A Shared-Constraint Approach to Multi-leader Multi-follower Games

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Abstract

Multi-leader multi-follower games are a class of hierarchical games in which a collection of leaders compete in a Nash game constrained by the equilibrium conditions of another Nash game amongst the followers. The resulting *equilibrium problem with equilibrium constraints* is complicated by nonconvex agent problems and therefore providing tractable conditions for existence of global or even local equilibria for it has proved challenging. Consequently, much of the extant research on this topic is either model specific or relies on weaker notions of equilibria. We consider a *modified* formulation in which every leader is cognizant of the equilibrium constraints of all leaders. Equilibria of this modified game *contain* the equilibria, if any, of the original game. The new formulation has a constraint structure called *shared constraints*, and our main result shows that if the leader objectives admit a potential function, the global minimizers of the potential function over the shared constraint are equilibria of the modified formulation. We provide another existence result using fixed point theory that does not require potentiality. Additionally, local minima, B-stationary, and strong-stationary points of this minimization are shown to be local Nash equilibria, Nash B-stationary, and Nash strong-stationary points of the corresponding multi-leader multi-follower game. We demonstrate the relationship between *variational equilibria* associated with this modified shared-constraint game and equilibria of the original game from the standpoint of the multiplier sets and show how equilibria of the original formulation may be recovered. We note through several examples that such potential multi-leader multi-follower games capture a breadth of application problems of interest and demonstrate our findings on a multi-leader multi-follower Cournot game.

1 Introduction

This paper concerns multi-leader multi-follower games where multiple Stackelberg leaders participate in a simultaneous move game, multiple followers particpate in a subsequent simultaneous move game taking the strategies of the leaders as given and leaders make decisions subject to the equilibrium conditions arising from the game between followers. This follower-level equilibrium need not be unique as a function of the leader strategy profile and leaders and followers may have a continuum of strategies, whereby an equilibrium of the game between leaders is characterized by an analytically difficult problem. This problem is popularly referred to as an **e**quilibrium **p**rogram with **e**quilibrium **c**onstraints (EPEC). We are concerned with the central question of the existence of an equilibrium to this problem.

Games described above arise organically in the modeling of a sequence of clearings such as the day-ahead and real-time clearings in power markets. Increases in the computing capability have enabled "rational" firms to make decisions with longer time horizons and by taking into account explicitly the situation that would emerge in later clearings. Consequently, models of these strategic interactions require a firm to be not just strategic with respect to other firms but also cognizant of the real-time market clearing to follow [38, 37, 32]. Technically such firms must be modeled as *leaders* that participate in a Nash game subject to the equilibrium amongst another set of participants called *followers*. The resulting game is a multi-leader multi-follower game.

While such models are indeed reasonable representations of the hierarchical competitive structure, general results on existence of equilibria are scarce. Definitive statements on the existence of equilibria have been obtained

mainly for multi-leader multi-follower games with specific structure [33, 34, 37] and for models arising from specific applications [4, 24]. Furthermore, these results are reliant on the follower-level problem having a clean structure, in most cases uniqueness of its equilibrium as a function of the leader strategies, so that upon substituting this equilibrium into the leader's objectives the resulting *implicit problem* has a form amenable to analysis via standard fixed-point theorems[1]. To the contrary, the main results of this paper impose no such requirements and probe EPECs from an entirely new perspective. Our contributions are as follows.

(i) We present a *modified* formulation of multi-leader multi-follower competition in which there exists a *common (or shared) constraint* that constrains each player's optimization problem [29, 7, 18]. The *conventional* formulation (which has been analyzed by the above surveyed results) of a multi-leader multi-follower game bears a close resemblance to shared-constraint game, but it is technically not a shared-constraint game, thereby motivating the need for a modified formulation. We show that if the leader objectives admit a potential function, then any minimizer of the potential function over a shared constraint is an equilibrium of the modified game. Furthermore, equilibria of the conventional formulation are equilibria of the modified formulation. Additionally, we show that local minimizers, B-stationary points, and strong-stationary points of this potential function over the shared constraint are local Nash equilibria, Nash B-stationary points, and Nash strong-stationary points of the modified game. We further show how the structure of shared constraints can be exploited in games that do not admit potential functions via advanced fixed-point theorems.

(ii) We present a clear understanding of the relationship between equilibria associated with the two formulations. First, it can be seen that modified game is a shared-constraint game that admits at least two sets of generalized Nash equilibria of interest: (i) Equilibria of the original game; and (ii) Equilibria characterized by a "common" or consistent Lagrange multiplier that can be viewed as *variational equilibria* [18] for which existence statements are available.

At a high level, this paper is motivated by the view that the competition between multiple Stackelberg leaders is not an obscure or pathological setting and may thereby admit a mathematical model that allows for a reasonably general existence theory. The shared-constraint model is an attempt in this direction. Our modified model can be viewed as either an alternative model or it can be seen as a vehicle for developing existence statements for the conventional model.

The remainder of the paper is organized into five sections. In Section 2, we introduce the conventional formulation and survey multi-leader multi-follower games studied in practice provide some background. In Section 3 we present the modification that leads to a shared-constraint game and present existence results for it. Recovery of equilibria of the original formulation is examined in Section 4. We apply our techniques towards the analysis of a hierarchical Cournot game in Section 5 and conclude in Section 6 with a brief summary.

2 Multi-leader multi-follower games: examples and background

This section begins with a general formulation for such games and the associated equilibrium problem in Section 2.1. In Section 2.2, we discuss several examples considered in literature with the intent of noting that in a majority of these instances, the associated objective functions of the leaders admit a potential function, thereby also noting the utility of this class in practice. Section 2.3 contains a few preliminaries for the results to follow.

2.1 Conventional formulation of multi-leader multi-follower games

Let $\mathcal{N} = \{1, 2, \ldots, N\}$ denote the set of leaders. In the conventional formulation of multi-leader multi-follower games, leader $i \in \mathcal{N}$ solves a parametrized optimization problem of the following kind

$$
\begin{array}{lll}
\text{L}_i(x^{-i}, y^{-i}) & \underset{x_i, y_i}{\text{minimize}} & \varphi_i(x_i, y_i; x^{-i}) \\[1em]
& & x_i \in X_i, \\
& \text{subject to} & y_i \in Y_i, \\
& & y_i \in \text{SOL}(G(x_i, x^{-i}, \cdot), K(x_i, x^{-i})),
\end{array}
$$

---

[1]A few other lines of work have shown the solvability of stationarity conditions of the problems of the leaders [16, 32, 26, 27, 21, 36].

where,
$$x^{-i} \triangleq (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \quad \text{and} \quad (\bar{x}_i, x^{-i}) \triangleq (x_1, \ldots, x_{i-1}, \bar{x}_i, x_{i+1}, \ldots, x_N).$$

In this formulation, the leader makes two decisions: his strategy denoted $x_i \in \mathbb{R}^{m_i}$, and his *conjecture* about the equilibrium of the followers, denoted y_i. The choice of the optimal strategy x_i can vary with the precise follower equilibrium that occurs; if this equilibrium is not unique, the leader must make a conjecture about which of the several follower equilibria will actually emerge. Since this conjecture y_i influences the choice of x_i, y_i is also a *decision*. The formulation above corresponds to an *optimistic* one since the leader picks the y_i that is most favorable to him [22]. The *pessimistic* formulation (more standard amongst control theorists [3]) involves a '$\min_{x_i} \max_{y_i}$'.

The set of equilibria of the game between followers are given as $\mathrm{SOL}(G(x_i, x^{-i}, \cdot), K(x_i, x^{-i}))$, which stands for the solution set of the variational inequality (VI), $\mathrm{VI}(G(x, \cdot), K(x))$, parametrized by the tuple of leader strategies $x = (x_1, \ldots, x_N)$. For each x, we let

$$\mathcal{S}(x) \triangleq \mathrm{SOL}(G(x_i, x^{-i}, \cdot), K(x_i, x^{-i})). \tag{1}$$

Throughout we assume that K is continuous as a set-valued map and G is a continuous mapping of all variables.

The sets X_i and Y_i are assumed to be closed convex sets. For each i, objective function $\varphi_i : X \times Y \to \mathbb{R}$, where $X \triangleq \prod_{i=1}^{N} X_i$ and $Y \triangleq \prod_{i=1}^{N} Y_i$, is assumed to be continuous. Let $y = (y_1, \ldots, y_N)$ and $\Omega_i(x^{-i}, y^{-i})$ be the feasible region of $\mathrm{L}_i(x^{-i}, y^{-i})$, given by

$$\Omega_i(x^{-i}, y^{-i}) \triangleq \left\{ (x_i, y_i) \in \mathbb{R}^{n_i} \ \middle| \ \begin{array}{ll} x_i & \in X_i, \\ y_i & \in Y_i, \\ y_i & \in \mathcal{S}(x) \end{array} \right\}, \tag{2}$$

where $\mathbb{R}^n$ is the ambient space of the tuple (x_i, y_i). Notice that, $\Omega_i(x^{-i}, y^{-i})$ is in fact independent of y^{-i}. However we use this notation to maintain consistency with other notation we introduce in the context of shared constraints. Let $\Omega(x, y)$ denote the Cartesian product of $\Omega_i(x^{-i}, y^{-i})$:

$$\Omega(x, y) \triangleq \prod_{i=1}^{N} \Omega_i(x^{-i}, y^{-i}). \tag{3}$$

An important object in our analysis is the set $\mathcal{F}$ defined as

$$\mathcal{F} \triangleq \left\{ (x, y) \in \mathbb{R}^n \ \middle| \ \begin{array}{ll} x_i & \in X_i, \\ y_i & \in Y_i, \\ y_i & \in \mathcal{S}(x), \qquad i = 1, \ldots, N \end{array} \right\}. \tag{4}$$

Clearly, $\mathcal{F} = \{(x, y) \in \mathbb{R}^n : (x, y) \in \Omega(x, y)\}$. We refer to Ω as the *feasible region mapping* and denote this multi-leader multi-follower game or EPEC by $\mathscr{E}$.

Definition 2.1 (Global Nash equilibrium) *Consider the multi-leader multi-follower game* $\mathscr{E}$*. The* global Nash equilibrium*, or* equilibrium*, of* $\mathscr{E}$ *is a point* $(x, y) \in \mathcal{F}$ *that satisfies the following:*

$$\varphi_i(x_i, y_i; x^{-i}) \leq \varphi_i(u_i, v_i; x^{-i}) \quad \forall (u_i, v_i) \in \Omega_i(x^{-i}, y^{-i}), \quad i = 1, \ldots, N. \tag{5}$$

Local notions of equilibria will be defined later in the paper.

2.2 Examples of multi-leader multi-follower games

The multi-leader multi-follower game is inspired by a strategic game in economic theory referred to as a Stackelberg game [35]. In such a game, the leader is aware of the strategic consequences of the follower's reaction and employs that knowledge in making a first move. The follower observes this move and responds as per its optimization problem. An extension to this regime was provided by Sherali et al. [34] where a set of followers compete in a Cournot game while a leader makes a decision constrained by the equilibrium of this game. While multi-leader generalizations were touched upon by Okuguchi [25], Sherali [33] presented amongst the first models for multi-leader multi-follower games in a Cournot regime. A majority of multi-leader multi-follower game-theoretic models appear to fall into three broad categories. We provide a short description of the games arising in each category:

Hierarchical Cournot games: In a hierarchical Cournot game, leaders compete in a Cournot game and are constrained by the reactions of a set of followers that also compete in a Cournot game. We discuss a setting comprising of N leaders and M followers, akin to that proposed by Sherali [33]. Suppose the ith leader's decision is denoted by x_i and the follower strategies conjectured by leader i are collectively denoted by $\{y_i^f\}_{f=1}^M$ where f denotes the follower index. Given the leaders' decisions, follower f participates in a Cournot game in which it solves the following parametrized problem:

$$
\text{F}(\bar{y}^{-f}, x) \qquad
\begin{aligned}
&\underset{y^f}{\text{minimize}} && \tfrac{1}{2}c_f(y^f)^2 - y^f p\left(\bar{y} + \bar{x}\right) \\
&\text{subject to} && y_f \geq 0,
\end{aligned}
$$

where $p(.)$ denotes the price function associated with the follower Cournot game, $\tfrac{1}{2}c_f(y^f)^2$ denotes firm f's quadratic cost of production, $\bar{x} \triangleq \sum_i x_i$, $\bar{y} \triangleq \sum_f y^f$, and $\bar{y}^{-f} \triangleq \sum_{j \neq f} y^j$. Leader i solves the following parametrized problem:

$$
\text{L}_i(x^{-i}, y^{-i}) \qquad
\begin{aligned}
&\underset{x_i, y_i}{\text{minimize}} && \tfrac{1}{2}d_i x_i^2 - x_i p\left(\bar{x} + \bar{y}_i\right) \\
&\text{subject to} && y_i^f = \text{SOL}(\text{F}(\bar{y}_i^{-f}, x_i, x^{-i})), \quad \forall\, f, \\
& && x_i \geq 0,
\end{aligned}
$$

where $y_i^f \in \mathbb{R}$ is leader i's conjecture of follower f's equilibrium strategy, $y_i \triangleq \{y_i^f\}_{f=1}^M$, $\tfrac{1}{2}d_i x_i^2$ denotes the cost of production of leader i, $x^{-i} \triangleq \{x_j\}_{j \neq i}$ and $y^{-i} \triangleq \{y_j^f\}_{j \neq i, f=1}^M$. The equilibrium of the resulting multi-leader multi-follower is given by $\{(x_i, y_i)\}_{i=1}^N$ where (x_i, y_i) is a solution of $\text{L}_i(x^{-i}, y^{-i})$ for $i = 1, \ldots, N$. In this regime, under identical leader costs, Sherali [33] proved the existence and uniqueness of the associated equilibrium. More recently, DeMiguel and Xu [4] extended this result to stochastic regimes wherein the price function is uncertain and the leaders solve expected-value problems.

