \alpha, x)) = (D, x).$$

For any coefficients $M : (\int_{\mathbb{C}} \varphi_!(G))^{\text{op}} \rightarrow \underline{\text{Ab}}$, there is a Grothendieck spectral sequence of the form

$$E_2^{p,q} = H^p(\int_{\mathbb{D}} G, R^q \epsilon_*(M)) \Rightarrow H^{p+q}(\int_{\mathbb{C}} \varphi_!(G), M);$$

see Appendix B. By (11) of Appendix B, the coefficient system occurring in the E_2 -term is the functor on $\int_{\mathbb{D}} G$ with as value at an object $(D, x) \in \int_{\mathbb{D}} G$ the abelian group $H^q(\epsilon/(D, x), \omega_{(D,x)}^*(M))$. (Here, as before, $\omega_{(D,x)}$ denotes the forgetful

functor $\epsilon/(D, x) \rightarrow \int_{\mathbb{C}} \varphi!(G)$.) But notice that the category $\epsilon/(D, x)$ has a terminal object $t_{(D,x)}$, given by the object $(\varphi(D), (D, id, x)) = \tilde{\varphi}(D, x)$ of $\int_{\mathbb{C}} \varphi!(G)$ and the identity map $\epsilon\tilde{\varphi}(D, x) \rightarrow (D, x)$. Hence

$$H^q(\epsilon/(D, x), \omega_{(D,x)}^*(M)) = \begin{cases} 0 & \text{if } q > 0 \\ M(\tilde{\varphi}(D, x)) & \text{if } q = 0, \end{cases}$$

and the spectral sequence collapses to the desired isomorphism.

COROLLARY 3.4. *Let $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ be a functor, as above. For any diagram of spaces \underline{Y} on \mathbb{D} and any local system of coefficients M on $\varphi!(\underline{Y})$, there is a local coefficient system $\tilde{\varphi}^*(M)$ on \underline{Y} and a natural isomorphism*

$$H^*(\underline{Y}, \tilde{\varphi}^*M) \cong H^*(\varphi!(\underline{Y}), M).$$

Proof. The twisted cohomology of \underline{Y} is defined as the cohomology of the diagram of categories $\Delta\underline{Y}$. The local system M on $\varphi!(\underline{Y})$ gives a similar system on $\varphi!(\Delta\underline{Y})$, by the weak equivalence $\Delta\varphi!(\underline{Y}) \simeq \varphi!(\Delta\underline{Y})$ of (5) above. From this we obtain a local system $\tilde{\varphi}^*(M)$ on $\Delta(\underline{Y})$, and an isomorphism $H^*(\varphi!(\Delta\underline{Y}), M) \cong H^*(\underline{Y}, \tilde{\varphi}^*M)$ as in Theorem 3.2. Finally, the Invariance Theorem 2.3 and the weak equivalence $\Delta\varphi!(\underline{Y}) \simeq \varphi!(\Delta\underline{Y})$ give an isomorphism $H^*(\varphi!(\Delta\underline{Y}), M) \cong H^*(\varphi!(\underline{Y}), M)$.

Next, consider a functor $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ as before, and a diagram of categories $F : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Cat}$. A coefficient system $M : (\int_{\mathbb{D}} \varphi^*(F))^{\text{op}} \rightarrow \mathbf{Ab}$ yields, for each object (C, x) of $\int_{\mathbb{C}} F$, a coefficient system $M_x : (\varphi/C)^{\text{op}} \rightarrow \mathbf{Ab}$, defined for each object $(D, \alpha : \varphi(D) \rightarrow C)$ of φ/C by

$$M_x(D, \alpha) = M(D, F(\alpha)(x)). \tag{6}$$

Hence there are cohomology groups $H^q(\varphi/C, M_x)$ (as in Appendix B), and the assignment $(C, x) \mapsto H^q(\varphi/C, M_x)$ defines a coefficient system on $\int_{\mathbb{C}} F$, denoted

$$\mathcal{H}^q(\varphi/(-), M) : (\int_{\mathbb{C}} F)^{\text{op}} \rightarrow \mathbf{Ab}. \tag{7}$$

With this definition, we have the following result:

THEOREM 3.5. *Let $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ be a functor, as before. For any diagram of categories F on \mathbb{C} and any local system of coefficients M on φ^*F , the coefficient system $\mathcal{H}^q(\varphi/(-), M)$ on F is local for each $q \geq 0$. Furthermore, there is a natural spectral sequence*

$$E_2^{p,q} = H^p(F, \mathcal{H}^q(\varphi/(-), M)) \Rightarrow H^{p+q}(\varphi^*F, M).$$

Proof. The functor $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ has an evident lifting

$$\bar{\varphi} : \int_{\mathbb{D}} \varphi^* F \rightarrow \int_{\mathbb{C}} F,$$

and we consider the Grothendieck spectral sequence (Appendix B) for this functor $\bar{\varphi}$. This spectral sequence converges to $H^*(\varphi^*(F), M)$, and the E_2 -term is $H^p(\int_{\mathbb{C}} F, R^q \bar{\varphi}_*(M))$. Moreover, by (11) of Appendix B, there is a natural isomorphism

$$R^q \bar{\varphi}_*(M)(C, x) \cong H^q(\bar{\varphi}/(C, x), \omega_{(C,x)}^* M),$$

for each object (C, x) of $\int_{\mathbb{C}} F$. Unraveling of the definitions reveals that

$$\bar{\varphi}/(C, x) = \int_{\varphi/C} F_x,$$

where $F_x : (\varphi/C)^{\text{op}} \rightarrow \mathbf{Cat}$ is the functor defined on objects by $F_x(D, \alpha) = F(\varphi(D))/F(\alpha)(x)$. In particular, each category $F_x(D, \alpha)$ has a terminal object. Thus the unique transformation $F_x \rightarrow 1$, from F_x to the constant diagram on φ/C with value the trivial one-object category 1 , is a weak equivalence of diagrams.

Next, the restriction of the local coefficient system M on $\int_{\mathbb{D}} \varphi^* F$, along the forgetful functor $\int_{\varphi/C} F_x = \bar{\varphi}/(C, x) \rightarrow \int_{\mathbb{D}} \varphi^* F$, gives a local coefficient system on the diagram F_x . It is easy to see that this system is isomorphic to $p_x^*(M_x)$, where M_x is the system on φ/C defined in (6) above, and $p_x : \int_{\varphi/C} F_x \rightarrow \varphi/C$ is the projection. Thus, by the invariance Theorem 2.3, the weak equivalence $F_x \rightarrow \underline{1}$ induces a natural isomorphism $H^*(\bar{\varphi}/(C, x), \omega_{(C,x)}^* M) \cong H^*(\varphi/C, M_x)$. This shows that

$$R^q \bar{\varphi}_*(M) \cong \mathcal{H}^q(\varphi/(-), M). \tag{8}$$

To conclude the proof, it therefore remains to be shown that the system $\mathcal{H}^q(\varphi/(-), M)$ is a local system on F . This will follow once we can show that, for any object $C \in \mathbb{C}$ and any arrow $u : x \rightarrow y$ in $F(C)$, the arrow $(id, u) : (C, x) \rightarrow (C, y)$ yields an isomorphism $H^q(\varphi/C, M_y) \rightarrow H^q(\varphi/C, M_x)$. But, by hypothesis, M is local on $\varphi^* F$ and so the map $M_y \rightarrow M_x$ is an isomorphism of coefficients on φ/C . Finally, the naturality of the spectral sequence (in M and, more interestingly, in F) follows from the naturality of the Grothendieck spectral sequence (Appendix B).

REMARK 3.6. The coefficient system $\mathcal{H}^q(\varphi/(-), M)$ of Theorem 3.5 has a simpler description if we assume that M is ‘‘constant’’. By this we mean that $M : (\int_{\mathbb{D}} \varphi^*(F))^{\text{op}} \rightarrow \mathbf{Ab}$ factors up to isomorphism through the projection ρ in the diagram below, say as $M \cong \rho^*(N)$.

$$\begin{array}{ccc} \int_{\mathbb{D}} \varphi^* F & \xrightarrow{\bar{\varphi}} & \int_{\mathbb{C}} F \\ \rho \downarrow & & \downarrow \pi \\ \mathbb{D} & \xrightarrow{\varphi} & \mathbb{C}. \end{array}$$

In this case it is clear from the definition (7) of \mathcal{H}^q that

$$\mathcal{H}^q(\varphi/(-), M) \cong \pi^*(R^q\varphi_*(N)) \quad (q \geq 0). \tag{9}$$

If, in addition, the induction functor φ_* in (1) is exact, the spectral sequence collapses to an isomorphism

$$H^p(F, \pi^*\varphi_*N) \cong H^p(\varphi^*F, \rho^*N) \quad (p \geq 0). \tag{10}$$

More generally, the functor $\bar{\varphi}$ induces a functor $\hat{\varphi}$, as in

$$\begin{array}{ccc} \int_{\mathbb{D}} \varphi^*(F) & \xrightarrow{\bar{\varphi}} & \int_{\mathbb{C}} F \\ \rho' \downarrow & & \downarrow \pi' \\ \Pi_{\mathbb{D}}(\varphi^*F) & \xrightarrow{\hat{\varphi}} & \Pi_{\mathbb{C}}(F). \end{array}$$

Now if we view a local coefficient system on $\varphi^*(F)$ as a functor

$M : \Pi_{\mathbb{D}}(\varphi^*F)^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$, then, as in the proof of the theorem, one can derive an isomorphism of the form (8) for the functor $\hat{\varphi}$:

$$R^q\hat{\varphi}_*(M) \cong \mathcal{H}^q(\varphi/(-), M).$$

Here $\mathcal{H}^q(\varphi/(-), M) : (\int_{\mathbb{C}} \Pi(F))^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$ is defined in the same way as \mathcal{H}^q in (7), and spelling out this definition yields that

$$R^q\hat{\varphi}_*((\rho')^*(M)) \cong (\pi')^*R^q\hat{\varphi}_*(M).$$

We will use this isomorphism in the next section for the proof of Proposition 1.3.

