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A SHARP CONDITION FOR THE WELL-POSEDNESS OF THE LINEAR KDV-TYPE EQUATION

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ABSTRACT. An initial value problem for a very general linear equation of KdV-type is considered. Assuming non-degeneracy of the third derivative coefficient, this problem is shown to be well-posed under a certain simple condition, which is an adaptation of the Mizohata-type condition from the Schrödinger equation to the context of KdV. When this condition is violated, ill-posedness is shown by an explicit construction. These results justify formal heuristics associated with dispersive problems and have applications to non-linear problems of KdV-type.

1. Introduction

This paper is concerned with the study of the equation

$$\begin{cases} \partial_t u + Lu = f \text{ for } (t,x) \in (0,T] \times \mathbb{R} \\ u(0,x) = u_0(x) \end{cases}, \text{ where } L = \sum_{j=0}^{3} a_j(t,x)\partial_x^j, \tag{1.1}$$

where a_j are real-valued functions.

This is the most general linear form of the KdV, one of the most studied dispersive equations, and is used as an important model in understanding the behavior of linear and non-linear waves. Such an equation with non-constant dispersive coefficient a_3 describes non-isotropic dispersion, and its study is of use for the quasi-linear analogues of (1.1).

Another motivation for the study of the well-posedness of (1.1) is understanding the relative strength of dispersive and non-dispersive effects present in the equation. In particular, from the geometrical optics expansion for the equation (cf. the classical book of Whitham [15]), the dispersive coefficient a_3 guides the propagation of the wave packets, while the term $a_2\partial_x^2$ can lead to the growth of the amplitudes of the wave packets of (1.1). In light of these heuristics, it is natural to expect that well-posedness requires *non-degeneracy* of a_3, which prevents the collapse of the wave packets, namely $0 < \varepsilon \leq |a_3| \leq \frac{1}{\varepsilon}$ for some ε, and a condition on a_2 to ensure that dispersion dominates anti-diffusion effects. Craig-Goodman [4] proved well-posedness in the Sobolev spaces H^s for $a_2 \equiv a_1 \equiv 0$ under the *non-degeneracy* of coefficient a_3 and ill-posedness for some degenerate cases of a_3. In a follow-up paper, Craig-Kappeler-Strauss [3] proved well-posedness with

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non-degenerate dispersion and $-a_2 \geq 0$, as well as extensions to the quasi-linear analogues. These results were extended in [1] to allow for "anti-diffusion" in a_2, as long as $\langle x \rangle^{\frac{1}{2}^+} |a_2| \leq C$, under some additional assumptions on other coefficients, and to systems of equations.

In the current paper, the condition on the diffusion coefficient a_2 is extended to a sharp one for the well-posedness in H^s, where well-posedness means *existence* of $C^0_{[0,T]}H^s$ distributional solutions of (1.1) that are *unique* and *depend continuously on data* in the $C^0_{[0,T]}H^s$ topology. Namely, a condition on the diffusion coefficient a_2 along the flow is obtained, one that separates well-posedness from ill-posedness (in the sense of violating continuous dependence) of (1.1) with non-degenerate dispersion. This is qualitatively similar to the necessity of a Mizohato condition $|\sup_{x,t|\omega|=1} \int_0^t \Im b(x+s\omega)\cdot \omega ds| < \infty$ for the well-posedness Schrödinger equation $\partial_t u + i\triangle u + b(x)\nabla u = 0$ in [9]; see also [5], [6], [8] and the references therein for more refined results on the variable coefficient Schrödinger equation. The well-posedness is proved by the "gauged energy method", and the condition on the gauge captures the a_2 condition. Ill-posedness is proved by an explicit geometrical optics construction.

While preparing this paper for publication, the author learned of a preprint by Ambrose-Wright [2] that treats an analogue of (1.1) in the periodic case. Their argument for the well-posedness is also based on the "gauged energy method"; however, in the case of $\mathbb{R}$ the smoothness of the coefficients does not imply integrability that is often needed. Additionally, this paper also proves that (1.1) possesses a local smoothing effect, which is not present in the periodic case. The ill-posedness result in [2] is done by a spectral method, which works only in the time independent case of (1.1). After this paper had been accepted for publication, the author learned of works of Tarama ([12], [13], [14]) and Mizuhara ([10]), who proved similar results in the case of constant dispersion ($a_3 \equiv 1$), including complex coefficients.