Spot-forward markets: Motivated by the need to investigate the role of forward transactions in power markets, there has been much interest in strategic models where firms compete in the forward market subject to equilibrium in the real-time market. Allaz and Vila [1] examined a forward market comprising of two identical Cournot firms and demonstrated that global equilibria exist in such markets. Su [37] extended these existence statements to a multi-player regime where firms need not have identical costs. In such an N-player setting, given the forward decisions of the players $\{x_i\}_{i=1}^N$, firm i solves the following parametrized problem in spot-market:

$$
\text{S}(z^{-i}, x) \qquad
\begin{aligned}
&\underset{z_i}{\text{minimize}} && c_i z_i - p\left(\bar{z}\right)(z_i - x_i) \\
&\text{subject to} && z_i \geq 0,
\end{aligned}
$$

$y_{i,j}$ is the where $c_i z_i$ is the linear cost of producing z_i units in the spot-market, $\bar{z} = \sum_j z_j$ and $p(.)$ is the price function in the spot-market. In the forward market, firm i's objective is given by its overall profit, which is given by $-p^f x_i - p(\bar{y}_i)(y_{i,i} - x_i) + c_i y_{i,i}$, where p^f denotes the price in the forward market $y_{i,j}$ is the anticipated equilibrium production by leader j and $\bar{y}_i = \sum_j y_{i,j}$. By imposing the no-arbitrage constraint that requires that $p^f = p(\bar{y}_i)$, the forward market objective reduces to $c_i y_{i,i} - p(\bar{y}_i) y_{i,i}$. Firm i's problem in the forward market is given by the following:

$$
\text{L}(x^{-i}) \qquad
\begin{aligned}
&\underset{x_i, y_i}{\text{minimize}} && c_i y_{i,i} - p(\bar{y}_i) y_{i,i} \\
&\text{subject to} && y_{i,j} \in \text{SOL}(\text{S}(\bar{y}_i^{-j}, x_i, x^{-i})), \quad \forall\, j.
\end{aligned}
$$

Note that while the spot-forward market problem is closely related to the hierarchical Cournot game, it has two key distinctions. First, leader i's cost is a function of forward and spot decisions. Second, every leader's revenue includes the revenue from the second-level spot-market sales. As a consequence, the problem cannot be reduced to the hierarchical Cournot game, as observed by Su [37]. In related work, Shanbhag, Infanger and Glynn [32] conclude the existence of local equilibria in a regime where each firm employs a conjecture of the forward price function. Finally, in a constrained variant of the spot-forward game examined by Allaz and Vila, Murphy and

Smeers [24] prove the existence of global equilibria when firm capacities are endogenously determined by trading on a capacity market and further discover that Allaz and Vila's conclusions regarding the benefits of forward markets may not necessarily hold. In electricity markets, there has been work beyond the papers mentioned above, in particular, by Henrion, Outrata, and Surowiec [14] and Escobar and Jofré [6].

We conclude this section with two observations. First, almost all of the existence results are model-specific and are not more generally applicable to the class of multi-leader multi-follower games. Second, in all of the instances surveyed above, the leader objectives admit a potential function. For instance, in hierarchical Cournot games, if the associated price functions are affine, then the resulting game is a potential multi-leader multi-follower game (cf. [23]). In the spot-forward games, the leader's objectives are dependent only on follower decisions; consequently, the payoffs are independent of competitive decisions and this can be immediately seen to be a potential multi-leader multi-follower game.

2.3 Preliminaries

This paper makes extensive use of *shared constraints* and *potential functions*. In this section we review these concepts.

2.3.1 Background on shared-constraint games

Shared-constraint games were introduced by Rosen [29] as a generalization of the classical Nash game. In a shared-constraint game, there exists a set $\mathbb{C}$ in the product space of strategies such that for any player i, and for any tuple of strategies of other players (denoted z^{-i}), the feasible strategies z_i for player i are those that satisfy $(z_i, z^{-i}) \in \mathbb{C}$. In an N-person shared-constraint Nash game with player payoffs denoted by $\{f_1, \ldots, f_N\}$, player i solves:

$$
\boxed{
\begin{array}{lll}
\mathrm{A}_i(z^{-i}) & \underset{z_i}{\text{minimize}} & f_i(z_i; z^{-i}) \\
& \text{subject to} & (z_i, z^{-i}) \in \mathbb{C}
\end{array}
}
$$

An equilibrium $z = (z_1, \ldots, z_N)$ satisfies the following:

$$
z \in \mathbb{C}, \quad f_i(z_1, \ldots, z_N) \leq f_i(z_1, \ldots, \bar{z}_i, \ldots, z_N) \quad \forall\, \bar{z}_i \text{ s.t. } (z_1, \ldots, \bar{z}_i, \ldots, z_N) \in \mathbb{C}, \quad \forall\, i \in \mathcal{N}. \tag{6}
$$

Equivalently, z is an equilibrium if $z \in \Omega^{\mathbb{C}}(z)$ and for all i

$$
f_i(z_1, \ldots, z_N) \leq f_i(z_1, \ldots, \bar{z}_i, \ldots, z_N) \quad \forall\, \bar{z}_i \in \Omega_i^{\mathbb{C}}(z^{-i}),
$$

where

$$
\Omega^{\mathbb{C}}(z) \triangleq \prod_{i=1}^{N} \Omega_i^{\mathbb{C}}(z^{-i}) \quad \text{and} \quad \Omega_i^{\mathbb{C}}(z^{-i}) \triangleq \left\{ \bar{z}_i \mid (\bar{z}_i; z^{-i}) \in \mathbb{C} \right\}.
$$

It is easy to show [18] that $z \in \mathbb{C} \iff z \in \Omega^{\mathbb{C}}(z)$

The feasible region mapping Ω defined in (3) (where $\Omega_i(x^{-i}, y^{-i})$ is the feasible region of $\mathrm{L}_i(x^{-i}, y^{-i})$) is a shared constraint if Ω has the following structure: for (x, y) in the domain of Ω,

$$
(u, v) \in \Omega(x, y) \iff (u_i, x^{-i}, v_i, y^{-i}) \in \mathcal{F} \quad \forall\, i \in \mathcal{N}. \tag{7}
$$

(Recall that $\mathcal{F}$ was defined in (4) and is the set of fixed points of Ω). It is easy to check that this condition does not hold in general for the mapping Ω, whereby $\mathscr{E}$ is in general not a shared constraint game.

Instead $\mathscr{E}$ is a *coupled constraint game* or *abstract economy* [2] where constraints of a player are dependent on the choices of other players, but it does not obey the form of (7). In such a game, an equilibrium is a point z such that

$$
z \in \prod_{i=1}^{N} \Omega_i^{\mathrm{NS}}(z^{-i}), \qquad f_i(z_1, \ldots, z_N) \leq f_i(z_1, \ldots, \bar{z}_i, \ldots, z_N) \quad \forall\, \bar{z}_i \in \Omega_i^{\mathrm{NS}}(z^{-i}), \quad \forall\, i \in \mathcal{N}.
$$

Here Ω_i^{NS} is *any* set-valued map, not necessarily of the form of a shared constraint. The key difference between Ω^{NS} and $\Omega^{\mathbb{C}}$ is that $\Omega^{\mathbb{C}}$ is completely defined by its fixed point set ($\mathbb{C}$), whereas Ω^{NS} is not. However in both

cases, the equilibrium is a point that lies in the fixed point set (given by $\cap_{i=1}^{N}\mathbb{C}_i$ for Ω^{NS}, where $\mathbb{C}_i$ is the graph of Ω_i^{NS}). The shared-constraint game is a special case of this with $\mathbb{C}_i = \mathbb{C}_j = \mathbb{C} = \cap_k \mathbb{C}_k$ for all i,j,k.

Shared constraint games arise naturally when players face a common constraint, e.g. in a bandwidth sharing game, and are an area of flourishing recent research; see [8, 18]. Less is known in literature about coupled constraint games without shared constraint even with convex constraints. On the contrary, much has been said about shared constraint games when the common constraint $\mathbb{C}$ is convex (see particularly, the works of Rosen [29], Facchinei et al. [7], Kulkarni and Shanbhag [18, 19, 17] and Facchinei and Pang [10]).

2.3.2 Potential games

Potential games were introduced by Monderer and Shapley [23]. In the context of $\mathscr{E}$, we say

Definition 2.2 (Potential game) *A multi-leader multi-follower game $\mathscr{E}$ where leaders have objective functions $\varphi_i, i \in \mathcal{N}$ is a potential game if there exists a function π, called potential function, such that for all $i \in \mathcal{N}$, for all $(x_i, x^{-i}) \in X, (y_i, y^{-i}) \in Y$ and for all $x_i' \in X_i, y_i' \in Y_i$*

$$\varphi_i(x_i, y_i; x^{-i}, y^{-i}) - \varphi_i(x_i', y_i'; x^{-i}, y^{-i}) = \pi(x_i, y_i; x^{-i}, y^{-i}) - \pi(x_i', y_i'; x^{-i}, y^{-i}). \tag{8}$$

If φ_i is a continuously differentiable function for $i = 1, \ldots, N$, then it follows [23] that π is continuously differentiable. In this case π is a potential function if and only if

$$\nabla_i \varphi_i(x_i, y_i; x^{-i}, y^{-i}) = \nabla_i \pi(x_i, y_i; x^{-i}, y^{-i}) \qquad \forall\, x, y, \forall\, i, \tag{9}$$

where $\nabla_i = \frac{\partial}{\partial(x_i, y_i)}$. i.e., if and only if the mapping

$$F \triangleq (\nabla_1 \varphi_1, \ldots, \nabla_N \varphi_N) \tag{10}$$

is integrable. The following lemma follows from a well known characterization of integrable mappings.

Proposition 2.1 *Consider a multi-leader multi-follower game in which the objective functions $\varphi_i, i \in \mathcal{N}$ of the leaders are continuously differentiable. Then the game is a potential game if and only if for all $(x, y) \in X \times Y$, the Jacobian $\nabla F(x, y)$ is a symmetric matrix.*

3 Existence statements for the shared-constraint formulation

In this section, we present a shared constraint modification of the conventional formulation and existence results for it. We begin with an illustrative example in Section 3.1 and provide a general formulation in Section 3.2. Sufficiency conditions for the existence of global and Nash-stationary equilibria are derived in Sections 3.3 and 3.4, respectively. The section concludes with Section 3.5 which provides an analysis of existence of global equilibria via fixed-point theory.

3.1 Motivation: The Pang and Fukushima example [28]

To motivate the modified model we recall the example Pang and Fukushima [28] presented to make the point that even simple multi-leader multi-follower games may not admit equilibria in pure strategies. We then analyze a modified version of this example that captures the spirit of the modified formulation we present.

Example 3.1. A modified version of the Pang and Fukushima example [28]: Pang and Fukushima consider a multi-leader multi-follower game comprising of two leaders and one follower [28]. The follower is solves the optimization problem

$$\min_{y \geq 0} \left\{ y(-1 + x_1 + x_2) + \tfrac{1}{2}y^2 \right\} = \max\left\{0, 1 - x_1 - x_2\right\}$$

Leaders solve the following optimization problems.

$L_1(x_2)$	$\underset{x_1, y_1}{\text{minimize}}$	$\varphi_1(x_1, y_1) = \frac{1}{2}x_1 + y_1$	$L_2(x_1)$	$\underset{x_2, y_2}{\text{minimize}}$	$\varphi_2(x_2, y_2) = -\frac{1}{2}x_2 - y_2$
	subject to	$x_1 \in [0,1]$ $y_1 = \max\{0, 1 - x_1 - x_2\}$		subject to	$x_2 \in [0,1]$ $y_2 = \max\{0, 1 - x_1 - x_2\}$

where $X_1 = X_2 = [0,1]$ and $Y = \mathbb{R}$. By substituting for y_1 (respectively, y_2), we find that $\mathrm{L}_1(x_2)$ is a convex problem for any x_2 but $\mathrm{L}_2(x_1)$ is not a convex problem in the space of x_2. Specifically, $\mathrm{L}_2(x_1)$ can be rewritten as

$$
\boxed{
\begin{aligned}
\mathrm{L}_2(x_1) \quad &\underset{x_2}{\text{minimize}} \quad \min\left(-\tfrac{1}{2}x_2, -1 + x_1 - \tfrac{1}{2}x_2\right) \\
&\text{subject to} \quad x_2 \in [0,1].
\end{aligned}
}
$$

The reaction maps $\mathcal{R}_1 : X_2 \to X_1, \mathcal{R}_2 : X_1 \to X_2$ that capture the best response for players $1, 2$ in the (x_1, x_2) space are given by the following:

$$
\mathcal{R}_1(x_2) = \{1 - x_2\} \ \ \forall x_2 \in [0,1] \qquad \text{and} \qquad \mathcal{R}_2(x_1) = \begin{cases} \{0\} & x_1 \in [0, \tfrac{1}{2}) \\ \{0,1\} & x_1 = \tfrac{1}{2} \\ \{1\} & x_2 \in (\tfrac{1}{2}, 0]. \end{cases}
$$

It is easy to see that $\mathcal{R} \triangleq \mathcal{R}_1 \times \mathcal{R}_2$ has no fixed point whereby this game has no equilibrium. Finally, note that this is game is a potential game and it admits a potential function in the (x, y) space given by:

$$
\pi(x, y) = \varphi_1(x_1, y_1) + \varphi_2(x_2, y_2) = \tfrac{1}{2}x_1 + y_1 - \tfrac{1}{2}x_2 - y_2.
$$