One can now use the invariance theorem to derive a “topological” version of Theorem 3.5, in the same way that Corollary 3.4 was derived from Theorem 3.2.

COROLLARY 3.7. *Let $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ be a functor, as before. For any diagram of spaces \underline{X} on \mathbb{C} , and any local system of coefficients M on $\varphi^*(\underline{X})$, there is a natural spectral sequence*

$$E_2^{p,q} = H^p(\underline{X}, \mathcal{H}^q(\varphi/(-), M)) \Rightarrow H^{p+q}(\varphi^*\underline{X}, M).$$

The coefficient system $\mathcal{H}^q(\varphi/(-), M)$ occurring in this corollary is defined in exactly the same way as that for diagrams of categories (cf. (7)).

Recall that, for a diagram \underline{X} of spaces on a category \mathbb{C} , the “fundamental groupoid” $\Pi_{\mathbb{C}}(\underline{X})$ is defined as the category $\int_{\mathbb{C}} \Pi\Delta(\underline{X})$. As noted in Remark 2.7, this is the same (up to a canonical equivalence of categories) as the total category $\int_{\mathbb{C}} \Pi(\underline{X})$ obtained from the diagram $\Pi(\underline{X})$ of all fundamental groupoids $\Pi(\underline{X}(C))$. We can now deduce the following Whitehead Theorem for diagrams of spaces.

COROLLARY 3.8. *Let \underline{X} and \underline{Y} be diagrams of spaces on a category \mathbb{C} . A map $f : \underline{Y} \rightarrow \underline{X}$ is a weak homotopy equivalence of diagrams iff f induces an equivalence of fundamental groupoids $\Pi_{\mathbb{C}}(\underline{Y}) \xrightarrow{\cong} \Pi_{\mathbb{C}}(\underline{X})$ as well as an isomorphism*

$$f^* : H^*(\underline{X}, M) \xrightarrow{\cong} H^*(\underline{Y}, f^*M)$$

for any local coefficient system on \underline{X} .

Proof. (\Rightarrow) This is an immediate consequence of the Invariance Theorem (cf. Corollary 2.6). (\Leftarrow) Suppose that $f : \underline{Y} \rightarrow \underline{X}$ gives an equivalence $\Pi_{\mathbb{C}}(f) : \Pi_{\mathbb{C}}(\underline{Y}) \rightarrow \Pi_{\mathbb{C}}(\underline{X})$ as well as isomorphisms in twisted cohomology. Since $\Pi_{\mathbb{C}}(f)$ is an equivalence, it follows for each object $C \in \mathbb{C}$ that $f(C) : \underline{Y}(C) \rightarrow \underline{X}(C)$ gives an equivalence of fundamental groupoids. We claim that $f(C)$ also induces isomorphisms in (ordinary) twisted cohomology of the spaces $\underline{X}(C)$ and $\underline{Y}(C)$. This would complete the proof since the “non-parametrized” classical Whitehead theorem then shows that $f(C)$ is a weak homotopy equivalence. To verify the claim, let A be a local system of coefficients on $\underline{X}(C)$ (thus A is a functor $(\Delta \underline{X}(C))^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$ which factors through the last vertex map $\Delta \underline{X}(C) \rightarrow \Pi \underline{X}(C)$). Consider the functor $\varphi : 1 \rightarrow \mathbb{C}$ which sends the unique object to $C \in \mathbb{C}$. Then $\varphi^*(\underline{X}) = \underline{X}(C)$, and by Corollary 3.7 we have a spectral sequence

$$E_2^{p,q}(\underline{X}) = H^p(\underline{X}, \mathcal{H}^q(\varphi/(-), A)) \Rightarrow H^{p+q}(\underline{X}(C), A), \tag{11}$$

which is natural in \underline{X} . Since $\mathcal{H}^q(\varphi/(-), A)$ is local, the map $f : \underline{Y} \rightarrow \underline{X}$ gives an isomorphism $E_2^{p,q}(\underline{X}) \rightarrow E_2^{p,q}(\underline{Y})$, by the assumption. It follows that the homomorphism $f_C^* : H^{p+q}(\underline{X}(C), A) \rightarrow H^*(\underline{Y}(C), f(C)^*(A))$ between the abutments is also an isomorphism, as required.

REMARK 3.9. Actually the spectral sequence (11) in the proof of Corollary 3.8 collapses. To see this, note that, by definition (7), the value of $\mathcal{H}^q(\varphi/(-), A)$ at an object $(D, \sigma : \Delta^n \rightarrow \underline{X}(D))$ of the category $\int_{\mathbb{C}} \Delta(\underline{X})$ is $\mathcal{H}^q(\varphi/D, A_\sigma)$. But φ/D is just the discrete category of arrows $\gamma : C \rightarrow D$, and A_σ sends such an arrow to $A(\underline{X}(\gamma) \circ \sigma)$. Thus $H^0(\varphi/D, A_\sigma) \cong \Pi_{\gamma:C \rightarrow D} A(\underline{X}(\gamma) \circ \sigma)$, and consequently $\mathcal{H}^0(\varphi/(-), ?)$ is an exact functor. This shows that its right derived functor $\mathcal{H}^q(\varphi/(-), ?)$ vanishes for $q > 0$. Hence we have an isomorphism

$$H^p(\underline{X}, \mathcal{H}^0(\varphi/(-), A)) \cong H^p(\underline{X}(C), A). \tag{12}$$

4. Applications to G-spaces

In this section we will apply our results concerning diagrams of spaces to spaces equipped with an action by a group G . Write $\mathcal{O}(G)$ for the category whose objects are the left G -sets G/H , for any subgroup $H \subseteq G$, and whose morphisms are the G -equivariant maps. For a G -space X one obtains an $\mathcal{O}(G)$ -diagram of spaces $\Phi(X)$ defined by

$$\Phi(X)(G/H) = X^H = \text{Map}_G(G/H, X).$$

We recall from [El, S] that there is also a functor C which associates to each diagram $\underline{Y} : \mathcal{O}(G)^{\text{op}} \rightarrow \underline{\mathbf{Top}}$ a G -space $C(\underline{Y})$, and that these functors are mutually inverse, up to weak homotopy equivalence. More precisely, for each G -space X and for

each $\mathcal{O}(G)$ -indexed diagram of spaces \underline{Y} , there are weak homotopy equivalences $C\Phi(X) \rightarrow X$ and $\Phi C(\underline{Y}) \rightarrow \underline{Y}$. Recall that, by definition, a map of G -spaces $X' \rightarrow X$ is a weak homotopy equivalence if for each subgroup $H \subseteq G$ the induced map $(X')^H \rightarrow X^H$ is an ordinary weak homotopy equivalence. Furthermore, recall from [MS] that, for a G -space X , its “fundamental groupoid” $\Pi_G(X)$ is the category $\int_{\mathcal{O}(G)} \Pi\Phi(X)$; this category is not identical to, but is equivalent to the category $\Pi_{\mathcal{O}(G)}\Phi(X) = \int_{\mathcal{O}(G)} \Pi\Delta\Phi(X)$ defined in Section 2. In [MS] we defined a local coefficient system on a G -space X to be a local system on the diagram $\Phi(X)$. For such a system M , we defined the cohomology groups $H_G^*(X, M)$ as $H^*(\Phi(X), M)$, and we showed that for “constant” M (i.e. $M : \mathcal{O}(G)^{\text{op}} \rightarrow \mathbf{Ab}$), these cohomology groups coincide with the Bredon cohomology groups (as defined in [Br]).

With these definitions, the preceding results on diagrams of spaces “translate” to the results on twisted Bredon cohomology for G -spaces which we stated in Section 1, as we will now explain in some detail. We begin with the equivariant Whitehead Theorem:

Proof of Theorem 1.1. By definition, a G -map $f : Y \rightarrow X$ is a weak G -homotopy equivalence if the induced map $\Phi(Y) \rightarrow \Phi(X)$ of diagram is a weak equivalence. So by the definition of $H_G^*(X, M)$ just quoted, Theorem 1.1 is a special case of Corollary 3.8.

Our next objective is to relate our induction along a group homomorphism to the usual induction in equivariant topology.

Let X be a G -space and let $\varphi : G \rightarrow K$ be a homomorphism of groups. The standard induction along φ is given by the K -space $K \times_G X$. Now $\varphi : G \rightarrow K$ induces a functor (again called)

$$\varphi : \mathcal{O}(G) \rightarrow \mathcal{O}(K),$$

defined on objects by $\varphi(G/H) = K/\varphi(H) = K \times_G (G/H)$. Thus there is an induction $\varphi_!$ from diagrams of spaces on $\mathcal{O}(G)$ to such diagrams on $\mathcal{O}(K)$. We claim that the fixed-point functor Φ respects these two kinds of induction in the following sense:

LEMMA 4.1. *Let $\varphi : G \rightarrow K$ be a homomorphism of groups. For any G -space X with the G -homotopy type of a G -CW-complex, there is a natural weak homotopy equivalence*

$$\varphi_!\Phi(X) \simeq \Phi(K \times_G X)$$

of diagrams of spaces on $\mathcal{O}(K)$.

(Note that the two operations Φ in 4.1 are different: one is for the group G and the other for K .)