The rest of the paper is organized as follows. In section 2 the main results of the paper are stated. Well-posedness is proved in section 3, and ill-posedness in section 4.

2. Main results

The following functional space notation is used. Let $\mathcal{B}^N_x = \{f(x) \in C^N(\mathbb{R}) : \partial^j_x f \in L^\infty \text{ for all } 0 \leq i \leq N\}$, $\mathcal{B} = \bigcap_N \mathcal{B}^N$, and $H^s = \{f \in \mathscr{S}' : \|f\|_{H^s} = \|\langle\xi\rangle^s \hat{f}(\xi)\|_{L^2} < \infty\}$, where $\langle x \rangle = \sqrt{1+|x|^2}$.

For $1 \leq p < \infty$, define

$$\|u\|_{L^p_{[0,T]}X_x} := (\int_0^T \|u(t)\|^p_{X_x} dt)^{\frac{1}{p}} \text{ and } X_T := \|u\|_{L^\infty_{[0,T]}X_x} := \operatorname{ess\,sup}_t \|u(t)\|_{X_x}$$

for one of spaces X above.

The following assumptions are made for the coefficients of (1.1):

(A1): Dispersive coefficient $a_3(t,x)$ is non-degenerate. That is, there are constants $\Lambda \geq \lambda > 0$ such that

$$\lambda \leq |a_3(t,x)| \leq \Lambda$$

uniformly for $(x,t) \in \mathbb{R} \times [0,T]$.

(A2): Regularity of the coefficients. For all $N \geq 0$:
- $a_3 \in C^0_{[0,T]}\mathcal{B}^{N+3}_x \cap C^1_{[0,T]}\mathcal{B}^1_x$.
- $a_2 \in C^0_{[0,T]}\mathcal{B}^{N+2}_x \cap C^1_{[0,T]}\mathcal{B}^0_x$.
- $a_1 \in C^0_{[0,T]}\mathcal{B}^{N+1}_x$.
- $a_0 \in C^0_{[0,T]}\mathcal{B}^{N}_x$.

(A3): Weak diffusion. $\int_0^x \frac{a_2(y,t)}{|a_3(y,t)|}dy \in C^1_{[0,T]}L^\infty_x$.

Note that by $(A1)$ and $(A2)$, a_3 has a constant sign.

For $N \geq 0$ define

$$C_N = \|a_3\|_{L^\infty_T} + \|\frac{1}{a_3}\|_{L^\infty_T} + \sum_{j=0}^{3}\|a_j\|_{\mathcal{B}^{N+i}_T} + \sum_{i=2}^{3}\|\partial_t a_j\|_{L^\infty_T} + \|\int_0^x \frac{a_2(y,t)}{|a_3(y,t)|}dy\|_{L^\infty_T}$$
$$+ \|\partial_t \int_0^x \frac{a_2(y,t)}{|a_3(y,t)|}dy\|_{L^\infty_T}.$$

For the well-posedness arguments, positive constants will depend on C_N for some N and will not be made explicit.

Theorem 2.1. *Suppose the coefficients of* (1.1) *satisfy* $(A1)$–$(A3)$*. Then for all* $s \in \mathbb{R}$*,* (1.1) *is well-posed in* H^s*. That is, for any* $(u_0, f) \in H^s \times L^1_{[0,T]}H^s$ *there exists a unique* $u \in C^0_{[0,T]}H^s$ *satisfying* (1.1) *in the sense of distributions. In addition, there exists* $C = C(s)$*:*

$$\sup_{0\leq t\leq T}\|u(t)\|_{H^s} \leq Ce^{CT}(\|u_0\|_{H^s} + \int_0^T \|f(t)\|_{H^s}dt). \tag{2.1}$$

Moreover, for any $\delta > \frac{1}{2}$*, the solution additionally satisfies* $u \in L^2_{[0,T]}H^{s+1}_{\langle x\rangle^{-2\delta}dx}$ *and there is a* $\tilde{C} = \tilde{C}(s,\delta)$*:*

$$\|\langle x\rangle^{-\delta}\,\partial_x u\|_{L^2_{[0,T]}H^s_x} \leq \tilde{C}(1+\sqrt{T})e^{\tilde{C}T}(\|u_0\|_{H^s} + \int_0^T \|f(t)\|_{H^s}dt). \tag{2.2}$$

Estimate (2.1) implies continuous dependence for (1.1), while estimate (2.2) is a manifestation of a *local smoothing effect* of (1.1).