Consider the following modification of this example. Leader 1 has an additional constraint, '$y_2 = \max\{0, 1 - x_1 - x_2\}$', which in the original problem appeared in leader 2's optimization problem. Likewise, leader 2 now has an additional constraint '$y_1 = \max\{0, 1 - x_1 - x_2\}$', which in the original problem, was in leader 1's optimization problem. More specifically,

- *both leaders* are constrained by *both equilibrium constraints*;

- Leader i's problem is parametrized the decisions of *rival leaders* (denoted by x^{-i}) and *the other leader's conjectures about the follower equilibrium* (denoted by y^{-i}).

$$
\boxed{
\begin{aligned}
\mathrm{L}_1(x_2, y_2) \quad &\underset{x_1, y_1}{\text{minimize}} \quad \varphi_1(x_1, y_1) = \tfrac{1}{2}x_1 + y_1 \\
&\qquad\qquad x_1 \in [0,1] \\
&\text{subject to} \quad y_1 = \max\{0, 1 - x_1 - x_2\} \\
&\qquad\qquad y_2 = \max\{0, 1 - x_1 - x_2\}
\end{aligned}
}
\qquad
\boxed{
\begin{aligned}
\mathrm{L}_2(x_1, y_1) \quad &\underset{x_2, y_2}{\text{minimize}} \quad \varphi_2(x_2, y_2) = -\tfrac{1}{2}x_2 - y_2 \\
&\qquad\qquad x_2 \in [0,1] \\
&\text{subject to} \quad y_1 = \max\{0, 1 - x_1 - x_2\} \\
&\qquad\qquad y_2 = \max\{0, 1 - x_1 - x_2\}
\end{aligned}
}
$$

We claim that $((x_1, x_2), (y_1, y_2)) = ((0, 1), (0, 0))$ is an equilibrium of this modified game. To see why this is true, observe that Leader 1 gets $\varphi_1(0, 0) = 0$ whereas leader 2 gets $\varphi_2(1, 0) = -\tfrac{1}{2}$. Leader 1's global minimum is 0 and he thus has no incentive to deviate from this strategy. Leader 2's strategy set at equilibrium reduces to a singleton containing only his equilibrium strategy. This is induced by the presence of leader 1's equilibrium constraint in his optimization problem (the constraint $y_1 = \max\{0, 1 - x_1, x_2\}$ is, at equilibrium, equivalent to $0 = \max\{0, 1 - x_2\}$; together with the constraint $x_2 \in [0,1]$ this implies $x_2 = 1$ and $y_2 = 0$.) $\qquad\square$

In the following section we generalize the approach adopted in this example. The modified game has shared constraints even while the original does not. We show (Theorem 3.2) that a potential game with shared constraints admits an equilibrium under mild conditions. Indeed global minimizers of the potential function over the shared constraint are equilibria of this game. In this game, the shared constraint is given by the set

$$
\mathcal{F}^{\mathrm{ae}} = \left\{ (x_1, x_2, y_1, y_2) \ \middle| \ (x_1, x_2) \in [0,1]^2, (y_1, y_2) \geq 0, \ \begin{array}{l} y_1 = \max\{0, 1 - x_1 - x_2\} \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array} \right\}.
$$

We will explain the notation $\mathcal{F}^{\mathrm{ae}}$ in the following sections. The global minimizer of π over $\mathcal{F}^{\mathrm{ae}}$ is

$$
\arg\min_{(x,y) \in \mathcal{F}^{\mathrm{ae}}} \tfrac{1}{2}x_1 + y_1 - \tfrac{1}{2}x_2 - y_2 = ((0,1), (0,0)),
$$

which is indeed the equilibrium.

3.2 Modification: Leaders sharing all equilibrium constraints

Consider the formulation in which the ith leader solves the following optimization problem.

$$
\begin{array}{lll}
\mathrm{L}_i^{\mathrm{ae}}(x^{-i}, y^{-i}) & \underset{x_i, y_i}{\text{minimize}} & \varphi_i(x_i, y_i; x^{-i}) \\
& & x_i \in X_i, \\
& \text{subject to} & y_i \in Y_i, \\
& & y_j \in \mathcal{S}(x), \qquad j = 1, \ldots, N.
\end{array}
$$

We denote this game by $\mathscr{E}^{\mathrm{ae}}$ and note that the difference between $\mathscr{E}^{\mathrm{ae}}$ and $\mathscr{E}$ is that *all* constraints $y_j \in \mathcal{S}(x)$, $j = 1, \ldots, N$ are now a part of *each* leader's optimization problem. In effect, each leader takes into account the conjectures regarding the follower equilibrium made by all other leaders. The result is that for any i, y_i satisfies the same constraints in problems L_i and $\mathrm{L}_i^{\mathrm{ae}}$, but x_i is constrained by additional constraints in $\mathrm{L}_i^{\mathrm{ae}}$.

For $y_j \in Y_j, x_j \in X_j$ for $j \neq i$, let $\Omega_i^{\mathrm{ae}}(x^{-i}, y^{-i})$ be the feasible region of $\mathrm{L}_i^{\mathrm{ae}}(x^{-i}, y^{-i})$ and let $\Omega^{\mathrm{ae}}, \mathcal{F}^{\mathrm{ae}}, \mathcal{S}^N$ and $\mathcal{G}$ be defined as

$$
\Omega^{\mathrm{ae}}(x, y) \triangleq \prod_{i=1}^{N} \Omega_i^{\mathrm{ae}}(x^{-i}, y^{-i}), \qquad\qquad \mathcal{F}^{\mathrm{ae}} \triangleq \{(x, y) \mid (x, y) \in \Omega^{\mathrm{ae}}(x, y)\}, \qquad (11)
$$

$$
\mathcal{S}^N(x) \triangleq \prod_{i=1}^{N} \mathcal{S}(x), \qquad\qquad \mathcal{G} \triangleq \{(x, y) \mid y \in \mathcal{S}^N(x)\}, \qquad (12)
$$

where $\mathcal{G}$ is the graph of $\mathcal{S}^N$ and $\mathcal{F}^{\mathrm{ae}}$ is the set of fixed points of Ω^{ae}. An equilibrium of $\mathscr{E}^{\mathrm{ae}}$ is a point

$$
(x, y) \in \mathcal{F}^{\mathrm{ae}}, \text{ such that } \quad \varphi_i(x_i, y_i; x^{-i}) \leq \varphi_i(\bar{x}_i, \bar{y}_i; x^{-i}) \qquad \forall\, (\bar{x}_i, \bar{y}_i) \in \Omega_i^{\mathrm{ae}}(x^{-i}, y^{-i}), \forall i.
$$

Proposition 3.1 *Consider the multi-leader multi-follower game defined by $\mathscr{E}^{\mathrm{ae}}$. Then the following hold:*

(i) The mapping $\Omega^{\mathrm{ae}}(x, y)$ is a shared constraint mapping satisfying (7);

(ii) A point (x, y) is a fixed point of Ω^{ae} if and only if it is a fixed point of Ω. i.e., $\mathcal{F} = \mathcal{F}^{\mathrm{ae}}$;

(iii) Every equilibrium of $\mathscr{E}$ is an equilibrium of $\mathscr{E}^{\mathrm{ae}}$.

(iv) $\mathcal{F}^{\mathrm{ae}}$ is a closed set.

Proof :

(i) It can be seen that for any i and any x^{-i}, y^{-i}, where $x_j \in X_j, y_j \in Y_j$ for all $j \neq i$, we have that

$$
\begin{aligned}
\Omega_i^{\mathrm{ae}}(x^{-i}, y^{-i}) &= \{x_i, y_i \mid x_i \in X_i, y_i \in Y_i, y_j \in \mathcal{S}(x) \text{ for } j = 1, \ldots, N\} \\
&= \{x_i, y_i \mid x_i \in X_i, y_i \in Y_i, y \in \mathcal{S}^N(x)\}.
\end{aligned}
$$

But $y_j \in Y_j, x_j \in X_j$ for $j \neq i$, implying that

$$
\begin{aligned}
\{x_i, y_i \mid x_i \in X_i, y_i \in Y_i, y \in \mathcal{S}^N(x)\} &= \{x_i, y_i \mid x_i \in X_i, y_j \in Y_j \text{ for } j = 1, \ldots, N, y \in \mathcal{S}^N(x)\} \\
&= \{x_i, y_i \mid x \in X, y \in Y, (x, y) \in \mathcal{G}\},
\end{aligned}
$$

where $\mathcal{G}$ is defined in (12). Thus Ω^{ae} is a shared constraint of the form dictated by (7).

(ii) It suffices to show that $\mathcal{F} = \mathcal{F}^{\mathrm{ae}}$. But, from (i) it follows that $\mathcal{F}^{\mathrm{ae}} = (X \times Y) \cap \mathcal{G}$. It is easy to see from the definition of $\mathcal{F}$ that $\mathcal{F} = (X \times Y) \cap \mathcal{G}$. The result follows.

(iii) An equilibrium (x, y) of $\mathscr{E}$ lies in $\mathcal{F}$ and thereby in $\mathcal{F}^{\mathrm{ae}}$. Since $\Omega_i^{\mathrm{ae}}(x^{-i}, y^{-i}) \subseteq \Omega_i(x^{-i}, y^{-i})$, the result follows.

(iv) The relation $y \in \mathcal{S}^N(x)$ is equivalent to the set of equations

$$\mathbf{F}^{\mathrm{nat}}(y_i; x) = 0, \quad \forall i \in \mathcal{N}$$

where $\mathbf{F}^{\mathrm{nat}}(\cdot; x)$ is the natural map [9] of $\mathrm{VI}(G(x, \cdot), K(x))$. Since K, G have been assumed continuous (cf. immediately following (1)), it follows that the zeros of $\mathbf{F}^{\mathrm{nat}}$ form a closed set.

$\blacksquare$

A special case of the conventional game $\mathscr{E}$ which is already a shared-constraint game is the case (denoted by $\mathscr{E}^{\mathrm{bl}}$) where leaders have disjoint set of followers. Effectively, each leader solves bilevel optimization problems as follows.

$\mathrm{L}_i^{\mathrm{bl}}(x^{-i})$	$\displaystyle\minimize_{x_i, y_i}$	$\varphi_i(x_i, y_i; x^{-i})$
		$x_i \in X_i,$
	subject to	$y_i \in \widehat{\mathcal{S}}_i(x_i),$
		$y_i \in Y_i.$

Since $y_i \in \widehat{\mathcal{S}}_i(x_i)$, there is no coupling of leader decisions in the constraints of leader problems. This is a special case of $\mathscr{E}$ with $\mathcal{S}(x) \equiv \prod_{i \in \mathcal{N}} \widehat{\mathcal{S}}_i(x_i)$, where $\widehat{\mathcal{S}}_i(x_i)$ is the solution of a variational inequality for each i and where the objective of leader i depends only on the equilibrium of $\widehat{\mathcal{S}}_i(x_i)$ and not on $\widehat{\mathcal{S}}_j(x_j)$ for $j \neq i$. With a slight abuse of our notation so far, we let y_i denote an element of the set $\widehat{\mathcal{S}}_i(x_i)$ and Y_i be the space of such y_i. Let Ω_i^{bl} be the feasible region of $\mathrm{L}_i^{\mathrm{bl}}(x^{-i}, y^{-i})$ and let $\mathcal{F}^{\mathrm{bl}}$ be the set of fixed points of $\Omega^{\mathrm{bl}} \triangleq \prod_{i=1}^{N} \Omega_i^{\mathrm{bl}}$. Since there is no coupling, it is easily seen that

$$\mathcal{F}^{\mathrm{bl}} = \Omega^{\mathrm{bl}} = \{(x, y) \mid x \in X, y \in \tilde{Y}, (x, y) \in \widehat{\mathcal{G}}\},$$

where $\widehat{\mathcal{G}} = \prod_{i=1}^{N} \widehat{\mathcal{G}}_i$, $\tilde{Y} \triangleq \prod_{i=1}^{N} Y_i$, and $\widehat{\mathcal{G}}_i$ is the graph of $\widehat{\mathcal{S}}_i$. If $(x_j, y_j) \in \Omega_j^{\mathrm{bl}}$, for $j \neq i$,

$$\Omega_i^{\mathrm{bl}} = \{(x_i, y_i) \mid x_i \in X_i, y_i \in Y_i, y_i \in \widehat{\mathcal{S}}_i(x_i)\} = \{(x_i, y_i) \mid (x, y) \in \mathcal{F}^{\mathrm{bl}}\}.$$

It follows that this game is a shared constraint game.

3.3   Existence of global equilibria

We now present existence results for the games $\mathscr{E}^{\mathrm{ae}}$, but since the only property we use is the shared constraint structure, our results apply also to $\mathscr{E}^{\mathrm{bl}}$. We emphasize that in our modified formulation $\mathscr{E}^{\mathrm{ae}}$, the optimization problem of each leader is indeed constrained by an equilibrium constraint – and it is thus a hard nonconvex problem in its own right.

Our main result relates the global minimizers of the following optimization problem to the equilibria of $\mathscr{E}^{\mathrm{ae}}$.