Proof. Write $\Delta_G(X)$ for the category $\int_{\mathcal{O}(G)} \Delta\Phi(X)$, and let $P_X : \Delta_G(X) \rightarrow \mathcal{O}(G) \subseteq (G\text{-spaces})$ be the evident projection functor. We refer to [MS] for the fact that there is a natural $K \times G$ -homotopy equivalence $\text{hocolim}_{\Delta_G(X)}(K \times P_X) = K \times \text{hocolim}_{\Delta_G(X)} P_X \rightarrow K \times X$. Under the assumption that X has the G -homotopy type of a G -CW-complex, this is an honest $K \times G$ -homotopy equivalence. So by factoring out the G -action, one obtains a K -homotopy equivalence

$$\text{hocolim}_{\Delta_G(X)}(K \times_G P_X) \simeq K \times_G X. \tag{1}$$

Next, we consider the fixed-point set of the space on the left-hand side of (1), for any subgroup $L \subseteq K$. We claim that there is a (weak) homotopy equivalence

$$\text{hocolim}_{\Delta_G(X)}(K \times_G P_X)^L \simeq B\varphi_!(\Delta\Phi(X))(K/L). \tag{2}$$

Indeed, since the G -spaces in the image of P_X are all discrete, the homotopy colimit on the left of (1) can be computed as the classifying space of the category

$$\int_{\Delta_G(X)} (K \times_G P_X)^L,$$

see [T]. For any object $(G/H, \alpha : \Delta^n \rightarrow X^H)$ of $\Delta_G(X)$, one has $(K \times_G P_X)(G/H, \alpha) = K \times_G G/H \cong K/\varphi(H)$. Moreover, for the L -fixed points, one has $(K/\varphi(H))^L = \text{Map}_K(K/L, K/\varphi(H)) = \text{Map}_K(K/L, \varphi(G/H))$. Thus

$$\begin{aligned} \left(\int_{\Delta_G(X)} K \times_G P_X\right)^L &= \int_{\Delta_G(X)} (K \times_G P_X)^L \\ &= \int_{\Delta_G(X)} \text{Map}_K(K/L, \varphi \circ P_X(-)). \end{aligned}$$

But this category is literally the same as the category

$$\int_{(K/L)/\varphi} \omega_{(K/L)}^*(\Delta\Phi(X)),$$

where $\omega_{K/L} : (K/L)/\varphi \rightarrow \mathcal{O}(G)$ is the forgetful functor, as before. By definition (3.2) of $\varphi_!$, this last category is exactly $\varphi_!(\Delta\Phi(X))(K/L)$. By taking classifying spaces, one thus obtains the desired homotopy equivalence (2). To conclude the proof, we observe that by (3.5), together with the fact that the functors B and Δ of diagram (2.1) are mutually weakly homotopy inverse (Appendix A), one has the following weak homotopy equivalences of diagrams of spaces:

$$B\varphi_!(\Delta\Phi(X)) \simeq B\Delta\varphi_!(\Phi(X)) \simeq \varphi_!(\Phi(X)).$$

By (1) and (2), finally, one obtains the desired weak equivalence $\varphi_!(\Phi(X)) \simeq \Phi(K \times_G X)$. This proves the lemma.

Proof of Proposition 1.2. As before, for a homomorphism of groups $\varphi : G \rightarrow K$ we also write φ for the associated functor $\mathcal{O}(G) \rightarrow \mathcal{O}(K)$. For a $G - CW$ -complex X and any local system of coefficients M on $K \times_G X$, Lemma 4.1 and Corollary 3.4 together give, for the induced system $\tilde{\varphi}^*(M)$ on X ,

$$\begin{aligned} H_K^*(K \times_G X, M) &\cong H^*(\Phi(K \times_G X), M) \cong H^*(\varphi_! \Phi(X), M) \\ &\cong H^*(\Phi(X), \tilde{\varphi}^* M) = H_G^*(X, \tilde{\varphi}^* M). \end{aligned}$$

Proof of Proposition 1.3. Let Y be a K -space and $\varphi : G \rightarrow K$ a group homomorphism. The restriction of $\Phi(Y)$ along $\varphi : \mathcal{O}(G) \rightarrow \mathcal{O}(K)$ is $\varphi^* \Phi(Y) = \Phi(Y)$, where, on the right-hand side, Y is considered as a G -space via $\varphi : G \rightarrow K$. Corollary 3.7 gives a spectral sequence

$$E_2^{p,q} = H_K^p(Y, \mathcal{H}^q(\varphi/(-), M)) \Rightarrow H_G^{p+q}(Y, M),$$

for any local system of coefficients M on Y (as G -space). By Remark 3.6, $\mathcal{H}^q(\varphi/(-), ?)$ is the right derived functor of the induction $\hat{\varphi}_*$ along the obvious functor $\hat{\varphi} : \Pi_G(Y) \rightarrow \Pi_K(Y)$. We claim that $\hat{\varphi}_*$ is exact. Indeed, by definition of $\hat{\varphi}_*$ (cf. (8) of Appendix B),

$$\hat{\varphi}_*(M)(K/L, y) = \text{Hom}(\mathbb{Z}[\Pi_K(Y)(\hat{\varphi}(-), (K/L, y))], M), \tag{3}$$

where the Hom is taken in the category $\underline{\text{Ab}}(\Pi_G(Y))$ of all functors $\Pi_G(Y)^{\text{op}} \rightarrow \underline{\text{Ab}}$, and $\mathbb{Z}[-]$ denotes the free abelian group. Let $K/L \cong \sum_i G/H_i$ be a decomposition of K/L as a G -set. Then $\Pi_K(Y)(\hat{\varphi}(-), (K/L, y)) = \sum_i \Pi_G(Y)(-, (G/H_i, y))$, and consequently

$$\mathbb{Z}[\Pi_K(Y)(\hat{\varphi}(-), (K/L, y))] = \sum_i \mathbb{Z}[\Pi_G(Y)(-, (G/H_i, y))]$$

This shows that $\mathbb{Z}[\Pi_K(Y)(\hat{\varphi}(-), (K/L, y))]$ is a projective object in $\underline{\text{Ab}}(\Pi_G(Y))$ and thus, by (3), $\hat{\varphi}_*$ is exact. This ends the proof of Proposition 1.3.

Proof of Proposition 1.4. The isomorphism of Proposition 1.4 is a special case of the isomorphism (12) in Remark 3.9, for the functor $\varphi : 1 \rightarrow \mathcal{O}(G)$ with value G/K . Note that, by the explicit description given there, the coefficient system $\text{ind}(A) : \Pi_G(X)^{\text{op}} \rightarrow \underline{\text{Ab}}$ occurring in Proposition 1.4 is given by

$$\text{ind}(A)(G/H, x) = \prod_f A(f^*(x)),$$

where the product is taken over all G -maps $f : G/K \rightarrow G/H$, while for $x \in X^H = \text{Map}_G(G/H, X)$, we write $f^*(x)$ for the induced point in X^K .

Proof of Proposition 1.5. Here we consider the inclusion functor $\varphi : \text{Aut}(G/1) \hookrightarrow \mathcal{O}(G)$. Notice that an $\text{Aut}(G/1)$ -diagram of spaces is the same as a space with a left action by G . Hence, for a left G -space X and any $M \in \underline{\text{Ab}}(\mathcal{O}(G))$, Corollary 3.4 gives an isomorphism

$$H^*(X, M(G/1)) \cong H^*(\varphi_!(X), M), \tag{4}$$

where the left-hand side is the cohomology of X as a G -diagram (i.e. $H^*(\int_G \Delta(X), M(G/1))$). Since $M(G/1)$ is morphism-inverting on $\int_G \Delta(X)$, we have, by (4) of Appendix B, that $H^*(\int_G \Delta(X), M(G/1)) \cong H^*_{tw}(B(\int_G(\Delta(X))), M(G/1))$, with a twisting coming from the map $\Pi B(\int_G \Delta(X)) \simeq \int_G \Pi(X) \rightarrow G$. Thomason’s weak homotopy equivalence τ (in Appendix A) gives

$$B(\int_G \Delta(X)) \xrightarrow{\tau} \text{hocolim}_G B\Delta X \simeq \text{hocolim}_G X \cong EG \times_G X.$$

This shows that the left-hand side of (4) equals $H^*_{tw}(EG \times_G X, M(G/1))$. To analyse the right-hand side of (4), notice that, if G acts freely on X , then

$$\varphi_!(X)(G/H) = \begin{cases} \emptyset & \text{if } H \neq 1 \\ EG \times X & \text{if } H = 1. \end{cases}$$

(One way to see that $\varphi_!(X)(G/1) = EG \times X$ is to compute the homotopy colimit involved as the classifying space of the topological category $\int_{(G/1)/\varphi} X$. This is the category with $G \times X$ as space of objects, and with exactly one arrow $(g, x) \rightarrow (h, y)$ whenever $g^{-1}y = h^{-1}x$. But this category is isomorphic to $\tilde{G} \times X$, where \tilde{G} has the elements of G as objects, and exactly one arrow between any two objects. Since the classifying space $B\tilde{G}$ of \tilde{G} is EG , we find that $B(\tilde{G} \times X) = EG \times X$.) Since $EG \times X \simeq X$, this shows that, for a free G -space X , there is a weak equivalence $\varphi_!(X) \rightarrow \Phi(X)$ of diagrams of spaces on the orbit category $\mathcal{O}(G)$. Consequently, the Invariance Theorem (Corollary 2.6) gives an isomorphism $H^*(\varphi_!(X), M) \cong H^*_G(X, M)$. This proves Proposition 1.5.

5. A remark on Mackey functors

In this short section we will show that our induction-restriction formalism for diagrams of spaces is also related to well-known splitting theorems for equivariant cohomology. More precisely, we will give a simple proof of Proposition 5.1 below. More abstract general splitting theorems occur e.g. in [LMS, §V.6], and in [O].

PROPOSITION 5.1. *Let G be a finite group, and let R be a commutative ring with unit such that $|G|$ is invertible in R . Then for any Mackey functor $M : \mathcal{O}(G)^{\text{op}} \rightarrow (R\text{-modules})$, there is a splitting of Bredon cohomology*

$$H^*_G(X, M) \cong \prod_{[G/H]} H^*(X^H, \underline{M}(G/H))^{WH}, \tag{1}$$

for any G -space X .