Remark 2.2. If, in addition, $f \in C^0_{[0,T]}H^{s-3}$, then for $s > 3\frac{1}{2}$ the unique solution from Theorem 2.1 is classical by the Sobolev embedding.

Remark 2.3. If the coefficients of (1.1), in addition, satisfy $(A1)$–$(A3)$ on $[-T, 0]$, then the transformation of the equation by $t \to -t$ changes the sign of all a_j, while again preserving all of the assumptions. Therefore, Theorem 2.1 extends to $[-T, 0]$.

Moreover, the transformation $x \to -x$ in (1.1) changes the sign of a_j for odd i, but preserves the assumptions $(A1)$–$(A3)$. Without loss of generality, $a_3 > 0$ will be assumed.

The ill-posedness result complements Theorem 2.1 and is proved by a different method.

Theorem 2.4. *Suppose the coefficients of* (1.1) *satisfy* $(A1)$, $(A2)$ *and*

(A3N): $\sup_{x>0}\int_0^x \frac{a_2(y,0)}{|a_3(y,0)|}dy = \infty$.

Then for all $T>0$ *and* $s\in\mathbb{R}$ (1.1) *is ill-posed in* $C^0_{[0,T]}H^s$ *forward in time. More precisely, there is no continuous function* $C(t,t_0)$ *for* $0\le t_0\le t\le T$ *such that*

$$\sup_{t_0\le t\le T}\|u(t)\|_{H^s}\le C(t,t_0)\|u(t_0)\|_{H^s} \tag{2.3}$$

whenever u *solves* (1.1) *on* $[0,T]$ *with* $f\equiv 0$. *Equivalently* (2.1) *fails on any* $[0,T]$.

Remark 2.5. The transformation $x\to -x$ shows that $(A3N)$ is equivalent to

$$\sup_{x<0}\int_x^0 \frac{a_2(y,0)}{|a_3(y,0)|}dy=\infty.$$

However, the equivalence breaks down if absolute values are removed from a_3 in $(A3)$. Thus $a_3>0$ can be assumed without loss of generality, as long as $(A3N)$ is replaced with

(A3N′): $a_3>0$. Furthermore,

$$\sup_{x>0}\int_0^x \frac{a_2(y,0)}{a_3(y,0)}dy=\infty \text{ or } \sup_{x<0}\int_x^0 \frac{a_2(y,0)}{a_3(y,0)}dy=\infty.$$

Remark 2.6. By reversing the time $t\to -t$ as in Remark 2.3, Theorem 2.4 shows that

$$\sup_{x>0}\int_0^x \frac{a_2(y,0)}{|a_3(y,0)|}dy=-\infty$$

leads to ill-posedness on $[-T,0]$. Thus the condition $\int_0^x \frac{a_2(y,0)}{|a_3(y,0)|}dy\in L^\infty$ is crucial for the well-posedness and condition $(A3)$ for Theorem 2.1 is sharp for well-posedness on $[-T,T]$.

3. Well-posedness

The main ingredient in the proof of Theorem 2.1 is stated as the following proposition, which is an *a priori* L^2 estimate for a slightly more general version of (1.1) that comes from commuting derivatives:

$$\begin{cases}\partial_t u+L_A u=f \text{ for } (t,x)\in(0,T]\times\mathbb{R}\\ u(0,x)=u_0(x)\end{cases}, \text{ where } L_A=L+A_0(t,x,\partial_x) \tag{3.1}$$

with L from (1.1). The following assumptions are made on $A_0\in C^0_{[0,T]}S^0$, the pseudo-differential operator of standard symbol class of order 0 (cf. Chapter VI of [11]):

(A4): The S^0 semi-norms of A_0 are bounded for $t\in[0,T]$, and their size depends on constants C_N from $(A1)$–$(A3)$.