P^{ae}	$\displaystyle\minimize_{x, y}$	$\pi(x, y)$
	subject to	$(x, y) \in \mathcal{F}^{\mathrm{ae}}.$

Theorem 3.2 (Minimizers of P^{ae} and Equilibria of $\mathscr{E}^{\mathrm{ae}}$) *Let $\mathscr{E}^{\mathrm{ae}}$ be a potential multi-leader multi-follower game with a potential function π. Then any global minimizer of π over $\mathcal{F}^{\mathrm{ae}}$ is an equilibrium of $\mathscr{E}^{\mathrm{ae}}$.*

Proof : Let $(x, y) \in \mathcal{F}^{\mathrm{ae}}$ be a global minimum of π over $\mathcal{F}^{\mathrm{ae}}$. Then, for each $i \in \mathcal{N}$

$$\pi(x_i, y_i, x^{-i}, y^{-i}) - \pi(u_i, v_i, x^{-i}, y^{-i}) \leq 0 \quad \forall (u_i, v_i) : (u_i, v_i, x^{-i}, y^{-i}) \in \mathcal{F}^{\mathrm{ae}}.$$

But, $(u_i, v_i, x^{-i}, y^{-i}) \in \mathcal{F}^{\mathrm{ae}}$ if and only if $(u_i, v_i) \in \Omega_i^{\mathrm{ae}}(x^{-i}, y^{-i})$, since Ω^{ae} is a shared constraint. Using this, together with the fact that π is a potential function, we obtain that for each i

$$\varphi_i(x_i, y_i; x^{-i}, y^{-i}) - \varphi_i(u_i, v_i; x^{-i}, y^{-i}) \leq 0 \quad \forall (u_i, v_i) \in \Omega_i^{\mathrm{ae}}(x^{-i}, y^{-i}).$$

This implies that for $i = 1, \ldots, N$, given (x^{-i}, y^{-i}), the vector (x_i, y_i) lies in the set of best responses for leader i. In other words, (x, y) is an equilibrium of $\mathscr{E}^{\mathrm{ae}}$. ∎

It now follows that if the minimizer of P^{ae} exists, the game $\mathscr{E}^{\mathrm{ae}}$ admits an equilibrium.

Theorem 3.3 (Existence of equilibria of $\mathscr{E}^{\mathrm{ae}}$) *Let $\mathscr{E}^{\mathrm{ae}}$ be a potential multi-leader multi-follower game with a potential function π. Suppose $\mathcal{F}^{\mathrm{ae}}$ is a nonempty set and $\varphi_i(x)$ is a continuous function for $i = 1, \ldots, N$. If the minimizer of P^{ae} exists (for example, if either π is a coercive function on $\mathcal{F}^{\mathrm{ae}}$ or if $\mathcal{F}^{\mathrm{ae}}$ is compact), then $\mathscr{E}^{\mathrm{ae}}$ admits an equilibrium.*

Proof : It is easy to see from (8) that π is continuous. By the hypothesis of the theorem, π achieves its global minimum on $\mathcal{F}^{\mathrm{ae}}$. This could, for instance, be deduced from the coercivity of π over a nonempty set $\mathcal{F}^{\mathrm{ae}}$ or by the compactness of $\mathcal{F}^{\mathrm{ae}}$ Based on Theorem 3.2, a global minimizer of π is an equilibrium of $\mathscr{E}^{\mathrm{ae}}$ and the result follows. ∎

Remark: Recall that $\mathcal{F}^{\mathrm{ae}} = \mathcal{F}$. Therefore P^{ae} is essentially a minimization of π over $\mathcal{F}$. Furthermore, since the closedness of $\mathcal{F}$ is already established, compactness follows from the boundedness of $X \times Y$. □

If the objectives of the leaders are independent of the strategies of other leaders, the sum of the objectives is a potential function, whereby any such game is a potential game. We thus have the following corollary.

Corollary 3.4 *Consider a multi-leader multi-follower game $\mathscr{E}^{\mathrm{ae}}$ for which $\mathcal{F}^{\mathrm{ae}}$ is nonempty and $\varphi_i, i \in \mathcal{N}$ are continuous. Assume further that for each $i \in \mathcal{N}$, $\varphi_i(x_i, y_i; x^{-i}) \equiv \varphi_i(x_i, y_i)$, i.e., assume that φ_i is independent of x^{-i}. If, either the functions $\varphi_i, i \in \mathcal{N}$ are coercive or if $\mathcal{F}^{\mathrm{ae}}$ is compact, the game $\mathscr{E}^{\mathrm{ae}}$ has an equilibrium.*

Proof : If $\varphi_i(x_i, y_i; x^{-i}) \equiv \varphi_i(x_i, y_i)$ for each i, $\pi = \sum_{i \in \mathcal{N}} \varphi_i$ is a potential function. Then by Theorem 3.2, the game has an equilibrium. ∎

3.4   Existence of Nash stationary equilibria

Since $\mathcal{F}^{\mathrm{ae}} = \mathcal{F}$ is characterized by equilibrium constraints, P^{ae} is an MPEC. In this section, we relate stationary points and local minimizers P^{ae} to their equilibrium counterparts in the context of $\mathscr{E}^{\mathrm{ae}}$. These relations assume relevance because, being an MPEC, the global minimization of π over $\mathcal{F}^{\mathrm{ae}}$ is hindered by the nonconvexity of $\mathcal{F}^{\mathrm{ae}}$ as well as the possible nonconvexity of π. When solved computationally, standard nonlinear programming solvers may only produce a suitably defined stationary point of P^{ae}. Traditionally, while a range of stationarity points are considered in the context of mathematical programs with equilibrium constraints [30], we focus on the notions Bouligand stationarity, local minima, strong stationarity and second-order strong stationarity. The proofs of these results are quite similar to those from our recent submission [20]; we therefore provide only a sketch of each proof.

3.4.1   B-stationary equilibria

Definition 3.1 (Nash B-stationary point) *A point $(x, y) \in \mathcal{F}^{\mathrm{ae}}$ is a Nash B-stationary point of $\mathscr{E}^{\mathrm{ae}}$ if for all $i \in \mathcal{N}$,*
$$\nabla_i \varphi_i(x, y)^\top d \geq 0 \quad \forall d \in \mathcal{T}((x_i, y_i); \Omega_i^{\mathrm{ae}}(x^{-i})),$$
where $\mathcal{T}(z; K)$, the tangent cone at $z \in K \subseteq \mathbb{R}^n$, is defined as follows:

$$\mathcal{T}(z; K) \triangleq \left\{ dz \in \mathbb{R}^n : \exists \{\tau_k\}, \{z_k\} \text{ such that } dz = \lim_{k \to \infty} \left(\frac{z_k - z}{\tau_k} \right), K \ni z_k \to z, 0 < \tau_k \to 0 \right\}.$$

Proposition 3.5 (B-Stationary points of P^{ae} and Nash B-stationary points of $\mathscr{E}^{\mathrm{ae}}$) *Let $\mathscr{E}^{\mathrm{ae}}$ be a potential multi-leader multi-follower game with potential function π and suppose $\{\varphi_i\}_{i \in \mathcal{N}}$ are continuously differentiable functions over $X \times Y$. If (x, y) is a B-stationary point of P^{ae}, then (x, y) is a Nash B-stationary point of $\mathscr{E}^{\mathrm{ae}}$.*

Proof : A stationary point (x, y) of π over $\mathcal{F}^{\mathrm{ae}}$ satisfies

$$\nabla_x \pi(x, y)^\top dx + \nabla_y \pi(x, y)^\top dy \geq 0, \quad \forall (dx, dy) \in \mathcal{T}((x, y); \mathcal{F}^{\mathrm{ae}}). \tag{13}$$

Fix some $i \in \mathcal{N}$ and consider an arbitrary $(dx_i', dy_i') \in \mathcal{T}(x_i, y_i; \Omega_i^{\mathrm{ae}}(x^{-i}, y^{-i}))$. Let $(u_{i,k}, v_{i,k}) \in \Omega^{\mathrm{ae}}(x^{-i}, y^{-i})$, $(u_{i,k}, v_{i,k}) \overset{k}{\to} (x_i, y_i)$ and $0 < \tau_k \overset{k}{\to} 0$ such that $\frac{u_{i,k} - x_i}{\tau_k} \overset{k}{\to} dx_i'$ and $\frac{v_{i,k} - y_i}{\tau_k} \overset{k}{\to} dy_i'$. It follows that the sequence $(\mathbf{x}_{i,k}, \mathbf{y}_{i,k}) \in \mathcal{F}^{\mathrm{ae}}$, where

$$\mathbf{x}_{i,k} = (x_1, \ldots, u_{i,k}, \ldots, x_N), \text{ and } \mathbf{y}_{i,k} = (y_1, \ldots, v_{i,k}, \ldots, y_N). \tag{14}$$

Therefore, the direction $(\mathbf{dx}_i, \mathbf{dy}_i) \in \mathcal{T}(z; \mathcal{F}^{\mathrm{ae}})$ where

$$\mathbf{dx}_i = (0, \ldots, dx_i', \ldots, 0) \text{ and } \mathbf{dy}_i = (0, \ldots, dy_i', \ldots, 0).$$

Taking $(dx, dy) = (\mathbf{dx}_i, \mathbf{dy}_i)$ in (13) and using (9) we get the required result. ∎

Proposition 3.6 (Local minimum of P^{ae} and local Nash equilibrium) *Consider the multi-leader multi-follower game $\mathscr{E}^{\mathrm{ae}}$ with potential function π. If (x, y) is a local minimum of P^{ae}, then (x, y) is a local Nash equilibrium of $\mathscr{E}^{\mathrm{ae}}$.*

Proof : The proof is analogous to that of Theorem 3.2. If (x, y) is a local minimum of P^{ae}, there exists a neighborhood of (x, y), denoted by $\mathcal{B}(x, y)$, such that

$$\pi(x, y) \leq \pi(x', y'), \qquad \forall\, (x', y') \in \mathcal{B}(x, y) \cap \mathcal{F}^{\mathrm{ae}}. \tag{15}$$

Consider an arbitrary $i \in \mathcal{N}$ and let $\mathcal{B}_i(x_i, y_i; x^{-i}, y^{-i}) := \left\{ (u_i, v_i) \,|\, (u_i, v_i, x^{-i}, y^{-i}) \in \mathcal{B}(x, y) \right\}$. Then it follows that

$$(u_i, v_i) \in \left(\Omega_i^{\mathrm{ae}}(x^{-i}, y^{-i}) \cap \mathcal{B}_i(x_i, y_i; x^{-i}, y^{-i}) \right) \iff (u_i, v_i, x^{-i}, y^{-i}) \in (\mathcal{F}^{\mathrm{ae}} \cap \mathcal{B}(x, y)).$$

Thus, using this relation in (15) and employing (8), we get

$$\varphi_i(x, y) \leq \varphi_i(u_i, v_i, x^{-i}, y^{-i}), \qquad \forall\, (u_i, v_i) \in \left(\Omega_i^{\mathrm{ae}}(x^{-i}, y^{-i}) \cap \mathcal{B}_i(x_i, y_i; x^{-i}, y^{-i}) \right).$$

In other words, (x_i, y_i) is a local minimizer of $\mathrm{L}_i^{\mathrm{ae}}(x^{-i}, y^{-i})$. This holds for each $i \in \mathcal{N}$, whereby (x, y) is a local Nash equilibrium. ∎

3.4.2 Strong stationarity

Here we relate other notions of stationarity for P^{ae} with weaker equilibrium notions of $\mathscr{E}^{\mathrm{ae}}$. When the algebraic form of the constraints are available, a *strong-stationary point* can be defined. Let $X_i = \{x_i | c_i(x_i) \geq 0\}, Y_i = \{y_i | d_i(y_i) \geq 0\}$, where c_i, d_i are continuously differentiable. Let $\mathcal{S}(x)$ be the solution of a complementarity problem: $y_i \in \mathcal{S}(x) \iff 0 \leq y_i \perp G(y_i; x) \geq 0$, and G is $\mathbb{R}^p$-valued and continuously differentiable. Thus P^{ae} can be written as

$$
\begin{array}{lll}
\mathrm{P}^{\mathrm{ae}} & \underset{x,y}{\text{minimize}} & \pi(x, y) \\[2mm]
& \text{subject to} & \left\{ \begin{array}{l} c_i(x_i) \geq 0 \\ d_i(y_i) \geq 0 \\ 0 \leq y_i \perp G(y_i, x) \geq 0 \end{array} \right\} \quad i = 1, \ldots, N.
\end{array}
$$

To define the stationarity conditions, we define the relaxed nonlinear program below which requires specifying the index sets $\tilde{\mathcal{I}}_{1i}$ and $\tilde{\mathcal{I}}_{2i}$ for $i = 1, \ldots, N$ where $\tilde{\mathcal{I}}_{1i}, \tilde{\mathcal{I}}_{2i} \subseteq \{1, \ldots, p\}$ and $\tilde{\mathcal{I}}_{1i} \cup \tilde{\mathcal{I}}_{2i} = \{1, \ldots, p\}$.

$$
\begin{array}{lll}
\mathrm{P}^{\mathrm{ae}}_{rnlp} & \underset{x,y}{\text{minimize}} & \pi(x, y) \\[2mm]
& \text{subject to} & \left\{\begin{array}{ll}
c_i(x_i) \geq 0 & \\
d_i(y_i) \geq 0 & \\
[y_i]_j = 0, & \forall j \in \tilde{\mathcal{I}}_{2i}^{\perp} \\
[G(y_i, x)]_j = 0, & \forall j \in \tilde{\mathcal{I}}_{1i}^{\perp} \\
[y_i]_j \geq 0, & \forall j \in \tilde{\mathcal{I}}_{1i} \\
[G(y_i, x)]_j \geq 0, & \forall j \in \tilde{\mathcal{I}}_{2i}
\end{array}\right\} & i = 1, \ldots, N,
\end{array}
$$

where $[\cdot]_j$ denotes the jth component of '$\cdot$' and $\tilde{\mathcal{I}}_{1i}^{\perp}, \tilde{\mathcal{I}}_{2i}^{\perp}$ denote the complements of $\tilde{\mathcal{I}}_{1i}, \tilde{\mathcal{I}}_{2i}$ respectively. Further, we refer to the both index sets collectively as $\tilde{\mathcal{I}}_i$ and the collection of index sets $\{\tilde{\mathcal{I}}_1, \ldots, \tilde{\mathcal{I}}_N\}$ by $\tilde{\mathcal{I}}$. Note that in accordance with [12], we define the index sets independent of the point (x, y). We may now state the strong stationarity conditions at a particular point (x, y).