In this proposition, $\underline{M}(G/H)$ is the R -module defined by

$$\underline{M}(G/H) = \bigcap_{f:G/K \rightarrow G/H} \text{Ker}(f^* : M(G/H) \rightarrow M(G/K)),$$

where f runs over all non-isomorphic maps. Furthermore, WH denotes the Weil group $WH = \text{Aut}(G/H) = N(H)/H$. Thus WH acts on both X^H and $\underline{M}(G/H)$, hence on $H^*(X^H, \underline{M}(G/H))$, by “conjugation”. The product in (1) ranges over all isomorphism classes of orbits $[G/H]$.

Our proof of Proposition 5.1 is based on the following splitting of the Mackey functor M . Recall that the group WH can be viewed as a one-object category. Let $i_H : WH \hookrightarrow \mathcal{O}(G)$ be the inclusion of WH as the orbit G/H with its automorphisms. For G, R and M as above, there is a splitting

$$M \cong \prod_{[G/H]} (i_H)_* \underline{M}(G/H), \tag{2}$$

where the product ranges over all isomorphism classes of orbits G/H . This isomorphism (2) has been observed by many authors, and published references can be traced back at least to [S1].

For the proof of 5.1, consider the abelian category $\underline{\text{Mod}}(R[WH])$ of $R[WH]$ -modules, or equivalently, of functors $WH \rightarrow (R\text{-modules})$, and the similar category $\underline{\text{Mod}}(R[\mathcal{O}(G)])$ of functors $\mathcal{O}(G) \text{ op} \rightarrow (R\text{-modules})$. Then induction along the functor $i_H : WH \rightarrow \mathcal{O}(G)$ can be viewed as a functor $(i_H)_* : \underline{\text{Mod}}(R[WH]) \rightarrow \underline{\text{Mod}}(R[\mathcal{O}(G)])$.

LEMMA 5.2. *For any $R[WH]$ -module A and any G -space X , there is a canonical isomorphism*

$$H_G^*(X, (i_H)_* A) \simeq H^*(\int_{WH} \Delta(X^H), A).$$

Proof. First we observe that the induction functor $\underline{\text{Mod}}(R[WH]) \rightarrow \underline{\text{Mod}}(R[\mathcal{O}(G)])$ is exact. Indeed, this is clear from the (last) description of $(i_H)_*$ in Appendix B (8); this description gives, for any $R[WH]$ -module A and any orbit G/K , that

$$(i_H)_*(A)(G/K) \cong \text{Hom}_{R[WH]}(R[\text{Hom}_{\mathcal{O}(G)}(G/H, G/K)], A),$$

where $R[-]$ is the free R -module on the set of all G -maps $G/H \rightarrow G/K$, viewed as a left $R[WH]$ -module via the action of WH on G/H . Since this module is free as an R -module and $|WH|$ is invertible in R , it is projective as an $R[WH]$ -module. By the exactness of $(i_H)_*$, it follows as in (10) of Remark 3.6 that, for the diagram $\Delta\Phi(X)$ on $\mathcal{O}(G)$, there is an isomorphism $H^p(\Delta\Phi(X), (i_H)_*(A)) \cong H^p(i_H^* \Delta\Phi(X), A)$. In other words, $H_G^p(X, i_{H*}(A)) \cong H^p(\int_{WH} \Delta(X^H), A)$.

LEMMA 5.3. *Let Γ be a finite group such that $|\Gamma|$ is invertible in R . Then for any Γ -space Y and any $R[\Gamma]$ -module A , there is a canonical isomorphism*

$$H^*\left(\int_{\Gamma} \Delta(Y), A\right) \cong H^*(Y, A)^{\Gamma}.$$

(On the left, we have identified A with the corresponding composite functor $\int_{\Gamma} \Delta(Y) \rightarrow \underline{B}\Gamma \xrightarrow{A} (R\text{-modules})$.)

Proof. First observe that, exactly as in the proof of Proposition 1.5 given in the preceding section, there is an isomorphism

$$H^*\left(\int_{\Gamma} \Delta(Y), A\right) \cong H_{tw}^{p+q}(E\Gamma \times_{\Gamma} Y, A),$$

where on the right, the twist comes from the projection $E\Gamma \times_{\Gamma} Y \rightarrow B\Gamma$. Now consider the spectral sequence for the covering projection $E\Gamma \times Y \rightarrow E\Gamma \times_{\Gamma} Y$,

$$E_2^{p,q} = H^p(\Gamma, H^q(Y, A)) \Rightarrow H_{tw}^{p+q}(E\Gamma \times_{\Gamma} Y, A).$$

Since $H^q(Y, A)$ is an R -module, multiplication by $|\Gamma|$ is an isomorphism on $H^p(Y, A)$. Thus $E_2^{p,q} = 0$ for $q > 0$ and this spectral sequence collapses to an isomorphism $H^q(Y, A)^{\Gamma} \cong H_{tw}^q(E\Gamma \times_{\Gamma} Y, A)$.

Proof of Proposition 5.1. The desired isomorphism (1) is obtained directly from (2), Lemma 5.2 for $A = \underline{M}(G/H)$, and Lemma 5.3 for $\Gamma = WH$ and $Y = X^H$.

EXAMPLES 5.4. (a) The constant functor $M = \text{con}(\mathbb{Q}) : \mathcal{O}(G)^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$, with value the rational numbers \mathbb{Q} , is a Mackey functor, with covariant action by a map $\alpha : G/H \rightarrow G/K$ in $\mathcal{O}(G)$ defined as multiplication by the cardinality $\#\alpha^{-1}(K)$ of the fiber of α . In this case we have $\underline{M}(G/H) = 0$ when $H \neq 1$, while $\underline{M}(G/1) \cong \mathbb{Q}$, so Proposition 5.1 together with (1.4) yield an isomorphism

$$H^*(X/G, \mathbb{Q}) \cong H^*(X, \mathbb{Q})^G$$

for any G -space X .

(b) Let R be a commutative ring so that $|G|$ is invertible in R , and consider the case $M = A \otimes R$ where A is the Burnside ring Mackey functor. Thus $(A \otimes R)(G/H) = A(G/H) \otimes R \cong A_H \times R$ (where A_H is the Burnside ring of H). In this case there is for any G -space X an isomorphism

$$H_G^*(X, A \otimes R) \cong \prod_{[G/H]} H^*(X^H, R)^{WH}.$$

Indeed, by Proposition 5.1 it suffices to show that for any subgroup $H \subseteq G$,

$$\underline{A \otimes R}(G/H) \cong R, \quad (\text{with trivial } WH\text{-action}). \tag{3}$$

To prove (3), recall first that by [tD1, Proposition 1.2.3], the usual injective ring homomorphism $A(G/H) = A_H \rightarrow \prod \mathbb{Z}$ becomes an isomorphism after tensoring with R ; here the product is over all H -conjugacy classes of subgroups of H , or equivalently, over all isomorphism classes of objects in $\mathcal{O}(G)/(G/H)$. Thus $A \otimes R(G/H)$ is isomorphic to the intersection of the kernels of all the maps

$$\alpha^* : \prod_{[\mathcal{O}(G)/(G/H)]} R \rightarrow \prod_{[\mathcal{O}(G)/(G/K)]} R,$$

for all non-isomorphisms $\alpha : G/K \rightarrow G/H$. Here α^* is the map

$$\alpha^*(r)_{[\beta:G/L \rightarrow G/K]} = r_{[\alpha \circ \beta]}$$

But for $r \in \prod_{\mathcal{O}(G)/(G/H)} R$, clearly $\alpha^*(r) = 0$ for all non-isomorphic α iff $r_\beta = 0$ for all $[\beta : G/L \rightarrow G/H]$ for which β is a non-isomorphism. Hence $A \otimes R(G/H) \cong R$, as claimed in (3).

6. Proof of the invariance theorem

The purpose of this section is to give a proof of Theorem 2.3. For a category \mathbb{C} , we will use $\Delta\mathbb{C}$ as a shorthand notation for $\int_\Delta N(\mathbb{C})$. (Since this notation will occur in the present section only, it should not lead to confusion with the analogous abbreviation $\Delta(X) = \int_\Delta S(X)$ for a topological space X .) Explicitly, if we write $[p]$ for the ordered set $\{0, \dots, p\}$ (for $p \geq 0$), then the category $\Delta\mathbb{C}$ has as objects all functors $\sigma : [p] \rightarrow \mathbb{C}$, and as arrows from $(\sigma : [p] \rightarrow \mathbb{C})$ to $(\tau : [q] \rightarrow \mathbb{C})$ all arrows $\alpha : [p] \rightarrow [q]$ in Δ such that $\sigma = \tau \circ \alpha$.

Let $F : \mathbb{B}^{\text{op}} \rightarrow \underline{\text{Cat}}$ be a diagram of categories, indexed by a fixed category \mathbb{B} . By composition with the functor $\Delta : \underline{\text{Cat}} \rightarrow \underline{\text{Cat}}$ just described, one obtains a new diagram ΔF on \mathbb{B} . In preparation for the proof of Theorem 2.3, we will first derive a spectral sequence for the cohomology of this diagram ΔF . To state this spectral sequence, some more notation is needed. For any p -simplex $\alpha = (B_0 \xrightarrow{\alpha_1} B_1 \rightarrow \dots \xrightarrow{\alpha_p} B_p)$ in the nerve of \mathbb{B} , the composition $\alpha_p \circ \alpha_{p-1} \circ \dots \circ \alpha_1$ gives a functor

$$\hat{\alpha} = \Delta F(\alpha_p \circ \dots \alpha_1) : \Delta F(B_p) \rightarrow \Delta F(B_0).$$

Furthermore, a local system of coefficients $M : (\int_{\mathbb{B}} \Delta F)^{\text{op}} \rightarrow \underline{\text{Ab}}$ on this diagram ΔF gives a local system M_{B_0} on $\Delta F(B_0)$, via restriction along the inclusion of this category into $\int_{\mathbb{B}} \Delta(F)$. Composing with $\hat{\alpha}$, we get a local system $\hat{\alpha}^*(M_{B_0})$ on $\Delta F(B_p)$. Using this notation we have, for each $q \geq 0$, a cosimplicial abelian group, defined in degree p by

$$\prod_{\alpha \in N_p(\mathbb{B})} H^q(\Delta F(B_p), \hat{\alpha}^*(M_{B_0})),$$

where $\alpha = (B_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_p} B_p)$, as before.