Proposition 3.1. *Suppose that the coefficients* a_j *of* (1.1) *satisfy* $(A1)$–$(A3)$ *and* A_0 *satisfies* $(A4)$. *Then there exists a constant* C, *and for any* $\delta>\frac{1}{2}$ *there is a constant* $\tilde{C}$ *such that for any* $u\in C^1_{[0,T]}L^2\cap C^0_{[0,T]}H^3$, *the triple* (u,u_0,f) *with* u_0

and f defined by (3.1) *satisfies*

$$\sup_{0\le t\le T}\|u(t)\|_{L^2}\le Ce^{CT}(\|u_0\|_{L^2}+\int_0^T\|f(t)\|_{L^2}dt), \tag{3.2}$$

$$\|\langle x\rangle^{-\delta}\partial_x u\|_{L^2_{[0,T]\times x}}\le \tilde{C}(1+\sqrt{T})e^{\tilde{C}T}(\|u_0\|_{L^2}+\int_0^T\|f(t)\|_{L^2}dt). \tag{3.3}$$

Remark 3.2. If $A_0\equiv 0$, then $N=0$ in $(A2)$ can be chosen for Proposition 3.1.

The proof of Proposition 3.1 is done by a change of variables (gauge) followed by the application of the energy estimates. The proof is broken into several preliminary results.

A gauge is a smooth invertible function, which for the purposes of the argument needs to have 3 bounded derivatives:

Definition 3.3. A function $\phi\in C^0_{[0,T]}\mathcal{B}^3_x\cap C^1_{[0,T]}\mathcal{B}^0$ is called a **gauge** if

- $\phi(x,t)>0$ with $\frac{1}{\phi}\in L^\infty_{[0,T]\times\mathbb{R}}$,
- $\|\frac{1}{\phi}\|_{L^\infty_{[0,T]\times\mathbb{R}}}+\|\phi\|_{\mathcal{B}^3_T}+\|\partial_t\phi\|_{L^\infty_T}\le C(C_0,\delta)$ with C_N from $(A1)$–$(A3)$.

Suppose that $\phi(x,t)$ is a *gauge.* Define

$$v=\phi^{-1}u.$$

Definition 3.3 implies that $v\in C^1_{[0,T]}L^2\cap C^0_{[0,T]}H^3$ if and only if $u\in C^1_{[0,T]}L^2\cap C^0_{[0,T]}H^3$ and substitution of v into (3.1) gives

$$\begin{cases}\partial_t v+L_\phi v=\phi^{-1}f,\\ v(x,0)=\phi^{-1}u_0,\end{cases} \tag{3.4}$$

where

$$\begin{aligned}L_\phi&=a_3\partial_x^3+\left(a_2+\phi^{-1}3a_3\partial_x\phi\right)\partial_x^2+\left(a_1+\phi^{-1}(2a_2\partial_x\phi+3a_3\partial_x^2\phi)\right)\partial_x\\&\quad+\left(a_0+\phi^{-1}(\partial_t\phi+a_1\partial_x\phi+a_2\partial_x^2\phi+a_3\partial_x^3\phi)\right)I+\phi^{-1}A_0(\phi\_).\end{aligned}$$

Lemma 3.4. *From the definition of the gauge,*

$$\|u\|_{L^2}\approx\|v\|_{L^2}\ \textit{and}\ \sum_{j=0}^1\|\langle x\rangle^{-\delta}\partial_x^j u\|_{L^2_{[0,T]\times x}}\approx\sum_{j=0}^1\|\langle x\rangle^{-\delta}\partial_x^j v\|_{L^2_{[0,T]\times x}} \tag{3.5}$$

with comparability constants dependent only on the constant in Definition 3.3. *Therefore, to prove Proposition* 3.1 *it suffices to prove* (3.2) *and* (3.3) *for v satisfying* (3.4).