Definition 3.2 (Strong-stationarity point of P^{ae}) *A point $(x, y) \in \mathcal{F}^{\mathrm{ae}}$ is a strong stationarity point of P^{ae} if there exist Lagrange multipliers η_i, μ_i, λ_i and $\beta_i, i \in \mathcal{N}$ such that the following conditions hold:*

$$
\left\{\begin{array}{r}
\nabla_{x_i} \pi(x, y) - \nabla_{x_i} c_i(x_i)^{\top} \eta_i - \sum_{k=1}^{N} \nabla_{x_i} G(y_k, x)^{\top} \beta_k = 0 \\[3mm]
\nabla_{y_i} \pi(x, y) - \nabla_{y_i} d_i(y_i)^{\top} \mu_i - \lambda_i - \nabla_{y_i} G(y_i, x)^{\top} \beta_i = 0 \\[2mm]
0 \leq \eta_i \perp c_i(x_i) \geq 0 \\[1mm]
0 \leq \mu_i \perp d_i(y_i) \geq 0 \\[1mm]
y_i \geq 0, \\[1mm]
[\lambda_i]_j [y_i]_j = 0, \quad \forall j \\[1mm]
G(y_i, x) \geq 0, \\[1mm]
[\beta_i]_j [G(y_i, x)]_j = 0, \quad \forall j \\[1mm]
[y_i]_j = 0 \text{ or } [G(y_i, x)]_j = 0, \quad \forall j \\[1mm]
\text{if } [G(y_i, x)]_j = 0 \text{ and } [y_i]_j = 0, \text{ then } [\lambda_i]_j, [\beta_i]_j \geq 0, \quad \forall j
\end{array}\right\}, \quad \forall i \in \mathcal{N}. \tag{16}
$$

Having defined the strong stationarity conditions, we are now in a position to define the second-order sufficiency conditions. These assume relevance in defining a local Nash equilibrium; loosely speaking, at a local Nash equilibrium, every agent's decision satisfies the **m**athematical **p**rograms with **e**quilibrium **c**onstraints-second-order sufficiency or the MPEC-SOSC conditions, given the decisions of its competitors. Furthermore, corresponding to a stationary point of $\mathrm{P}^{\mathrm{ae}}_{rnlp}$, we may prescribe an active set $\tilde{\mathcal{A}}(x, y)$ such that $\tilde{\mathcal{A}}(x, y) \triangleq \{\tilde{\mathcal{A}}_1(x, y), \ldots, \tilde{\mathcal{A}}_N(x, y)\}$, where $\tilde{\mathcal{A}}_i(x, y)$ denotes the set of active constraints corresponding to the set of constraints

$$
\left\{\begin{array}{ll}
c_i(x_i) \geq 0 & \\
d_i(y_i) \geq 0 & \\
[y_i]_j = 0, & \forall j \in \tilde{\mathcal{I}}_{2i}^{\perp} \\
[G(y_i, x)]_j = 0, & \forall j \in \tilde{\mathcal{I}}_{1i}^{\perp} \\
[y_i]_j \geq 0, & \forall j \in \tilde{\mathcal{I}}_{1i} \\
[G(y_i, x)]_j \geq 0, & \forall j \in \tilde{\mathcal{I}}_{2i}
\end{array}\right\}.
$$

Suppose $\tilde{\mathcal{A}}_i(x, y) = \{\tilde{\mathcal{A}}_i^c(x, y), \tilde{\mathcal{A}}_i^d(x, y), \tilde{\mathcal{A}}_i^e(x, y)\}$, where $\tilde{\mathcal{A}}_i^c, \tilde{\mathcal{A}}_i^d$ and $\tilde{\mathcal{A}}_i^e$ denote the active sets associated with $c_i(x_i) \geq 0$, $d_i(y_i) \geq 0$, and the remaining constraints, respectively. The specification of the active set allows us to

define the critical cone $S^*(x, y)$ as

$$S^*(x,y) \triangleq \left\{ s : s \neq 0, \nabla \pi(x,y)^\top s = 0, a_j^\top s = 0, j \in \tilde{\mathcal{A}}(x,y), a_j^\top s \geq 0, j \notin \tilde{\mathcal{A}}(x,y) \right\}, \tag{17}$$

where a_j denotes the constraint gradients of the jth constraint.

Definition 3.3 (Second-order Strong-stationarity point of P^{ae}**)** *A point (x,y) of the optimization problem* P^{ae} *is a* second-order strong stationary point *of* P^{ae} *if it is a strong stationary point with Lagrange multipliers* $(\eta, \mu, \lambda, \beta)$ *and* $s^T \nabla^2_{x,y} \mathcal{L} \, s > 0$ *for* $s \in S^*(x,y)$*, where* $S^*(x,y)$ *is given by (17) and* $\nabla^2_{x,y} \mathcal{L}$ *denotes the Hessian of the Lagrangian of* $\mathrm{P}^{\text{ae}}_{rnlp}$ *with respect* (x,y) *evaluated at* $(x, y, \eta, \mu, \lambda, \beta)$*.*

Next, we provide a formal definition of *Nash strong-stationary* and *Nash second-order strong-stationary points* of $\mathscr{E}^{\text{ae}}$, which requires defining the critical cone $S^*_i(x,y)$ for each leader $i = 1, \ldots, N$:

$$S^*_i(x,y) \triangleq \left\{ s_i : s_i \neq 0, \nabla_i \varphi_i(x,y)^\top s_i = 0, a_j^\top s_i = 0, j \in \mathcal{A}_i(x,y), a_j^\top s_i \geq 0, j \notin \mathcal{A}_i(x,y) \right\}, \tag{18}$$

where $\mathcal{A}_i(x,y)^2$ denotes the active set utilized in defining the relaxed nonlinear program associated with $\mathrm{L}^{\text{ae}}_i(x^{-i}, y^{-i})$ and a_j denotes the constraint gradient associated with j^{th} constraint.

Definition 3.4 (Nash strong-stationary and Nash second-order strong-stationary points) *A point $(x,y) \in \mathcal{F}^{\text{ae}}$ is a* Nash strong-stationary point *of* $\mathscr{E}^{\text{ae}}$ *if for* $i = 1, \ldots, N$*, there exist Lagrange multipliers* $\bar{\eta}_i, \bar{\mu}_i, \bar{\lambda}_i$ *and* $\bar{\beta}^k_i, k = 1, \ldots, N$*, such that the following conditions hold:*

$$\left\{ \begin{array}{r} \nabla_{x_i} \varphi_i(x,y) - \nabla_{x_i} c_i(x_i)^\top \bar{\eta}_i - \sum_{k=1}^{N} \nabla_{x_i} G(y_k, x)^\top \bar{\beta}^k_i = 0 \\[2mm] \nabla_{y_i} \varphi_i(x,y) - \nabla_{y_i} d_i(y_i)^\top \bar{\mu}_i - \bar{\lambda}_i - \nabla_{y_i} G(y_i, x)^\top \bar{\beta}^i_i = 0 \\[2mm] 0 \leq \bar{\eta}_i \perp c_i(x_i) \geq 0 \\[2mm] 0 \leq \bar{\mu}_i \perp d(y_i) \geq 0 \\[2mm] y_i \geq 0, \\[2mm] [\bar{\lambda}_i]_j [y_i]_j = 0, \quad \forall j \\[2mm] G(y_i, x) \geq 0, \\[2mm] [\bar{\beta}^k_i]_j [G(y_i, x)]_j = 0, \quad \forall k \in \mathcal{N}, \forall j \\[2mm] [y_i]_j = 0 \text{ or } [G(y_i, x)]_j = 0, \quad \forall j \\[2mm] \text{if } [G(y_i, x)]_j = 0 \text{ and } [y_i]_j = 0, \text{ then } [\bar{\lambda}_i]_j, [\bar{\beta}^k_i]_j \geq 0, \quad \forall k \in \mathcal{N}, \forall j \end{array} \right\}, \quad \forall i \in \mathcal{N}. \tag{19}$$

Furthermore, (x,y) is a Nash second-order strong stationary point *of* $\mathscr{E}^{\text{ae}}$ *if (x,y) is a Nash strong stationary point of* $\mathscr{E}^{\text{ae}}$ *and if for* $i = 1, \ldots, N$*,* $s_i^T \nabla^2_{x_i, y_i} \mathcal{L}_i(x,y) s_i > 0$ *for* $s_i \in S^*_i(x,y)$ *where* $S^*_i(x,y)$ *is given by (18), where* $\nabla^2_{x_i, y_i} \mathcal{L}_i$ *denotes the Hessian of the Lagrangian function of* $\mathrm{L}^{\text{ae}}_i(x^{-i}, y^{-i})$ *with respect to* (x_i, y_i) *evaluated at* $(x_i, y_i, \bar{\eta}_i, \bar{\mu}_i, \bar{\lambda}_i, \bar{\beta}^1_i, \ldots, \bar{\beta}^N_i)$ *if* $\mathscr{E}^{\text{ae}} = \mathscr{E}^{\text{ae}}$ *or at* $(x_i, y_i, \bar{\eta}_i, \bar{\mu}_i, \bar{\lambda}_i, \bar{\beta}_i)$ *if* $\mathscr{E}^{\text{ae}} \in \{\mathscr{E}^{\text{ind}}, \mathscr{E}^{\text{bl}}\}$ *or at* $(x_i, y_i, \bar{\eta}_i, \bar{\mu}_i, \bar{\lambda}_i, \bar{\gamma}_i, \bar{\beta}_i)$ *if* $\mathscr{E}^{\text{ae}} = \mathscr{E}^{\text{cc}}$*.*

Having defined the relevant objects, we now show that a strong-stationary point of P^{ae} is a Nash strong-stationary point of $\mathscr{E}^{\text{ae}}$ and a second-order strong-stationary point of P^{ae} is a second-order strong-stationary point of $\mathscr{E}^{\text{ae}}$. For $i = 1, \ldots, N$, one may define a corresponding relaxed NLP associated with the ith leader's problem, namely $\mathrm{L}^{\text{ae}}_i(x^{-i}, y^{-i})$, by employing the index sets $\mathcal{I}_i$. These index sets are defined using $\tilde{\mathcal{I}}$ and are given by[3] $\mathcal{I}_i = \{\tilde{\mathcal{I}}_1, \ldots, \tilde{\mathcal{I}}_N\}$.

---

[2]The active set associated with $\mathrm{P}^{\text{ae}}_{rnlp}$ is denoted by $\tilde{\mathcal{A}}$ while the active set associated with leader i's problems is denoted by $\tilde{\mathcal{A}}_i$.

[3]The index sets associated with $\mathrm{P}^{\text{ae}}_{rnlp}$ are denoted by $\tilde{\mathcal{I}}$ while the index sets employed for specifying leader i's relaxed NLP are denoted by $\mathcal{I}_i$. Note that the cardinality of $\tilde{\mathcal{I}}_i$ and $\mathcal{I}_i$ differs when considering the relaxed NLPs corresponding to $\mathscr{E}^{\text{ae}}$ since every leader level problem contains equilibrium constraints of all the leaders.

Proposition 3.7 [*Strong stationary points of* $\mathrm{P^{ae}}$ *and Nash strong stationary points of* $\mathscr{E}^{\mathrm{ae}}$] *Consider the multi-leader multi-follower game with shared constraints* $\mathscr{E}^{\mathrm{ae}}$. *Suppose* (x, y) *is a strong-stationary point of* $\mathrm{P^{ae}}$ *and satisfies* (16) *with Lagrange multipliers* $(\eta_i, \mu_i, \lambda_i, \beta_i)_{i=1}^N$. *Then* (x, y) *is a Nash strong-stationary point of* $\mathscr{E}^{\mathrm{ae}}$ *and for* $i = 1, \ldots, N$, (x, y) *satisfies* (19) *with Lagrange multipliers defined as* $\left(\bar{\eta}_i, \bar{\mu}_i, \bar{\lambda}_i, (\bar{\beta}_i^k)_{k=1}^N\right) = \left(\eta_i, \mu_i, \lambda_i, (\beta_k)_{k=1}^N\right)$ *Furthermore, if* (x, y) *is a second-order strong stationary point of* $\mathrm{P^{ae}}$ *with multipliers* $(\eta_i, \mu_i, \lambda_i, \beta_i)_{i=1}^N$, *then* (x, y) *is a Nash second-order strong stationary point of* $\mathscr{E}^{\mathrm{ae}}$ *with firm* i's *multipliers given by* $\left(\bar{\eta}_i, \bar{\mu}_i, \bar{\lambda}_i, (\bar{\beta}_i^k)_{k=1}^N\right)$.

Proof : Suppose (x, y) is a strong stationary point $\mathrm{P^{ae}}$, i.e., suppose there exist multipliers η, μ, λ and β such that for (x, y), system (16) holds. For each kind of $\mathscr{E}^{\mathrm{ae}}$, we show that (x, y) is a Nash strong stationary point of $\mathscr{E}^{\mathrm{ae}}$. One may then construct Lagrange multipliers to satisfy (19). By comparison of (16) and (19), we see that (19) admits a solution (x, y) with multipliers $\bar{\eta}_i = \eta_i, \bar{\mu}_i = \mu_i, \bar{\lambda}_i = \lambda_i$ and $\bar{\beta}_i^k = \beta_k$ for all i, k.