LEMMA 6.1. *Let $F : \mathbb{B}^{\text{op}} \rightarrow \underline{\text{Cat}}$ be a diagram of categories on \mathbb{B} , with associated diagram ΔF as above. For any local system M on ΔF there is a natural spectral sequence*

$$E_1^{p,q} = \prod_{\alpha \in N_p(\mathbb{B})} H^q(\Delta F(B_p), \hat{\alpha}^*(M_{B_0})) \Rightarrow H^{p+q}(\Delta F, M).$$

Proof. A p -simplex of $N(\int_{\mathbb{B}} \Delta F)$ is a sequence

$$\xi = ((B_0, [n_0] \xrightarrow{\sigma_0} F(B_0)) \xrightarrow{(\alpha_1, u_1)} \dots \xrightarrow{(\alpha_p, u_p)} (B_p, [n_p] \xrightarrow{\sigma_p} F(B_p))),$$

such that

$$\begin{array}{ccc} [n_i] & \xrightarrow{\sigma_i} & F(B_i) \\ u_{i+1} \downarrow & & \uparrow F(\alpha_{i+1}) \\ [n_{i+1}] & \xrightarrow{\sigma_{i+1}} & F(B_{i+1}) \end{array}$$

commutes, for all $0 \leq i \leq p - 1$. Thus given the α_i , the u_i and σ_p , all the σ_i for $i < p$ are determined. Consequently, we can write

$$N_p(\int_{\mathbb{B}} \Delta F) \cong \sum_{\alpha \in N_p(\mathbb{B})} N_p(\Delta F(B_p)),$$

where $\alpha = (B_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_p} B_p)$ as before. Recall from Appendix B our standard notation $\alpha(0)$ for B_0 ; similarly for a $\xi \in N(\int_{\mathbb{B}} \Delta F)$ we write $\xi(0)$ for $(B_0, [n_0] \xrightarrow{\sigma_0} F(B_0))$. With this notation, the standard cosimplicial group used to compute $H^*(\int_{\mathbb{B}} \Delta F, M)$ (see appendix B, (1)) takes the form

$$\prod_{\xi \in N_p(\int_{\mathbb{B}} \Delta F)} M(\xi(0)) = \prod_{\alpha \in N_p(\mathbb{B})} \prod_{u \in N_p(\Delta F(B_p))} \hat{\alpha}^*(M_{\alpha(0)})(u(0)).$$

But note that this is the diagonal of the bicosimplicial abelian group $C^{\cdot\cdot}(\Delta F, M)$ with (p, q) -simplices

$$C^{p,q} = \prod_{\alpha \in N_p(\mathbb{B})} \prod_{u \in N_q(\Delta F(B_p))} \hat{\alpha}^*(M_{\alpha(0)})(u(0)),$$

with cosimplicial structure coming from the nerves of \mathbb{B} and of $\Delta F(-)$. Filtration of the associated double complex by vertical lines gives a spectral sequence with the desired E_1 -term, abutting to $H^*(\text{Tot}C^{\cdot\cdot}(\Delta F, M))$. Here Tot denotes the usual total complex of a double complex, as in [DP, p 212]. By a basic result in [DP] (see p 213), this total complex has the same cohomology as the diagonal of the double complex, and the latter cohomology has just been identified as $H^*(\Delta F, M)$. This proves the lemma.

Next, let F be a diagram of categories on \mathbb{B} as before, and consider the natural transformation

$$\lambda_F : \Delta F = \int_{\Delta} N(F) \rightarrow F,$$

obtained by applying the “last vertex” map λ of Appendix A for each of the categories $F(B)$ in the diagram. Of course, λ_F is a weak equivalence of diagrams. We first prove the following special case of the invariance theorem:

LEMMA 6.2. *For any local system of coefficients M on F , the map λ_F induces an isomorphism*

$$H^*(F, M) \cong H^*(\Delta F, \lambda_F^*(M)).$$

Proof. By integration, we obtain a functor $\int_{\mathbb{B}} \lambda_F : \int_{\mathbb{B}} \Delta F \rightarrow \int_{\mathbb{B}} F$. For the purpose of this proof (and the next one), we agree to denote this functor simply by λ or by λ_F . The Grothendieck spectral sequence (Appendix B) of this functor λ has the form

$$E_2^{p,q} = H^p\left(\int_{\mathbb{B}} F, R^q \lambda_* (\lambda^* M)\right) \Rightarrow H^{p+q}\left(\int_{\mathbb{B}} \Delta F, \lambda^* M\right). \tag{1}$$

Here, as in Appendix B, $R^q \lambda_* (\lambda^* M)$ is (isomorphic to) the coefficient system sending an object (B, x) of $\int_{\mathbb{B}} F$ to the abelian group $H^q(\lambda/(B, x), \omega^* \lambda^* M)$, where $\omega : \lambda/(B, x) \rightarrow \int_{\mathbb{B}} \Delta F$ is the forgetful functor. Now note that

$$\lambda/(B, x) = \int_{\mathbb{B}/B} \Delta(F/x),$$

where $F/x : (\mathbb{B}/B) \text{ op} \rightarrow \underline{\text{Cat}}$ is the functor sending an object $\alpha : B' \rightarrow B$ of \mathbb{B}/B to $F(B')/F(\alpha)(x)$. Applying Lemma 6.1 to the diagram of categories $\Delta(F/x)$, we get a spectral sequence with

$$E_1^{p,q} = \prod_{\alpha \in N_p(\mathbb{B}/B)} H^q(\Delta(F/x)(B_p \rightarrow B), \hat{\alpha}^*(\omega^* \lambda^*(M)_{B_0 \rightarrow B})) \tag{2}$$

which converges to $H^{p+q}(\lambda/(B, x), \omega^* \lambda^* M)$.

But $(F/x)(B_p \rightarrow B) = F(B_p)/F(B_p \rightarrow B)(x)$ has a terminal object $(F(B_p \rightarrow B)(x), id)$. Consequently, the category $(F/x)(B_p \rightarrow B)$ has trivial cohomology groups. Since the last vertex functor $\Delta(F/x)(B_p \rightarrow B) \rightarrow (F/x)(B_p \rightarrow B)$ is a weak equivalence (see Appendix A), and since $\hat{\alpha}^*(\omega^* \lambda^*(M)_{B_0 \rightarrow B})$ is morphism-inverting on $\Delta(F/x)(B_p \rightarrow B)$, it follows by (5) of Appendix B that

$$H^q(\Delta(F/x)(B_p \rightarrow B), \hat{\alpha}^*(\omega^* \lambda^*(M)_{B_0 \rightarrow B})) = \begin{cases} 0 & q > 0 \\ M(B_0, F(B_0 \rightarrow B)(x)), & q = 0. \end{cases}$$

Thus $E_1^{p,q} = 0$ for $q > 0$ in (2), and the E_2 -term can be computed as follows:

$$\begin{aligned} E_2^{p,0} &= H^p\left(\prod_{\alpha \in N_p(\mathbb{B}/B)} M(B_0, F(B_0 \rightarrow B)(x))\right) \\ &= H^p(\mathbb{B}/B, M^B), \end{aligned}$$

where M^B is the obvious coefficient system on \mathbb{B}/B . Now \mathbb{B}/B has a terminal object, so $H^p(\mathbb{B}/B, M^B) = 0$ for $p > 0$, while $H^0(\mathbb{B}/B, M^B) = M^B(B \xrightarrow{id} B) = M(B, F(B \xrightarrow{id} B)(x)) = M(B, x)$. This proves that the spectral sequence (2) collapses to an isomorphism

$$H^p(\lambda/(B, x), \omega^* \lambda^* M) \cong \begin{cases} 0, & p > 0 \\ M(B, x), & p = 0. \end{cases}$$

Thus the spectral sequence reduces to an isomorphism

$$H^*(\int_{\mathbb{B}} F, M) \cong H^*(\Delta F, \lambda_F^* M), \text{ as desired. This proves Lemma 6.2.}$$

Proof of Theorem 2.3. Let $\nu : G \rightarrow F$ be a weak equivalence of diagrams, and consider the commutative square of categories and functors

$$\begin{array}{ccc} \int_{\mathbb{B}} \Delta G & \xrightarrow{\tilde{\nu}} & \int_{\mathbb{B}} \Delta F \\ \lambda_G \downarrow & & \downarrow \lambda_F \\ \int_{\mathbb{B}} G & \xrightarrow{\nu} & \int_{\mathbb{B}} F \end{array}$$

where $\tilde{\nu} = \int_{\mathbb{B}} \Delta \nu$. Applying the spectral sequence of Lemma 6.1, for both F and G , we find that $\tilde{\nu}$ induces an isomorphism in cohomology with local coefficients. By Lemma 6.2, the same is true for λ_F and λ_G , and consequently for ν . This proves Theorem 2.3.

REMARK 6.3. By combining Lemma 6.1 and 6.2, we see that, for a diagram of categories $F : \mathbb{B}^{op} \rightarrow \underline{\text{Cat}}$ and a local system of coefficients M on F , there is a spectral sequence

$$E_1^{p,q} = \prod_{\alpha \in N_p(\mathbb{B})} H^q(F(B_p), \hat{\alpha}^*(M_{B_0})) \Rightarrow H^{p+q}(F, M)$$

Here $\alpha = (B_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_p} B_p)$ as before, M_{B_0} is the system on $F(B_0)$ induced by M , and $\hat{\alpha}^*(M_{B_0})$ is its restriction along $\hat{\alpha} = F(\alpha_0 \circ \dots \circ \alpha_1) : F(B_p) \rightarrow F(B_0)$.