Proof. It suffices to show one-sided inequalities in (3.5) as ϕ^{-1} satisfies the same estimates as ϕ. The first comparability follows from $\|u\|_{L^2}\le\|\phi\|_{L^\infty}\|v\|_{L^2}$. For the second, a similar computation and Cauchy-Schwartz implies

$$(\sum_{j=0}^1\|\langle x\rangle^{-\delta}\partial_x^j u\|_{L^2_{[0,T]\times x}})^2\le 2(\|\phi\|^2_{L^\infty_T}+\|\partial_x\phi\|^2_{L^\infty_T})(\sum_{j=0}^1\|\langle x\rangle^{-\delta}\partial_x^j v\|_{L^2_{[0,T]\times x}})^2.$$

It is clear from (3.5) that (3.2) is equivalent for u and v, whereas using (3.5) for the estimate

$$\|\langle x\rangle^{-\delta}\partial_x u\|_{L^2_{[0,T]\times x}}\le C(\sqrt{T}\sup_{0\le t\le T}\|v(t)\|_{L^2_x}+\|\langle x\rangle^{-\delta}\partial_x v\|_{L^2_{[0,T]\times x}})$$

implies (3.3) for u if (3.2) and (3.3) hold for v. □

The energy method involves multiplying (3.4) by v to estimate $\partial_t\|v\|^2_{L^2}$ by $\|v\|^2_{L^2}$:

$$\partial_t \int |v|^2 = -2Re(L_\phi v, v) + (f, \phi v).$$

The following lemma summarizes the energy estimates for L or L_ϕ:

Lemma 3.5. *Consider an operator $L = a_3\partial_x^3 + a_2\partial_x^2 + a_1\partial_x + a_0$, where a_3–a_0 satisfy (A2). Then for $v \in C^0_{[0,T]}H^3$,*

$$Re(Lv, v) = (\left[-a_2 + \frac{3}{2}\partial_x a_3\right] \partial_x v, \partial_x v) + (b_0 v, v)$$

for $b_0 = a_0 - \frac{1}{2}(\partial_x a_1 - \partial_x^2 a_2 + \partial_x^3 a_3)$, where (u, v) is an L^2_x pairing.

Proof of Lemma 3.5. The computation is immediate by computing the adjoint L^* of L using the calculus of PDO. Alternatively, as L is a differential operator, the same computation can also be done by a repeated integration by parts. Indeed, the operator ∂_x^k is skew-adjoint for odd k, which implies that principal parts of odd order terms are eliminated by integration by parts. For example,

$$(a_1\partial_x v, v) = -(v, a_1\,\partial_x v) - (\partial_x a_1\, v, v) = -\overline{(a_1\,\partial_x v, v)} - (\partial_x a_1\, v, v).$$

An identical computation shows

$$Re(a_3\partial_x^2 v, \partial_x v) = -\frac{1}{2}(\partial_x a_3 \partial_x v, \partial_x v) \text{ and } Re(\partial_x^2 a_3 \partial_x v, v) = -\frac{1}{2}(\partial_x^3\, a_3 v, v).$$

Using these identities and more integration by parts establishes

$$Re(a_3\partial_x^3 v, v) = \frac{3}{2}(\partial_x a_3\,\partial_x v, \partial_x v) - \frac{1}{2}(\partial_x^3\, a_3 v, v).$$

A similar analysis for $Re(a_2\,\partial_x^2 v, v)$ completes the proof. $\square$

Applying Lemma 3.5 to L_ϕ shows that the only term of order higher than 0 is $\left([2a_2 + \frac{6a_3\partial_x\phi}{\phi} - 3\partial_x a_3]\partial_x v, \partial_x v\right)$. Thus, if this term were negative, an *a priori* estimate would be obtained for v. This motivates the choice of a gauge ϕ that should satisfy

$$2a_2 + \phi^{-1}6a_3\partial_x\phi - 3\partial_x a_3 \le 0.$$

A choice of equality in this equation can be made, and this choice is enough for the estimate (3.2), but by exploiting the inequality the local smoothing estimate (3.3) is proved. The exact choice of a gauge is summarized in the following lemma.

Lemma 3.6. *For $\delta > \frac{1}{2}$, let $\phi(x, t)$ be a solution of the ODE*

$$\begin{cases} 6a_3\partial_x\phi = \left(3\partial_x a_3 - c_\delta\langle x\rangle^{-2\delta} - 2a_2\right)\phi, \\ \phi(t, 0) = 1, \end{cases}$$

where $c_\delta = 0$ or 1. Then ϕ is a gauge in the sense of Definition 3.3 and is independent of δ if $c_\delta = 0$.