Now assume that (x, y) is a second-order strong stationary point of $\mathrm{P^{ae}}$. To show that (x, y) is a Nash second-order strong stationary point of $\mathscr{E}^{\mathrm{ae}}$, we construct Lagrange multipliers as above. It is easy to see, that by construction, $\nabla_{x_i, y_i} \mathcal{L} = \nabla_{x_i, y_i} \mathcal{L}_i$ and $\nabla^2_{x_i, y_i} \mathcal{L}_i = \nabla^2_{x_i, y_i} \mathcal{L}$ for all $i \in \mathcal{N}$, where $\mathcal{L}_i$ is the Lagrangian of $\mathrm{L}_i^{\mathrm{ae}}$ evaluated at (x, y) and the above constructed Lagrange multipliers. Furthermore, by comparing the feasible region of $\mathrm{L}_i^{\mathrm{ae}}$ with $\mathcal{F}^{\mathrm{ae}}$, we observe that the active sets of $\mathrm{L}_i^{\mathrm{ae}}$ can be defined as $\mathcal{A}_i(x, y) = \left\{ \tilde{\mathcal{A}}_i^c(x, y), \tilde{\mathcal{A}}_i^d(x, y), \tilde{\mathcal{A}}_1^e(x, y), \ldots, \tilde{\mathcal{A}}_N^e(x, y) \right\}$.

Given the specification of the active set, we may now define a relaxed NLP corresponding to this active set as well as define the corresponding critical cone $S_i^*(x, y)$. To prove the claim, we proceed by contradiction. If (x, y) is not a Nash second-order strong stationary point, then for some $i \in \{1, \ldots, N\}$, the point (x_i, y_i) does not satisfy second-order strong stationary conditions, given (x^{-i}, y^{-i}). Then there exists a w_i such that $w_i \in S_i^*(x, y)$ such that $w_i^\top \nabla^2_{x_i, y_i} \mathcal{L}_i^* w_i \leq 0$. We may now define $\mathbf{w}$ such that

$$\mathbf{w} \triangleq (w_1, \ldots, w_N),$$

where $w_j = 0, j \neq i$. Since $w_i \in S_i^*(x, y)$, it follows that

$$0 = w_i^\top \nabla_{x_i, y_i} \varphi_i(x, y) = w_i^\top \nabla_{x_i, y_i} \pi(x, y).$$

By definition of $\mathbf{w}$, it follows that $\mathbf{w}^\top \nabla_{x, y} \pi(x, y) = 0$. From the definition of $\mathbf{w}$ and by noting the constructions of $\mathcal{A}_i(x, y)$, it can be seen that $\mathbf{w} \in S^*(x, y)$. As a consequence, we have that

$$0 \geq w_i^T \nabla^2_{x_i, y_i} \mathcal{L}_i w_i = \mathbf{w}^T \nabla^2_{x_i, y_i} \mathcal{L} \ \mathbf{w}.$$

But this contradicts the hypothesis that (x, y) is a second-order strong stationary point of $\mathrm{P^{ae}}$ and the result follows. ∎

Notice that the form of equilibrium constraints was only used when considering stationarity concepts. The global equilibrium results did not use the explicit form of the equilibrium constraints and as such are applicable even for extensive form games.

As a final note, we recall it is not entirely necessary to employ the algebraic characterization of the constraints in articulating strong stationarity (cf. [11, 13]). For instance, the authors examine the optimality conditions of a disjunctive program defined as

$$
\begin{aligned}
\min_x \quad & f(x) \\
\text{subject to} \quad & x \in \Lambda \triangleq \bigcup_{i=1}^m \Lambda_i,
\end{aligned}
\tag{20}
$$

where Λ_i is a convex polyhedron for $i = 1, \ldots, m$. Such a problem captures most MPEC models considered in the research literature. The authors proceed to show that if the generalized Guignard constraint qualification and a suitably defined intersection property holds at a local minimizer z, then z is a strong stationary point. Note that the definition of strong stationarity relies on using the Frećhet normal cone associated with Λ_i rather than the algebraic characterization of the sets.

3.5 Existence of global equilibria via fixed-point arguments

The reaction map of the multi-leader multi-follower game $\mathscr{E}$ does not have the properties required for applying fixed point theorems. However, the modified formulation $\mathscr{E}^{\mathrm{ae}}$, because of its shared constraint structure, allows for the construction of a *modified reaction map* whose fixed points are equilibria of $\mathscr{E}^{\mathrm{ae}}$ and has properties that are more favorable for the application of fixed point theory. This leads to an existence result for $\mathscr{E}^{\mathrm{ae}}$ that uses fixed point theory and does not assume the existence of a potential function. We touch upon this topic in this section.

To define the modified reaction map let $\Psi : (X \times Y) \times (X \times Y) \to \mathbb{R}$ be given by

$$\Psi(x, y, \bar{x}, \bar{y}) \triangleq \sum_{i=1}^{N} \varphi_i(\bar{x}_i, \bar{y}_i; x^{-i}) \qquad \forall (x, y), (\bar{x}, \bar{y}) \in X \times Y. \tag{21}$$

and consider the *modified reaction map* $\Upsilon^{\mathrm{ae}} : X \times Y \to 2^{\mathcal{F}^{\mathrm{ae}}}$, defined as

$$\Upsilon^{\mathrm{ae}}(x, y) \triangleq \left\{ (\bar{x}, \bar{y}) \in \mathcal{F}^{\mathrm{ae}} \mid \Psi(x, y, \bar{x}, \bar{y}) = \inf_{(u,v) \in \mathcal{F}^{\mathrm{ae}}} \Psi(x, y, u, v) \right\}. \tag{22}$$

We show below that a fixed point of Υ^{ae} is an equilibrium of $\mathscr{E}^{\mathrm{ae}}$. The map Υ^{ae} is analogous to that used by Rosen [29, Theorem 1].

Theorem 3.8 (Fixed points of Υ^{ae} and equilibria of $\mathscr{E}^{\mathrm{ae}}$) *Consider the game multi-leader multi-follower game $\mathscr{E}^{\mathrm{ae}}$ with a feasible region mapping Ω^{ae}. If Υ^{ae} admits a fixed point, the game $\mathscr{E}^{\mathrm{ae}}$ admits an equilibrium.*

Proof : Assume that the claim is false, i.e., assume there exists an $(x, y) \in \Upsilon^{\mathrm{ae}}(x, y)$ such that for some $(u, v) \in \Omega^{\mathrm{ae}}(x, y)$ and an index $i \in \{1, \ldots, N\}$ we have

$$\varphi_i(u_i, v_i; x^{-i}) < \varphi_i(x_i, y_i; x^{-i}).$$

Since $(u, v) \in \Omega^{\mathrm{ae}}(x, y)$, and since Ω^{ae} satisfies (7), we must have $(u_i, x^{-i}, v_i, y^{-i}) \in \mathcal{F}$. But this means

$$\Psi(x, y, (u_i, x^{-i}), (v_i, y^{-i})) < \Psi(x, y, x, y),$$

a contradiction to $(x, y) \in \Upsilon^{\mathrm{ae}}(x, y)$. ■

We further have that Υ^{ae} is upper semicontinuous under mild conditions.

Lemma 3.1 *Let Ψ be continuous on $X \times Y$ and assume that $X \times Y$ is compact. Then Υ^{ae} is upper semicontinuous. If Υ^{ae} is single-valued, then it is continuous (as a single-valued function).*

Proof : By compactness of $X \times Y$ the infimum in the definition of Υ^{ae} is achieved. Upper semi-continuity of Υ^{ae} follows from classical stability results (see e.g., Hogan [15]). The last claim follows as a special case of upper semicontinuity of set-valued maps for single-valued maps. ■

By using the Eilenberg-Montgomery fixed point theorem [5], we obtain an existence result.

Theorem 3.9 *Consider the multi-leader multi-follower game $\mathscr{E}^{\mathrm{ae}}$ where the objective functions $\varphi_i, i \in \mathcal{N}$ are continuous. Suppose $X \times Y$ is nonempty, compact and convex. Suppose if Υ^{ae} satisfies one of the following:*

(i) Single-valued on $X \times Y$;

(ii) Multi-valued on $X \times Y$ with contractible images.

Then Υ^{ae} admits a fixed point and $\mathscr{E}^{\mathrm{ae}}$ admits an equilibrium.

Proof : Υ^{ae} may be taken to be a mapping from the compact convex set $X \times Y$ to subsets of $X \times Y$. If Υ^{ae} is single-valued, Lemma 3.1 implies that it is continuous. Consequently, by Brouwer's fixed point theorem there exists a fixed point of Υ. If Υ^{ae} is multi-valued, then Lemma 3.1 shows that Υ^{ae} is upper semicontinous. Then since Υ^{ae} is contractible-valued, by the Eilenberg-Montgomery fixed point theorem [5, Theorem 1], there exists a

fixed point of Υ^{ae}. In each of these cases, since there exists a fixed point of Υ^{ae}, from Theorem 3.8, $\mathscr{E}^{\mathrm{ae}}$ admits an equilibrium. ∎

A natural question is when such conditions are useful. In general, convexity or contractibility of images of Υ^{ae} is not immediate; however, if there are specific settings where such claims can be made, then the aforementioned results are powerful in that they do not require leader payoffs to admit potential functions. It is not true that every equilibrium of the multi-leader multi-follower game with shared constraints is a fixed point of Υ^{ae}; existence of a fixed point to Υ^{ae} is only a sufficient condition for such an equilibrium to exist. This can be checked easily by considering a hypothetical case with convex $\mathcal{F}^{\mathrm{ae}}$, wherein it is well known that fixed points of the reaction map and the modified reaction map can be very different. In [18], Kulkarni and Shanbhag discuss these issues in detail for convex shared-constraint games; in general, there exist equilibria that are not fixed points of Υ^{ae} and also games for which there are equilibria, but no fixed points to Υ^{ae}.

Remark 3.1. **(Relationship to variational equilibria in convex shared constraint games)** When $\mathcal{F}^{\mathrm{ae}}$ is convex, and $\varphi_i(x_i; x^{-i})$ is convex in x_i for all x^{-i} and all i, Lemma 3.8 and the map Υ^{ae} also has an interesting connection with the "variational equilibrium" [8, 10, 18] in games with convex shared constraints. In this setting, these games are typically referred to as generalized Nash games and equilibria of such games are referred to as generalized Nash equilibria (GNE). The variational equilibrium is defined as the solution of the variational inequality VI$(\mathcal{F}^{\mathrm{ae}}, F)$, where $F = (\nabla_1 \varphi_1, \ldots, \nabla_N \varphi_N)$. By convexity, it can be easily seen that VI$(\mathcal{F}^{\mathrm{ae}}, F)$ equals the set of fixed points of Υ^{ae}. The variational equilibrium is the generalized Nash equilibrium at which the Lagrange multipliers, corresponding to the shared constraints, are identical across players. These multipliers can be interpreted as the shadow prices of the associated constraints. Furthermore, when these prices are equal, the equilibria can be viewed as corresponding to a uniform auction price while disparities in prices are a consequence of discriminatory prices. The above observations form the basis of a detailed study of the VE and the GNE [18] where we show that under general conditions, if a GNE exists, a VE also exists, in which case the VE is said to be a *refinement* of the GNE [31, 3]. Furthermore, for potential games with potential function π, $F \equiv \nabla \pi$, whereby VI$(\mathcal{F}^{\mathrm{ae}}, F)$ is equivalent to VI$(\mathcal{F}^{\mathrm{ae}}, \nabla \pi)$. Thus in a potential game with shared constraints, every VE is also a stationary point of the potential function over the shared constraint. Coming back to the game $\mathscr{E}^{\mathrm{ae}}$, a stationary point of P^{ae} is therefore akin to a something like a VE of this formulation. □

Theorem 3.9 did not invoke the existence of a potential function. Nonetheless, there is a close relation between the minimizer of the potential function, i.e. the solution of problem P^{ae}, and the fixed points of Υ^{ae}. We formalize it through the following definition.

Definition 3.5 *A* stationary fixed point *of* Υ^{ae} *is a point* $(x, y) \in \mathcal{F}^{\mathrm{ae}}$ *with the property that* (x, y) *is satisfies the stationarity conditions of the minimization of* $\Psi(x, y, u, v)$ *over* $(u, v) \in \mathcal{F}$ *i.e.,*

$$\bar{\nabla} \Psi(x, y, x, y)^{\top} d \geq 0 \qquad \forall\, d \in \mathcal{T}((x, y); \mathcal{F}^{\mathrm{ae}}),$$

where $\bar{\nabla} \Psi(x, y, x, y) \triangleq \left. \frac{\partial}{\partial(u,v)} \Psi(x, y, u, v) \right|_{(u,v)=(x,y)}$.

If $\mathscr{E}^{\mathrm{ae}}$ is a potential game with potential function π, then we have $\nabla \pi(x, y) \equiv \bar{\nabla} \Psi(x, y, x, y)$. Consequently, we have the following relation.

Proposition 3.10 *Let* $\mathscr{E}^{\mathrm{ae}}$ *be a potential game with potential function* π. $(x, y) \in \mathcal{F}^{\mathrm{ae}}$ *is a stationary point of the minimization* π *over* $\mathcal{F}^{\mathrm{ae}}$ *if and only if* (x, y) *is a stationary fixed point of* Υ^{ae}.

4   Recovery of equilibria of $\mathscr{E}$

The prior section has concentrated on the development of existence statements for the equilibria associated with the shared-constraint (modified) equilibrium problem, denoted by $\mathscr{E}^{\mathrm{ae}}$. While it has been shown that the equilibria of $\mathscr{E}$ are indeed equilibria of $\mathscr{E}^{\mathrm{ae}}$, it remains unclear as to how one may obtain the equilibria of $\mathscr{E}$. In Section 4.1, we consider settings where the Nash-stationary equilibria of $\mathscr{E}$ may indeed be recovered. A more refined statement is provided in Section 4.2 under the assumption that follower equilibria are unique as a function of leader-level decisions.