REMARK 6.4. We will now determine the E_2 -term of the spectral sequence of the preceding remark in the special case where the coefficient system is constant, i.e., factors through the projection functor $\int_{\mathbb{B}} F \rightarrow \mathbb{B}$. We will view M as a functor $M : \mathbb{B}^{op} \rightarrow \underline{\text{Ab}}$, so that in this case $\hat{\alpha}^*(M_{B_0})$ in 6.3 is simply the group $M(B_0)$.

Now consider the category $\underline{\mathbf{Ab}}(\mathbb{B} \times \mathbb{B}^{\text{op}})$ of all contravariant abelian group functors on $\mathbb{B} \times \mathbb{B}^{\text{op}}$ (we use the notation of Appendix B here). Define the object

$$\mathbb{Z}\text{Hom}_{\mathbb{B}} \in \underline{\mathbf{Ab}}(\mathbb{B} \times \mathbb{B}^{\text{op}})$$

for each pair of objects B, B' of \mathbb{B} by letting $\mathbb{Z}\text{Hom}_{\mathbb{B}}(B, B')$ be the free Abelian group on the set of all arrows $B \rightarrow B'$ in \mathbb{B} . We claim that, for any other object $L \in \underline{\mathbf{Ab}}(\mathbb{B} \times \mathbb{B}^{\text{op}})$, the group $\text{Ext}^p(\mathbb{Z}\text{Hom}_{\mathbb{B}}, L)$ can be computed as the p -th cohomology of the cochain complex defined in degree p by $\prod_{\alpha \in N_p(\mathbb{B})} L(B_0, B_p)$. To see this, define, for each $p \geq 0$, an object $R^p \in \underline{\mathbf{Ab}}(\mathbb{B} \times \mathbb{B}^{\text{op}})$ by

$$R^p = \sum_{\alpha \in N_p(\mathbb{B})} \mathbb{Z}[\mathbb{B}^{\text{op}} \times \mathbb{B}((B_0, B_p), -)].$$

Each such object R^p is projective, being a sum of free groups on representable functors $\mathbb{B}^{\text{op}} \times \mathbb{B}((B_0, B_p), -)$. Furthermore, one defines coboundary operators

$$\dots \xrightarrow{d} R^1 \xrightarrow{d} R^0$$

in the evident way, using alternating sums of the face operators of the nerve $N(\mathbb{B})$. There is a canonical augmentation

$$R^0 \rightarrow \mathbb{Z}\text{Hom}_{\mathbb{B}},$$

defined for each pair of objects $(C, D) \in \mathbb{B}^{\text{op}} \times \mathbb{B}$ by using the composition in the category \mathbb{B} :

$$\begin{aligned} R^0(C, D) &= \sum_{B \in \mathbb{B}} \mathbb{Z}[\mathbb{B}^{\text{op}} \times \mathbb{B}((B, B), (C, D))] = \sum_{B \in \mathbb{B}} \mathbb{Z}[\mathbb{B}(C, B) \times \mathbb{B}(B, D)] \\ &\quad \downarrow \\ \mathbb{Z}\text{Hom}_{\mathbb{B}}(C, D) &= \mathbb{Z}[\mathbb{B}(C, D)] \end{aligned}$$

One readily checks that this augmented complex $\dots \rightarrow R^1 \rightarrow R^0 \rightarrow \mathbb{Z}\text{Hom}_{\mathbb{B}}$ is exact. Thus we obtain a projective resolution of $\mathbb{Z}\text{Hom}_{\mathbb{B}}$, and hence $\text{Ext}^p(\mathbb{Z}\text{Hom}_{\mathbb{B}}, L)$ can be computed as the cohomology of the complex $\text{Hom}(R^*, L)$ (where the Hom is in the category $\underline{\mathbf{Ab}}(\mathbb{B}^{\text{op}} \times \mathbb{B})$). But $\text{Hom}(R^p, L) = \prod_{\alpha \in N_p(\mathbb{B})} L(B_0, B_p)$, thus proving our claim. Applying this to the special case where L is the functor $H^q(F(-), M(?))$, one finds that, when the coefficient system M is constant, the E_2 -term in the spectral sequence 6.3 is

$$E_2^{p,q} = \text{Ext}^p(\mathbb{Z}\text{Hom}_{\mathbb{B}}, H^q(F(-), M(?))).$$

Appendix A. Spaces, simplicial sets and categories

The purpose of this appendix is to review the basic constructions by which one can pass from one to another of the three categories of spaces, simplicial sets and (small) categories. We will denote these categories by $\underline{\mathbf{Top}}$, $\underline{\mathbf{Sset}}$ and $\underline{\mathbf{Cat}}$ respectively.

We begin with the definition of the functors in diagram (2.1), repeated here for convenience

$$\begin{array}{ccc}
 & \underline{\text{Cat}} & \\
 f_{\Delta} \nearrow & & \nwarrow \Delta \\
 \text{Sset} & \begin{array}{c} N \quad B \\ | \cdot | \\ \xrightarrow{S} \end{array} & \underline{\text{Top.}} \\
 & \xleftarrow{S} &
 \end{array} \tag{1}$$

For a simplicial set Z , its geometric realization is denoted $|Z|$. For a topological space X , its singular complex is denoted $S(X)$. These functors $|\cdot|$ and S are extensively discussed in any standard text on simplicial topology [GZ, L, M]. In particular, we recall that there are, for any simplicial set Z and any space X , natural maps

$$\eta : Z \rightarrow S|Z| \quad \text{and} \quad \epsilon : |SX| \rightarrow X.$$

These maps are the unit and counit of the adjunction $\text{Hom}(Z, SX) \cong \text{Hom}(|Z|, X)$. Both η and ϵ are weak homotopy equivalences. Thus the functors $|\cdot|$ and S are mutually inverse, up to natural weak homotopy equivalence.

For a small category \mathbb{C} , its nerve is denoted $N(\mathbb{C})$. This is the simplicial set with as n -simplices all composable strings $C_0 \xrightarrow{\alpha_1} C_1 \rightarrow \dots \xrightarrow{\alpha_n} C_n$ of arrows in \mathbb{C} . In the other direction, one can construct for each simplicial set Z a category $\int_{\Delta} Z$. The objects of this category are pairs (n, z) where $n \geq 0$ and $z \in Z_n$. An arrow $(n, z) \rightarrow (n', z')$ is an order-preserving map $\alpha : [n] \rightarrow [n']$ with the property that $\alpha^*(z') = z$; here $[n]$ denotes the ordered set $\{0, \dots, n\}$, as usual. The category of all these ordered sets $[n]$ and order-preserving maps between them is denoted Δ . When we view a simplicial set Z as a functor $\Delta^{\text{op}} \rightarrow \underline{\text{Set}}$, then the category $\int_{\Delta} Z$ is a special instance of the Grothendieck construction, defined in Section 2 above.

The other two functors Δ and B between $\underline{\text{Cat}}$ and $\underline{\text{Top}}$ in the above diagram are defined by composition: for a small category \mathbb{C} and a topological space X ,

$$B\mathbb{C} = |N\mathbb{C}| \quad \text{and} \quad \Delta(X) = \int_{\Delta} S(X). \tag{2}$$

$B\mathbb{C}$ is called the *classifying space* of \mathbb{C} . By definition, a functor $\mathbb{C} \rightarrow \mathbb{C}'$ between small categories is said to be a weak homotopy equivalence iff it induces a weak equivalence $B\mathbb{C} \rightarrow B\mathbb{C}'$.

With this definition, the functors \int_{Δ} and N are also mutually inverse, up to weak homotopy equivalence. Indeed, for each simplicial set Z there is a natural map

$$\rho = \rho_Z : N\left(\int_{\Delta} Z\right) \rightarrow Z, \tag{3}$$

sending a p -simplex $((n_0, z_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_p} (n_p, z_p))$ of the nerve of $\int_{\Delta} Z$ to the p -simplex $\tilde{\alpha}^*(z_p) \in Z_p$; here $\tilde{\alpha} : [p] \rightarrow [n_p]$ is defined by

$$\tilde{\alpha}(i) = \alpha_p \circ \dots \circ \alpha_{i+1}(n_i), \quad 0 \leq i \leq p \text{ (so } \tilde{\alpha}(p) = n_p).$$

We refer to [I, p 21] or [W, p 359] for the observation that ρ is a weak equivalence. In the other direction, there is, for each small category \mathbb{C} , a natural “last vertex” functor

$$\lambda = \lambda_{\mathbb{C}} : \int_{\Delta} N(\mathbb{C}) \rightarrow \mathbb{C}. \tag{4}$$

It is defined, for any object $(n, C_0 \rightarrow \dots \rightarrow C_n)$ of $\int_{\Delta} N(\mathbb{C})$, by

$$\lambda(n, C_0 \rightarrow \dots \rightarrow C_n) = C_n.$$

This definition of λ can be extended to the arrows of $\int_{\Delta} N(\mathbb{C})$ in the evident way. The map λ is a weak homotopy equivalence of categories, because $N(\lambda_{\mathbb{C}}) = \rho_{N(\mathbb{C})}$. Note that both functors $N : \underline{\mathbf{Cat}} \rightarrow \underline{\mathbf{Sset}}$ and $\int_{\Delta} : \underline{\mathbf{Sset}} \rightarrow \underline{\mathbf{Cat}}$ preserve weak equivalences; the first by definition, and hence also the second since it is inverse to N up to natural weak homotopy equivalence.

It follows that the functors B and $\Delta : \underline{\mathbf{Cat}} \rightleftarrows \underline{\mathbf{Top}}$ are also mutually inverse, up to natural weak homotopy equivalence, and that these functors B and Δ both preserve weak equivalences. To see this, one uses, for any category \mathbb{C} and any space X , the weak equivalences

$$\sigma : B\Delta X \rightarrow X \quad \text{and} \quad \Delta B\mathbb{C} \leftarrow \int_{\Delta} N(\mathbb{C}) \rightarrow \mathbb{C} \tag{5}$$

obtained from the compositions:

$$\sigma = \epsilon \circ |\rho| : B\Delta(X) = |N(\int_{\Delta} S(X))| \rightarrow |S(X)| \rightarrow X, \tag{6}$$

and

$$\int_{\Delta} \eta : \int_{\Delta} N(\mathbb{C}) \rightarrow \int_{\Delta} S|N(\mathbb{C})| = \Delta B\mathbb{C}.$$

together with $\lambda : \int_{\Delta} N(\mathbb{C}) \rightarrow \mathbb{C}$.