Proof. The ODE for ϕ is solved explicitly as

$$\phi(x, t) = \sqrt{\frac{a_3(x, t)}{a_3(t, 0)}}\, e^{-\int_0^x \frac{a_2(y,t)}{3a_3(y,t)}dy} e^{-\int_0^x \frac{c_\delta\, dy}{6a_3(y,t)\langle y\rangle^{2\delta}}}.$$

By (A3) $e^{-\int_0^x \frac{a_2(y,t)}{3a_3(y,t)}dy} \approx 1$. (A1) implies $\sqrt{\frac{a_3(x,t)}{a_3(t,0)}} \approx 1$.

Finally, as $\delta > \frac{1}{2}$,

$$e^{-\int_0^x \frac{c_\delta dy}{6a_3(y,t)\langle y\rangle^{2\delta}}} = \begin{cases} 1, & \text{if } c_\delta = 0, \\ \text{function in } \mathcal{B}^N, & \text{if } c_\delta = 1. \end{cases}$$

A computation for $\partial_t\phi$ and $\partial_x^j\phi$ for $j = 1$, 2 and 3 and using $(A1)$–$(A3)$ finishes the proof. □

Proof of Proposition 3.1. By Lemma 3.4 it suffices to prove the proposition for v satisfying (3.4).

Applying Lemma 3.5 for L_ϕ implies that

$$\partial_t \int |v|^2 dx = (\left[2a_2 + \frac{6a_3\partial_x\phi}{\phi} - 3\partial_x a_3\right] \partial_x v, \partial_x v) + (\tilde{b}_0 v, v) \\ - 2Re(A_0(\phi v), \phi v) + (f, \phi v),$$

where $\tilde{b}_0$ is obtained from Lemma 3.5 applied to L_ϕ. With ϕ chosen from Lemma 3.6, this implies

$$\partial_t \int |v|^2 dx \le -c_\delta(\langle x\rangle^{-2\delta} v, v) + (\tilde{b}_0 v, v) - 2Re(A_0(\phi v), \phi v) + (f, \phi v).$$

By $(A4)$, $A_0 : L^2 \to L^2$ is bounded. Moreover, by Definition 3.3 and $(A2)$, $\phi \in L^\infty$ and $\tilde{b}_0 \in L^\infty$. Hence

$$\partial_t \int |v|^2 \le C(\int |v|^2 dx + \|v\|_{L^2}\|f\|_{L^2}) - \|\langle x\rangle^{-\delta} \partial_x v\|_{L^2}^2.$$

For $c_\delta = 0$ an application of the Grownwall Lemma implies (3.2) for v.

Moreover, moving the $\partial_x v$ term to the left hand side for $c_\delta = 1$ and integrating in time give

$$\int_0^T \|\langle x\rangle^{-\delta} \partial_x v\|^2 dt \le C \int_0^T (\int |v|^2 dx + \|v\|_{L^2}\|f\|_{L^2})dt + \|v_0\|_{L^2}^2 - \|v\|_{L^2}^2 \\ \le (C(1+T) - 1) \sup_{0\le t\le T} \|v(t)\|_{L^2}^2 + \|v_0\|_{L^2}^2 + (\int_0^T \|f(t)\|_{L^2} dt)^2.$$

Using (3.2) completes the proof of (3.3). □

Proposition 3.1 can be strengthened to an H^s estimate.

Proposition 3.7. *Let L be as in* (1.1)*, whose coefficients a_j satisfy* $(A1)$–$(A3)$*. Then for any $s \in \mathbb{R}$ there exist constants $C(s)$ and $\tilde{C}(s,\delta)$ for any $\delta > \frac{1}{2}$ such that for any $u \in C^1_{[0,T]}H^s \cap C^0_{[0,T]}H^{s+3}$ the following estimates hold:*

$$\sup_{0\le t\le T} \|u(t)\|_{H_x^s} \le Ce^{CT}(\|u(0)\|_{H_x^s} + \int_0^T \|\partial_t u + Lu\|_{H_x^s} dt), \tag{3.6}$$