4.1 Obtaining Nash stationary points of $\mathscr{E}$

In this section, we begin by providing an intuition about the relationship between the equilibria of the original game and its shared constraint modification by considering a convex generalized Nash game. Consider a Nash game in which player i has strategies x_i, y_i, objective $f_i(x_i, y_i; x^{-i})$ and a nonlinear constraint $h(x, y_i) \geq 0$, where for any x^{-i}, $h(x_i, y_i; x^{-i})$ and $f_i(x_i, y_i; x^{-i})$ are concave and convex in x_i, y_i, respectively. Specifically, player i solves $A_i(x^{-i})$, defined next.

$$
\boxed{
\begin{array}{lll}
A_i(x^{-i}) & \underset{x_i, y_i}{\text{minimize}} & f_i(x_i, y_i; x^{-i}) \\
& \text{subject to} & h(x, y_i) \geq 0. \qquad (\lambda_1)
\end{array}
}
$$

We refer to this game as G and corresponds to $\mathscr{E}$. Suppose $A_i(x^{-i})$ is a convex optimization problem for each x^{-i}. The shared constraint modification of this problem akin to the "ae" formulation is given by the following:

$$
\boxed{
\begin{array}{lll}
A_i^{\mathrm{ae}}(x^{-i}, y^{-i}) & \underset{x_i, y_i}{\text{minimize}} & f_i(x_i, y_i; x^{-i}) \\
& & h(x, y_1) \geq 0 \quad (\lambda_{11}) \\
& \text{subject to} & \vdots \\
& & h(x, y_N) \geq 0. \quad (\lambda_{1N})
\end{array}
}
$$

Notice that since $h(\cdot, \cdot)$ is concave in x_i, y_i for all x^{-i}, $A_i^{\mathrm{ae}}(x^{-i}, y^{-i})$ is a convex optimization problem. Denote this game as G^{ae}. Our first result relates equilibria of G to that of G^{ae}:

Lemma 4.1 *The point (x, y) is an equilibrium of G with multipliers $\lambda_1, \ldots, \lambda_N$ if and only if (x, y) is an equilibrium of G^{ae} with*

$$
\lambda_{ii} = \lambda_i \text{ and } \lambda_{ij} = 0, \forall j \neq i. \tag{23}
$$

Proof : By the same logic as in Prop. 3.1 (iii), any equilibrium of G is an equilibrium G^{ae}. Since problems $A_i, i \in \mathcal{N}$, and $A_i^{\mathrm{ae}}, i \in \mathcal{N}$, are convex optimization problems, the aggregated KKT conditions of individual players are necessary and sufficient for (x, y) to be an equilibrium of G and G^{ae}, respectively. (23) now follows from a examination of the KKT conditions. ∎

A more general statement is available in the context of $\mathscr{E}$ and $\mathscr{E}^{\mathrm{ae}}$ by examining the strong stationarity conditions (the proof is straightforward; we skip it).

Proposition 4.1 (Strong stationary points of $\mathscr{E}^{\mathrm{ae}}$ and $\mathscr{E}$) *Consider the multi-leader multi-follower games $\mathscr{E}$ and $\mathscr{E}^{\mathrm{ae}}$. Then the following hold:*

1. *(x, y) is a Nash strong-stationary point of $\mathscr{E}^{\mathrm{ae}}$ satisfying (19) with Lagrange multipliers $\left(\bar{\eta}_i, \bar{\mu}_i, \bar{\lambda}_i, (\bar{\beta}_i^k)_{k=1}^N\right)$ and $(\bar{\beta}_k^i) = 0, \forall k \neq i, \forall i$ if and only if (x, y) is a Nash strong-stationary point of $\mathscr{E}$ with multipliers $\left(\bar{\eta}_i, \bar{\mu}_i, \bar{\lambda}_i, \bar{\beta}_i^i\right)$.*

2. *Furthermore, (x, y) is a Nash second-order strong-stationary point of $\mathscr{E}^{\mathrm{ae}}$ with firm i's multipliers given by $\left(\bar{\eta}_i, \bar{\mu}_i, \bar{\lambda}_i, (\bar{\beta}_i^k)_{k=1}^N\right)$ and $(\bar{\beta}_k^i) = 0, \forall k \neq i, \forall i$ if and only if (x, y) is a Nash second-order strong-stationary point of $\mathscr{E}$ with multipliers $\left(\bar{\eta}_i, \bar{\mu}_i, \bar{\lambda}_i, \bar{\beta}_i^i\right)$.*

In effect, one can inspect the multipliers of an equilibrium of the shared-constraint game to ascertain whether indeed such a point is an equilibrium of the original equilibrium problem. Yet, much more can be drawn from this observation. Through the shred-constraint modification, a generalized Nash game is converted to a shared-constraint (generalized) Nash game. Furthermore, this modified game admits a set of generalized Nash equilibria (GNE) that contain two important sets of equilibria (each of which may be empty):

Equilibria of $\mathscr{E}$: These equilibria correspond to points characterized by multipliers that display a precise form as specified by Prop. 3.7.

Variational equilibria of $\mathscr{E}^{\mathrm{ae}}$: These equilibria are defined by a common Lagrange multiplier across every agent. Such equilibria have proved to be particularly relevant in the context of convex shared-constraint Nash games where the *common* Lagrange multiplier is seen as the *uniform auction price* [18]. Moreover, in the context of convex Nash games, VE may be obtained through the solution of a variational inequality problem [18, 19]. In the current context, under an assumption of potentiality on the leader-level problems, such equilibria may be derived by the solution to a suitably defined optimization problem, such as P^{ae}.

We believe that viewing the equilibria of $\mathscr{E}$ as particular equilibria of $\mathscr{E}^{\mathrm{ae}}$ allows for two important directions:

1. *Pathways to existence statements:* The shared constraint formulation admits a larger set of equilibria, that includes equilibria of the conventional formulation (if they exist). We have seen that under mild assumptions, existence of equilibrium akin to variational equilibria can be guaranteed. This may be a stepping stone towards developing an approach for claiming existence of equilibria of $\mathscr{E}$.

2. *Tools for equilibrium computation and selection:* Equilibrium computation is a crucial concern in the design of markets, a realm where such problems routinely arise. Yet, such designs are plagued by a key challenge in that equilibria are not readily computable. If the objectives admit a potential function, this formulation provides two crucial benefits. First, it allows for computing global variational equilibria through the solution of a single optimization problem. Second, if one takes the view that the conventional formulation is the "correct" formulation, the modification may provide a means to arriving at an equilibrium of the conventional formulation, provided it exists.

4.2 Recovery of equilibria when follower equilibria are unique

Assume that for every $x \in X$, $\mathcal{S}(x)$ is a singleton belonging to $\cap_{i \in \mathcal{N}} Y_i$ in which case we have $\mathcal{F} = \{(x,y) \mid x \in X, y = \mathcal{S}^N(x)\} = \mathcal{F}^{\mathrm{ae}}$. Substituting the follower equilibrium tuple y in terms of x in the definition of Υ^{ae}, we define $\Gamma^{\mathrm{ae}} : X \to 2^X$, as follows:

$$\Gamma^{\mathrm{ae}}(x) \triangleq \arg\min_{u \in X} \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}^N(u)).$$

If x is a fixed point of Γ^{ae}, then $(x, \mathcal{S}^N(x))$ an equilibrium of $\mathscr{E}^{\mathrm{ae}}$. For simplicity of exposition we will refer to fixed points of Γ^{ae} as "equilibria" of $\mathscr{E}^{\mathrm{ae}}$. Now consider the original formulation $\mathscr{E}$ and rewrite the leader problem L_i in the following form.

$\mathrm{L}_i(x^{-i})$	$\begin{aligned} \underset{x_i}{\text{minimize}} \quad & \varphi_i(x_i, \mathcal{S}(x); x^{-i}) \\ \text{subject to} \quad & x_i \in X_i, \end{aligned}$

It is easy to see that an "equilibrium" of this game is the same as a fixed point of $\Gamma : X \to 2^X$, where

$$\Gamma(x) \triangleq \arg\min_{u \in X} \sum_{i=1}^N \varphi_i(u_i, \mathcal{S}(u_i, x^{-i}); x^{-i}) = \arg\min_{u \in X} \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}(u_1, x^{-1}), \dots, \mathcal{S}(u_N, x^{-N})).$$

This follows from noting that X is a Cartesian product of $X_1, \dots, X_N$. The next theorem exploits the similarity between Γ^{ae} and Γ to develop conditions under which the fixed points of Γ^{ae} are also fixed points of Γ.

Theorem 4.2 *Suppose for all $x \in X$, $\mathcal{S}(x)$ is a singleton lying in $\cap_{i \in \mathcal{N}} Y_i$ and let the objectives of players be such that*

$$\Psi(x, \mathcal{S}^N(x), u, \mathcal{S}^N(u)) \leq \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}(u_1, x^{-1}), \dots, \mathcal{S}(u_N, x^{-N})), \qquad \forall\, u, x \in X.$$

Then every fixed point of Γ^{ae} is also a fixed point of Γ and thus an equilibrium of $\mathscr{E}$. In particular, if Γ^{ae} admits a fixed point, the conventional formulation $\mathscr{E}$ admits an equilibrium.

Proof : If x is a fixed point of Γ^{ae},

$$\Psi(x, \mathcal{S}^N(x), x, \mathcal{S}^N(x)) \leq \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}^N(u)) \quad \forall u \in X.$$

By the hypothesis of the theorem, we have

$$\Psi(x, \mathcal{S}^N(x), x, \mathcal{S}^N(x)) \leq \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}(u_1, x^{-1}), \dots, \mathcal{S}(u_N, x^{-N})),$$

which means x is a fixed point of Γ. ∎

Remark 4.2. Notice the difference between Γ and Γ^{ae}. Importantly, observe that a fixed point of one is not necessarily a fixed point of the other. This may come as a surprise, considering that Proposition 3.1 shows that equilibria of $\mathscr{E}$ are also equilibria of $\mathscr{E}^{\mathrm{ae}}$. But this "contradiction" can be explained by noticing that a fixed point of Γ *is* an equilibrium of $\mathscr{E}^{\mathrm{ae}}$, but such an equilibrium need not be a fixed point of Γ^{ae}. Since the fixed point formulation through Υ^{ae} or Γ^{ae} is only a sufficient condition for the existence of equilibria of $\mathscr{E}^{\mathrm{ae}}$, there may exist equilibria of these games that are not necessarily fixed points of the Γ^{ae}. □

5 An example: a hierarchical Cournot game

In this section, we present a multi-leader multi-follower game from [33] which when formulated in the conventional form has an equilibrium. Through this game, we will demonstrate the validity of Propositions 3.1 and Theorem 3.2.

Below, we modify this game in the form of $\mathscr{E}^{\mathrm{ae}}$ and show that the claim made in Proposition 3.1 holds: the equilibrium of this game is also an equilibrium of its modification. The example shows that equilibrium conditions of $\mathscr{E}^{\mathrm{ae}}$ have more variables than the conditions of $\mathscr{E}$, and thus allow for more "degrees of freedom" for their satisfaction. Cournot games, as noted in Section 2.2, admit potential functions. We then calculate the minimizer of the potential function of this game (i.e., the solution of P^{ae}) and show that it is an equilibrium of the modified game, thereby verifying Theorem 3.2.

Example 5.2. Let $\mathscr{E}$ be a game with N *identical* leaders and n *identical* followers. The follower strategies conjectured by leader i are denoted by $\{y_i^f\}_{f=1,\dots,n}$ (we use f to index followers) and we let $\bar{y}_i^{-f}$ denote $\sum_{j \neq f} y_i^j$. Leader i solves the following parametrized problem:

$$
\boxed{
\begin{array}{lll}
\mathrm{L}_i(x^{-i}, y^{-i}) & \underset{x_i, y_i}{\text{minimize}} & \frac{1}{2}cx_i^2 - x_i\left(a - b(x_i + \sum_{j \neq i} x_j + \sum_{f=1}^n y_i^f)\right) \\[2mm]
& \text{subject to} & y_i^f = \mathrm{SOL}(\mathrm{F}(\bar{y}_i^{-f}, x_i, x^{-i})), \quad \forall\, f, \\[1mm]
& & x_i \geq 0,
\end{array}
}
$$

where $y_i^f \in \mathbb{R}$ is the conjecture of leader i of the equilibrium strategy of follower f. Follower f solves the problem $(\mathrm{F}(\bar{y}^{-f}, x))$:

$$
\boxed{
\begin{array}{lll}
\mathrm{F}(\bar{y}^{-f}, x) & \underset{y^f}{\text{minimize}} & \frac{1}{2}c(y^f)^2 - y^f\left(a - b(y^f + \sum_{j \neq f} y^j + \sum_{i \in \mathcal{N}} x_i)\right) \\[2mm]
& \text{subject to} & y_f \geq 0,
\end{array}
}
$$

where constants a, b, c are positive real numbers. Since these constants are the same for all followers, equilibrium strategies of all followers are equal. Consequently the follower equilibrium tuple conjectured by leader i is given by $y_i = (\widehat{y}_i, \dots, \widehat{y}_i)$, where $\widehat{y}_i$ satisfies $\widehat{y}_i \in \mathrm{SOL}(\mathrm{F}((n-1)\widehat{y}_i, x))$ (since $\bar{y}^{-f} = (n-1)\widehat{y}_i$). For any x, there is a unique $\widehat{y}_i$ that satisfies this relation, given by

$$
\widehat{y}_i = \begin{cases} (a - b\sum_j x_j)/(c + b(n+1)) & \text{if } 0 \leq \sum_j x_j \leq a/b, \\ 0 & \text{if } \sum_j x_j > a/b. \end{cases} \tag{24}
$$