These constructions also apply to diagrams of spaces, simplicial sets, and categories, and are natural in the index category, at least up to weak homotopy equivalence. In Section 3, this was used in particular for the functors B and Δ . To be more explicit, write $\underline{\mathbf{Top}}(\mathbb{C})$ for the category of diagrams of spaces on \mathbb{C} , i.e. functors $\mathbb{C}^{\text{op}} \rightarrow \underline{\mathbf{Top}}$, and natural transformations between them. Similarly, write

$\underline{\text{Cat}}(\mathbb{C})$ for the category of diagrams $\mathbb{C}^{\text{op}} \rightarrow \underline{\text{Cat}}$ of categories. For any functor $\varphi : \mathbb{D} \rightarrow \mathbb{C}$, we then obtain a diagram of categories and functors

$$\begin{array}{ccc} \underline{\text{Cat}}(\mathbb{C}) & \begin{array}{c} \xrightarrow{\varphi^*} \\ \xrightarrow{\varphi_!} \end{array} & \underline{\text{Cat}}(\mathbb{D}) \\ B \downarrow \uparrow \Delta & & B \downarrow \uparrow \Delta \end{array} \tag{7}$$

$$\underline{\text{Top}}(\mathbb{C}) \begin{array}{c} \xrightarrow{\varphi^*} \\ \xrightarrow{\varphi_!} \end{array} \underline{\text{Top}}(\mathbb{D}).$$

Here the functors B and Δ for diagrams are defined by applying the earlier functors B and $\Delta : \underline{\text{Cat}} \rightleftarrows \underline{\text{Top}}$ “pointwise”. Thus for diagrams $\underline{X} : \mathbb{C}^{\text{op}} \rightarrow \underline{\text{Top}}$ and $F : \mathbb{C}^{\text{op}} \rightarrow \underline{\text{Cat}}$, we have $\Delta(\underline{X})(C) = \Delta(\underline{X}(C))$ and $B(F)(C) = B(F(C))$, for any object $C \in \mathbb{C}$. The functor φ^* is defined simply by composition with $\varphi : \mathbb{D} \rightarrow \mathbb{C}$. Thus for \underline{X} and F as before, we have $\varphi^*(\underline{X})(D) = \underline{X}(\varphi D)$ and $\varphi^*(F)(D) = F(\varphi D)$, for any object $D \in \mathbb{D}$. Clearly, there are identities

$$B\varphi^*(F) = \varphi^*(BF) \quad \text{and} \quad \Delta\varphi^*(\underline{X}) = \varphi^*\Delta(\underline{X}).$$

Recall from Section 3 that the functors $\varphi_!$ are defined, for diagrams $G : \mathbb{D}^{\text{op}} \rightarrow \underline{\text{Cat}}$ and $\underline{Y} : \mathbb{D}^{\text{op}} \rightarrow \underline{\text{Top}}$ and for any object $C \in \mathbb{C}$, by

$$\begin{aligned} \varphi_!(G)(C) &= \int_{C/\varphi} \omega_C^*(G), \\ \varphi_!(\underline{Y})(C) &= \text{hocolim}_{(C/\varphi)} \omega_C^*(\underline{Y}). \end{aligned}$$

Here $\omega_C : C/\varphi \rightarrow \mathbb{D}$ is the evident forgetful functor. To compare the compositions $B\varphi_!$ and $\varphi_!B$ in (7), as well as $\Delta\varphi_!$ and $\varphi_!\Delta$, we use the result from [T] which states that for a diagram of small categories $G : \mathbb{D}^{\text{op}} \rightarrow \underline{\text{Cat}}$, there is a natural weak equivalence

$$\tau : \text{hocolim}_{\mathbb{D}} BG \xrightarrow{\cong} B\left(\int_{\mathbb{D}} G\right).$$

(In other words, for categories the Grothendieck construction acts as a homotopy colimit.) This map τ , applied to each index category C/φ , gives a natural weak homotopy equivalence

$$\tau : \varphi_!(BG) \xrightarrow{\cong} B(\varphi_!G). \tag{8}$$

Finally, for a diagram of spaces $\underline{Y} : \mathbb{D}^{\text{op}} \rightarrow \underline{\text{Top}}$, we can now compare $\varphi_!(\Delta\underline{Y})$ and $\Delta\varphi_!(\underline{Y})$ by using the weak homotopy equivalences discussed so far. Indeed, for any object $C \in \mathbb{C}$, there are the following maps, all natural in C :

$$\int_{C/\varphi} \omega_C^*(\Delta\underline{Y}) \xleftarrow{\lambda} \int_{\Delta} N\left(\int_{C/\varphi} \omega_C^*(\Delta\underline{Y})\right) \xrightarrow{\int \Delta^\eta} \Delta B \int_{C/\varphi} \omega_C^*(\Delta\underline{Y})$$

and, with σ as in (6),

$$\Delta(\text{hocolim}_{(C/\varphi)} \omega_C^*(\underline{Y})) \xrightarrow{\Delta\sigma} \Delta \text{hocolim}_{(C/\varphi)} \omega_C^*(B\Delta\underline{Y}) \xrightarrow{\Delta\tau} \Delta B(\int_{C/\varphi} \omega_C^*(\Delta\underline{Y})).$$

By definition of $\varphi_!(\Delta\underline{Y})(C)$ as $\int_{C/\varphi} \omega_C^*(\Delta\underline{Y})$, and of $\Delta\varphi_!(\underline{Y})(C)$ as $\Delta(\text{hocolim} \omega_C^*(\underline{Y}))$, this gives a zig-zag of weak homotopy equivalences

$$\varphi_!(\Delta\underline{Y}) \xrightarrow{\sim} \cdot \xrightarrow{\sim} \cdot \xrightarrow{\sim} \cdot \xrightarrow{\sim} \Delta\varphi_!(\underline{Y}).$$

We remark (but will not use) that, for diagrams of categories, as well as for diagrams of spaces, the induction functors are homotopy left-adjoint to the restriction functors φ^* , in the following sense. In the case of diagrams of spaces, denote by $\text{HoTop}(\mathbb{C})$ the category obtained from $\text{Top}(\mathbb{C})$ by formally inverting all weak homotopy equivalences, and similarly for $\text{HoTop}(\mathbb{D})$. Then for diagrams of spaces \underline{X} on \mathbb{C} and \underline{Y} on \mathbb{D} there is a natural bijection between arrows $\underline{Y} \rightarrow \varphi^*(\underline{X})$ in $\text{HoTop}(\mathbb{D})$ and arrows $\varphi_!(\underline{Y}) \rightarrow \underline{X}$ in $\text{HoTop}(\mathbb{C})$. In the case of diagrams of categories, the adjointness of $\varphi_!$ and φ^* is proved as follows: for diagrams $F \in \underline{\text{Cat}}(\mathbb{C})$ and $G \in \underline{\text{Cat}}(\mathbb{D})$ there are evident morphisms

$$\varphi_! \varphi^*(F) \rightarrow F \quad \text{and} \quad G \leftarrow id_!(G) \rightarrow \varphi^* \varphi_!(G),$$

where $id : \mathbb{D} \rightarrow \mathbb{D}$ denotes the identity functor. (The map $id_!(G) \rightarrow G$ is an instance of the first map $\varphi_! \varphi^*(F) \rightarrow F$, for φ the identity.) This map $id_!(G) \rightarrow G$ is a weak equivalence, so in the homotopy category $\text{Ho } \underline{\text{Cat}}(\mathbb{D})$ we obtain a morphism $G \rightarrow \varphi^* \varphi_!(G)$. One can now check that these morphisms $\varphi_! \varphi^*(F) \rightarrow F$ and $G \rightarrow \varphi^* \varphi_!(G)$ satisfy the triangular identities for an adjunction. The adjointness for diagrams of spaces is a consequence of that for categories. Indeed, since the functors B and Δ in (7) are mutually inverse up to weak equivalence, they induce an equivalence of categories at the level of homotopy categories, and one obtains from (7) a diagram

$$\begin{array}{ccc} \text{Ho } \underline{\text{Cat}}(\mathbb{C}) & \begin{array}{c} \xrightarrow{\varphi^*} \\ \xrightarrow{\varphi_!} \end{array} & \text{Ho } \underline{\text{Cat}}(\mathbb{D}) \\ \simeq \downarrow & & \downarrow \simeq \\ \text{Ho } \underline{\text{Top}}(\mathbb{C}) & \begin{array}{c} \xrightarrow{\varphi^*} \\ \xrightarrow{\varphi_!} \end{array} & \text{Ho } \underline{\text{Top}}(\mathbb{D}). \end{array}$$

Appendix B. Cohomology of Categories

In this appendix we will review some of the standard definitions and constructions related to the cohomology of small categories. In particular, we will recall a suitable version of the Grothendieck spectral sequence.