By considering only the first of above cases in (24), we get a restricted game where leader i solves

$$
\boxed{
\begin{array}{lll}
\mathrm{L}_i'(x^{-i}, \widehat{y}^{-i}) & \underset{x_i, \widehat{y}_i}{\text{minimize}} & \frac{1}{2}cx_i^2 - x_i\left[a - b\left(x_i + \sum_{j \neq i} x_j + n\widehat{y}_i\right)\right] \\[2mm]
& & \widehat{y}_i = \frac{a - b\sum_j x_j}{c + b(n+1)}, \qquad : \bar{\lambda}_i \\[2mm]
& \text{subject to} & \sum_j x_j \leq a/b, \qquad\qquad : \bar{\mu}_i \\[1mm]
& & x_i \geq 0.
\end{array}
}
$$

This is a generalized Nash game with coupled but not shared constraints. However, since the optimization problems of the leaders are convex (this is not obvious; see [33, Lemma 1] for a proof), we may use the first-order KKT conditions to derive an equilibrium. Let $\bar{\lambda}_i$ be the Lagrange multiplier corresponding the constraint "$\widehat{y}_i = \frac{a - b \sum_j x_j}{c + b(n+1)}$". The equilibrium conditions of this game are

$$
\left\{
\begin{aligned}
& 0 \leq x_i \perp (c+b)\, x_i - a + b \left(\sum_j x_j + n \widehat{y}_i \right) + \frac{b}{c + b(n+1)} \bar{\lambda}_i + \bar{\mu}_i \geq 0, \\
& \quad \widehat{y}_i = \frac{a - b \sum_j x_j}{c + b(n+1)}, \\
& 0 \leq \bar{\mu}_i \perp a/b - \sum_j x_j \geq 0, \\
& \quad 0 = nbx_i + \bar{\lambda}_i.
\end{aligned}
\right\}
\quad \forall\, i \in \mathcal{N}.
\tag{25}
$$

We can verify that the tuple $x = x^*$ where all leaders play the same strategy $\widehat{x}$, i.e. $x_i^* = \widehat{x}$ for all i with $\widehat{x}$ given by

$$
\widehat{x} = \frac{a(b+c)}{b(b+c)(N+1) + c(b+c) + bcn},
$$

satisfies equilibrium conditions for the restricted game $\{L_i'\}_{i \in \mathcal{N}}$. The optimal Lagrange multiplier is given by $\bar{\lambda}_i^* = -nbx_i^*, \bar{\mu}_i^* = 0$. It can then be verified that this equilibrium also satisfies the requirement $\sum_i x_i^* < a/b$, whereby it is an equilibrium of the original game. The other case of $\widehat{y}_i = 0$ does not result in an equilibrium that satisfies $\sum_i x_i > a/b$, and consequently x^* is the only equilibrium.

Verifying Proposition 3.1 (An equilibria of $\mathscr{E}$ is an equilibrium of $\mathscr{E}^{\mathrm{ae}}$): Let us now consider this game modified as $\mathscr{E}^{\mathrm{ae}}$.

$$
\boxed{
\begin{aligned}
& \mathrm{L}_i^{\mathrm{ae}}(x^{-i}, y^{-i}) \quad & \underset{x_i, y_i}{\text{minimize}} \quad & \tfrac{1}{2} c x_i^2 - x_i (a - b(x_i + \textstyle\sum_{j \neq i} x_j + \sum_{f=1}^n y_i^f)) \\
& & \text{subject to} \quad & y_k^f = \mathrm{SOL}(\mathrm{F}(\bar{y}_k^{-f}, x_k, x^{-k})), \quad \forall\, f, \forall\, k = 1, \ldots, N \\
& & & x_i \geq 0.
\end{aligned}
}
$$

Notice that the equilibrium constraint is now for all f and for all k. For any k, the equilibrium constraint may be simplified using (24), giving an equation in $\widehat{y}_k$. It is easy to check that this game also admits no equilibrium with $\sum_j x_j > a/b$. Thus, this game is equivalent to the game where $\sum_j x_j$ is constrained to be in $[0, a/b]$. For such values of $\sum_j x_j$, the first case of (24) applies, and it gives us a game where leader i solves

$$
\boxed{
\begin{aligned}
& \mathrm{L}_i^{\mathrm{ae}}(x^{-i}, \widehat{y}^{-i}) \quad & \underset{x_i, y_i}{\text{minimize}} \quad & \tfrac{1}{2} c x_i^2 - x_i \left[a - b \left(x_i + \textstyle\sum_{j \neq i} x_j + n \widehat{y}_i \right) \right] \\
& & \text{subject to} \quad & \widehat{y}_k = \frac{a - b \sum_j x_j}{c + b(n+1)}, \quad : \lambda_i^k, \quad k = 1, \ldots, N \\
& & & \textstyle\sum_j x_j \leq a/b, \quad\quad\;\; : \mu_i, \\
& & & x_i \geq 0.
\end{aligned}
}
$$

This is a generalized Nash game with (convex) shared constraints and convex optimization problems for leaders. Let λ_i^k be the Lagrange multiplier corresponding to the constraint "$\widehat{y}_k = \frac{a - b \sum_j x_j}{c + b(n+1)}$" in the problem L_i. The

equilibrium conditions for the generalized Nash equilibrium (see [18]) of this game are

$$
\begin{cases}
0 \le x_i \perp (b+c)x_i - a + b\left(\displaystyle\sum_j x_j + n\widehat{y}_i\right) + \dfrac{b}{c+b(n+1)}\displaystyle\sum_{j=1}^{N}\lambda_i^j + \mu_i \ge 0, \\[3mm]
\quad \widehat{y}_i = \dfrac{a - b\sum_j x_j}{c + b(n+1)}, \\[3mm]
0 \le \mu_i \perp a/b - \displaystyle\sum_j x_j \ge 0, \\[3mm]
\quad 0 = nbx_i + \lambda_i^i,
\end{cases} \qquad \forall\, i \in \mathcal{N}. \qquad (26)
$$

Notice that the Lagrange multipliers λ_i^j for $j \ne i$ are unconstrained barring their presence in the first condition of (26). Comparing (26) and (25), we see that if $\bar{\lambda}^*, x^*$ solve system (25), then $x = x^*$ and $\lambda_i^j = \bar{\lambda}_i^* \mathbb{I}_{\{j=i\}}$ for all $i, j \in \mathcal{N}$ gives a solution to system (26). Consequently, an equilibrium of the original game $\mathscr{E}$ is an equilibrium of $\mathscr{E}^{\mathrm{ae}}$. We have thereby verified Proposition 3.1 for this problem.

The presence of surplus Lagrange multipliers provides us with more variables than the number of equations, whereby existence of solutions is easier to guarantee. An equilibrium of $\mathscr{E}$ is an equilibrium of the modified game $\mathscr{E}^{\mathrm{ae}}$ with a specific configuration of the vector of Lagrange multipliers. Consequently, if an equilibrium exists to $\mathscr{E}^{\mathrm{ae}}$, there is no guarantee that there exists one to the original game $\mathscr{E}$.

Verifying Theorem 3.2 (Global minimizer of π is an equilibrium of $\mathscr{E}^{\mathrm{ae}}$): Applying the same arguments as before, we can effectively consider the strategies of leader i in game $\mathscr{E}^{\mathrm{ae}}$ as x_i and $\widehat{y}_i$. Further, suppose the function π is given by

$$
\pi(x,\widehat{y}) = \tfrac{1}{2}c\sum_i x_i^2 - a\sum_i x_i + b\left(\sum_i x_i^2 + \sum_{i<j} x_i x_j\right) + nb\sum_i x_i \widehat{y}_i,
$$

where $\widehat{y} \triangleq (\widehat{y}_1, \ldots, \widehat{y}_N)$. Notice that the map F (cf., Lemma 2.1) is given by

$$
F(x,\widehat{y}) = \begin{pmatrix} \frac{\partial \varphi_i}{\partial x_i} \\[2mm] \frac{\partial \varphi_i}{\partial \widehat{y}_i} \end{pmatrix}_{i \in \mathcal{N}} = \begin{pmatrix} (b+c)x_i - a + b\left(\sum_j x_j + n\widehat{y}_i\right) \\[2mm] nbx_i \end{pmatrix}_{i \in \mathcal{N}},
$$

and that $\nabla \pi \equiv F$, whereby π is a potential function for $\mathscr{E}^{\mathrm{ae}}$. The set $\mathcal{F}^{\mathrm{ae}}$ for this game is

$$
\mathcal{F}^{\mathrm{ae}} = \{(x,\widehat{y}) \mid x \ge 0,\ \widehat{y}_i \text{ satisfies } (24)\ \forall\, i\}.
$$

We now determine the global minimizer of π over $\mathcal{F}^{\mathrm{ae}}$. A significant difficulty in characterizing the global minimizer of π is that π is not necessarily convex (despite the convexity of the objectives of leaders in their own variables).

We argue as follows. By membership of $(x,\widehat{y})$ in $\mathcal{F}^{\mathrm{ae}}$, we either have $\widehat{y}_i = \frac{a - b\sum_j x_j}{c+b(n+1)}$ for all i or we have $\widehat{y}_i = 0$ for all i. Substituting for $\widehat{y}$, we can write π as a function only of x (with a slight abuse of notation)

$$
\pi(x) = \begin{cases} \tfrac{1}{2}c\sum_i x_i^2 - a\sum_i x_i + b\left(\sum_i x_i^2 + \sum_{i<j} x_i x_j\right) + nb\frac{a-b\sum_i x_i}{c+b(n+1)}\sum_i x_i & \text{if } 0 \le \sum_i x_i \le a/b, \\[3mm] \tfrac{1}{2}c\sum_i x_i^2 - a\sum_i x_i + b\left(\sum_i x_i^2 + \sum_{i<j} x_i x_j\right) & \text{if } \sum_i x_i > a/b. \end{cases}
$$

By symmetry, the values x_i that minimize π are equal for all i. Let $x_i = x'$ for all i be the minimizer. Then,

$$
\pi(x') = \begin{cases} \tfrac{1}{2}Ncx'^2 - aNx' + b\left(Nx'^2 + \frac{N(N-1)}{2}x'^2\right) + nb\frac{a-bNx'}{c+b(n+1)}Nx' & \text{if } 0 \le x' \le a/(Nb), \\[3mm] \tfrac{1}{2}Ncx'^2 - aNx' + b\left(Nx'^2 + \frac{N(N-1)}{2}x'^2\right) & \text{if } x' > a/(Nb). \end{cases}
$$

The right hand derivative of π at $x' = a/b$ is positive, $\nabla \pi(x')^+|_{x'=a/b} = N[\frac{ac}{Nb} - a + a(N+1)] > 0$. Furthermore π is increasing and coercive for $x' > a/(Nb)$, and consequently the minimizer of π lies in $[0, a/(Nb)]$. Since x'

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is a global minimizer of π, x' necessarily satisfies the first-order KKT conditions for the minimization of π over $[0, a/(Nb)]$:

$$0 \le x' \perp N\left((b+c)x' - a + bNx' + nb\frac{a - bNx'}{c + b(n+1)} - \frac{nb^2Nx'}{c + b(n+1)}\right) + \mu' \ge 0, \qquad (27)$$
$$0 \le \mu' \perp a/(Nb) - x' \ge 0,$$

where μ' is the Lagrange multiplier for the constraint '$a/(Nb) - x' \ge 0$'. If x', μ' is a solution of system (27), then $x_i = x'$, $\mu_i = \mu'$ and $\lambda_i^j = -nbx_j = -nbx'$ for all $i, j \in \mathcal{N}$, solves system (26) for the equilibrium of $\mathscr{E}^{\text{ae}}$. Consequently $x = (x', \dots, x')$ is an equilibrium of $\mathscr{E}^{\text{ae}}$. This verifies Theorem 3.2.

It should be emphasized that we have claimed that a solution to the concatenated first-order KKT conditions of the minimization of π over $\mathcal{F}^{\text{ae}}$ is a global equilibrium of $\mathscr{E}^{\text{ae}}$, a claim that is valid because the leader problems in $\mathscr{E}^{\text{ae}}$ have been reduced to convex problems. In the case where $\mathcal{S}$ is single-valued (as it was in this example), this is possible because we could argue that for values (x, y) of interest, the equation $y = \mathcal{S}(x)$ is linear in x, y. $\square$

6   Conclusions

To summarize, while general existence results for the original formulation of multi-leader multi-follower games (more generally EPECs) are rare, we observed that a modified formulation in which each player is constrained by the equilibrium constraints of all players contains the equilibria of the original game when this game does indeed admit equilibria. This modified game admits a shared constraint structure and when the leaders' objectives admit a potential function, the set of global minimizers of the potential function over the shared constraint are the equilibria of the modified multi-leader multi-follower game. Similar statements were made relating the stationary points of such a problem to the associated Nash B-stationary and strong-stationary equilibria. In effect, the above results reduced the question of the existence of an equilibrium to that of the solvability of an optimization problem, in particular a mathematical program with equilibrium constraints. This solvability can be claimed under fairly standard conditions that are tractable and verifiable – e.g., coercive objective over a nonempty feasible region – and existence of a global equilibrium is seen to follow.

It was further seen that the equilibria of the original formulation can be viewed as equilibria of the shared-constraint modification in which the associated Lagrange multipliers take on a specific form. This understanding may have much potential in deriving existence statements as well as computational schemes for the equilibria arising from the original formulation. We concluded with an application of our findings on a multi-leader multi-follower symmetric Cournot game.

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