Let \mathbb{C} be a small category, and let $A : \mathbb{C}^{\text{op}} \rightarrow \underline{\text{Ab}}$ be a functor into the category of abelian groups. One way to introduce the cohomology groups $H^n(\mathbb{C}, A)$ is as follows. Let $N(\mathbb{C})$ be the nerve of \mathbb{C} , i.e. the simplicial set with as n -simplices all sequences $\beta = (C_0 \xrightarrow{\beta_1} C_1 \rightarrow \dots \xrightarrow{\beta_n} C_n)$. We also write $\beta(i)$ for C_i ($i = 0, \dots, n$). Define a cochain complex $C^*(\mathbb{C}, A)$ in degree p by

$$C^p(\mathbb{C}, A) = \prod_{\beta \in N_p(\mathbb{C})} A(\beta(0)).$$

thus an element of $C^p(\mathbb{C}, A)$ is a sequence $a = (a_\beta)_{\beta \in N_p(\mathbb{C})}$. For such a sequence, the coboundary operator $d : C^p(\mathbb{C}, A) \rightarrow C^{p+1}(\mathbb{C}, A)$ is given by

$$(da)_\beta = \beta_1^*(a_{d_0\beta}) + \sum_{i=1}^{p+1} (-1)^i a_{d_i\beta},$$

where $\beta = (C_0 \xrightarrow{\beta_1} \dots \xrightarrow{\beta_{p+1}} C_{p+1})$. Now define

$$H^p(\mathbb{C}, A) = H^p(C^*(\mathbb{C}, A)). \tag{1}$$

This definition is part of general homological algebra. Indeed, write $\underline{\text{Ab}}(\mathbb{C})$ for the abelian category of all such functors $A : \mathbb{C}^{\text{op}} \rightarrow \underline{\text{Ab}}$. This abelian category has enough injectives [G, p 135]. Consider now the functor

$$\Gamma : \underline{\text{Ab}}(\mathbb{C}) \rightarrow \underline{\text{Ab}}$$

defined for each A in one of the following equivalent ways:

$$\Gamma(A) := \varprojlim_{\mathbb{C}} A(C) = \text{Hom}(\mathbb{Z}, A). \tag{2}$$

Here Hom is taken in the category $\underline{\text{Ab}}(\mathbb{C})$, and $\mathbb{Z} \in \underline{\text{Ab}}(\mathbb{C})$ is the constant functor with value \mathbb{Z} . Then the above cohomology groups are exactly the right derived functors of Γ :

$$H^p(\mathbb{C}, A) = R^p(\Gamma(A)). \tag{3}$$

(One way to see this is to observe that $\underline{\text{Ab}}(\mathbb{C})$ also has enough projectives, and to consider the projective resolution $\dots \rightarrow P^1 \rightarrow P^0 \rightarrow \mathbb{Z}$ where

$$P^n(C) = \sum_{\beta \in N_n(\mathbb{C})} \mathbb{Z}[\mathbb{C}(-, \beta(0))].$$

Here $\mathbb{C}(-, \beta(0)) : \mathbb{C}^{\text{op}} \rightarrow \underline{\text{Sets}}$ is the representable functor, and $\mathbb{Z}[-]$ denotes the free abelian group on a set. Then $\text{Hom}(P^*, A)$ is exactly the complex $C^*(\mathbb{C}, A)$.) The cohomology groups $H^*(\mathbb{C}, A)$ are functorial in the usual way, contravariant in \mathbb{C} and covariant in A .

For general coefficients A the groups $H^n(\mathbb{C}, A)$ cannot be expressed in terms of the classifying space $B\mathbb{C}$. However, if $A : \mathbb{C}^{\text{op}} \rightarrow \underline{\text{Ab}}$ sends every morphism

of \mathbb{C} to an isomorphism, then A gives rise to a twisted coefficient system (again called A) on the classifying space $B\mathbb{C}$, and there is a natural isomorphism

$$H^*(\mathbb{C}, A) \cong H_{tw}^*(B\mathbb{C}, A), \tag{4}$$

see [Q, p 91]. In particular, it follows that if $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ is a weak homotopy equivalence between categories (i.e., if φ induces a (weak) homotopy equivalence $B\varphi : B\mathbb{D} \rightarrow B\mathbb{C}$ between classifying spaces), then φ yields an isomorphism

$$H^*(\mathbb{C}, A) \xrightarrow{\cong} H^*(\mathbb{D}, \varphi^* A) \tag{5}$$

provided A is morphism-inverting.

Next, consider, for a functor $\varphi : \mathbb{D} \rightarrow \mathbb{C}$, the associated restriction functor

$$\varphi^* : \underline{\mathbf{Ab}}(\mathbb{C}) \rightarrow \underline{\mathbf{Ab}}(\mathbb{D}), \tag{6}$$

defined by composition: for any $A : \mathbb{C}^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$ and any object $D \in \mathbb{D}$, one has $\varphi^*(A)(D) = A(\varphi D)$. This functor φ^* has a right adjoint

$$\varphi_* : \underline{\mathbf{Ab}}(\mathbb{D}) \rightarrow \underline{\mathbf{Ab}}(\mathbb{C}), \tag{7}$$

which can be defined, for each $B \in \underline{\mathbf{Ab}}(\mathbb{D})$ and each object $C \in \mathbb{C}$, in one of the following equivalent ways:

$$\begin{aligned} \varphi_*(B)(C) &= \lim_{\longleftarrow \varphi/C} \omega_C^*(B) \\ &= H^0(\varphi/C, \omega_C^*(B)) \\ &= \text{Hom}(\mathbb{Z}[\mathbb{C}(\varphi -, C)], B). \end{aligned} \tag{8}$$

Here φ/C is the usual comma-category with as objects pairs $(D, \alpha : \varphi(D) \rightarrow C)$ and $\omega_C : \varphi/C \rightarrow \mathbb{D}$ is the forgetful functor; the Hom at the end of (8) is taken in the category $\underline{\mathbf{Ab}}(\mathbb{D})$. We note that, by the adjunction $\text{Hom}(\varphi^* A, B) \cong \text{Hom}(A, \varphi_* B)$ and the fact that φ^* is left exact, it follows that φ_* preserves injectives.

Observe that, for the special case where φ is the functor $\varphi : \mathbb{C} \rightarrow 1$ into the trivial category 1 , one has $\underline{\mathbf{Ab}}(1) = \underline{\mathbf{Ab}}$ and $\psi_* = \Gamma : \underline{\mathbf{Ab}}(\mathbb{C}) \rightarrow \underline{\mathbf{Ab}}$, defined in (2) above. For an arbitrary functor $\varphi : \mathbb{D} \rightarrow \mathbb{C}$, the composition of the functors

$$\underline{\mathbf{Ab}}(\mathbb{D}) \xrightarrow{\varphi_*} \underline{\mathbf{Ab}}(\mathbb{C}) \xrightarrow{\Gamma} \underline{\mathbf{Ab}} \tag{9}$$

is (naturally isomorphic to) $\Gamma : \underline{\mathbf{Ab}}(\mathbb{D}) \rightarrow \underline{\mathbf{Ab}}$. Thus, by [G, p 148; HS, p 299], there is a Grothendieck spectral sequence $E_2^{p,q} = R^p\Gamma(R^q\varphi_*(B)) \Rightarrow R^{p+q}\Gamma(B)$ for this composition of functors. By (3) above, this spectral sequence can be written as

$$H^p(\mathbb{C}, R^q\varphi_*(B)) \Rightarrow H^{p+q}(\mathbb{D}, B). \tag{10}$$

The coefficient system $R^q\varphi_*(B) : \mathbb{C}^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$, occurring in (10), can equivalently be described, for each object $C \in \mathbb{C}$, by

$$R^q\varphi_*(B)(C) \cong H^q(\varphi/C, \omega_C^*(B)). \tag{11}$$

Indeed, using the uniqueness of derived functors, this follows easily from the description of $\varphi_*(B)$ as $H^0(\varphi/-, \omega_-^*(B))$ in (8). Thus the Grothendieck spectral sequence can be written in this context as

$$H^p(\mathbb{C}, H^q(\varphi/-, \omega_-^*(B))) \Rightarrow H^{p+q}(\mathbb{D}, B). \tag{12}$$

To conclude this appendix, we remark that it readily follows from the explicit constructions that this Grothendieck spectral sequence is natural in φ in the following sense. Consider, in addition to the functor $\varphi : \mathbb{D} \rightarrow \mathbb{C}$, a functor $\sigma : \mathbb{C}' \rightarrow \mathbb{C}$, and construct the pullback

$$\begin{array}{ccc} \mathbb{D}' & \xrightarrow{\varphi'} & \mathbb{C}' \\ \tau \downarrow & & \downarrow \sigma \\ \mathbb{D} & \xrightarrow{\varphi} & \mathbb{C}. \end{array}$$

The pair of functors τ and σ together induce, for each object $C' \in \mathbb{C}'$, an evident functor (τ/σ) making the following square commute:

$$\begin{array}{ccc} \varphi'/C' & \xrightarrow{(\tau/\sigma)} & \varphi/\sigma C' \\ \omega_{C'} \downarrow & & \downarrow \omega_{\sigma C'} \\ \mathbb{D}' & \xrightarrow{\tau} & \mathbb{D}. \end{array}$$

Thus, for each $B \in \underline{\text{Ab}}(\mathbb{D})$, there is a canonical map

$$H^q(\varphi/\sigma(C'), \omega_{\sigma(C')}^*(B)) \rightarrow H^q(\varphi'/C', \omega_{C'}^* \tau^*(B)).$$

These maps, for all C' , give a map

$$\theta : \sigma^*(H^q(\varphi/-, \omega_-^*(B))) \rightarrow H^q(\varphi'/-, \omega_-^* \tau^*(B)),$$

or equivalently (cf. (11)), a map

$$\theta : \sigma^* R^q \varphi_*(B) \rightarrow R^q \varphi'_*(\tau^* B).$$

Thus we obtain a map

$$\theta \circ \sigma^* : H^p(\mathbb{C}, H^q(\varphi/-, B)) \rightarrow H^p(\mathbb{C}', H^q(\varphi'/-, \tau^* B)),$$

between the E_2 -terms of the spectral sequences for φ and for φ' respectively. The naturality of the Grothendieck spectral sequence in this context means that these maps converge to the map τ^* , as in

$$\begin{array}{ccc} H^p(\mathbb{C}, H^q(\varphi/-, B)) & \Rightarrow & H^{p+q}(\mathbb{D}, B) \\ \downarrow \theta \circ \sigma^* & & \downarrow \tau^* \\ H^p(\mathbb{C}', H^q(\varphi'/-, \tau^* B)) & \Rightarrow & H^{p+q}(\mathbb{D}', \tau^* B). \end{array}$$

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