$$\sup_{0\le t\le T} \|u(t)\|_{H_x^s} \le Ce^{CT}(\|u(T)\|_{H_x^s} + \int_0^T \|-\partial_t u + L^* u\|_{H_x^s} dt),$$

where L^* is the adjoint of L. Moreover,

$$\|\langle x\rangle^{-\delta} \partial_x u\|_{L^2_{[0,T]}H_x^s} \le \tilde{C}(1+\sqrt{T})e^{\tilde{C}T}(\|u_0\|_{H^s} + \int_0^T \|f(t)\|_{H^s} dt).$$

Corollary 3.8. *By Theorem* 23.1.2 *on page* 387 *in* [7]*, the proof of Theorem* 2.1 *reduces to Proposition* 3.7*.*

Proposition 3.7 is reduced to Proposition 3.1. Observe that

$$f = \partial_t u + Lu \text{ if and only if } J^s f = \partial_t J^s u + LJ^s u + [J^s L]J^{-s} J^s u,$$

where J^s is a pseudo differential operator with the symbol $\langle\xi\rangle^s$. Therefore to prove (3.6) it suffices to show that Proposition 3.1 applies to the operator $\tilde{L} = L + [J^s L]J^{-s}$.

Lemma 3.9. *Let $\tilde{L} = L + [J^s L]J^{-s}$ with L from* (1.1) *that satisfies* $(A1)$ *and* $(A2)$. *Then*

$$\begin{aligned} &\tilde{L} = a_3\partial_x^3 + a_0 + \sum_{i=1}^{2}(a_j + \tilde{a}_j)\partial_x^j + A_s(t,x,\partial_x) \\ &\text{with} \quad \tilde{a}_2 = s\partial_x a_3 \quad \text{and} \quad \tilde{a}_1 = s\partial_x a_2 + \frac{s(s-1)}{2}\partial_x^2 a_3, \end{aligned} \tag{3.7}$$

where $A_s \in S^0$, whose semi-norms depend on the coefficient bounds $(A2)$ *for $N = N(s)$ and hence satisfies* $(A4)$.

Furthermore, the coefficients $\tilde{a}_j$ for $i = 1, 2$ satisfy $(A2)$–$(A3)$.

Proof. From the first term in the calculus of PDO, $[J^s L]J^{-s} \in S^2$. A further expansion of $[J^s, a_3\partial_x^3]$ gives

$$\begin{aligned} \sigma([J^s, a_3\partial_x^3]) &= \sum_{1\le|\alpha|\le 2} \frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha \langle\xi\rangle^s \partial_x^\alpha (a_3(i\xi)^3) \mod S^s \\ &= s\partial_x a_3 (i\xi)^2 \langle\xi\rangle^s + \frac{s(s-1)}{2}\partial_x^2 a_3 i\xi \langle\xi\rangle^s \mod S^s, \end{aligned}$$

where the substitution $\xi^2 = \langle\xi\rangle^2 - 1$ was used and the terms of order s were absorbed into the remainder. Performing a similar computation for the remaining terms in $[J^s L]$ and composition with J^{-s} verifies (3.7).

It is immediate from (3.7) that $\tilde{a}_j$ satisfies $(A2)$. To verify $(A3)$ observe that

$$\int_0^x \frac{\tilde{a}_2(y,t)}{|a_3(y,t)|}dy = s\,\text{sign}(a_3)\log\frac{a_3(x,t)}{a_3(0,t)} \in C^1_{[0,T]}L^\infty_x$$

by $(A1)$ and $(A2)$. □

Remark 3.10. A simple computation shows that the adjoint L^* of the operator L from (1.1) is

$$\begin{aligned} L^* = &- a_3\partial_x^3 + (a_2 - 3\partial_x a_3)\partial_x^2 + (a_1 + 2\partial_x a_2 - 3\partial_x^2 a_3)\partial_x \\ &+ (a_0 - \partial_x a_1 + \partial_x^2 a_2 - \partial_x^3 a_3), \end{aligned}$$

whereas a substitution $t \to T - t$ transforms (1.1) to

$$\begin{cases} -\partial_t u(T-t) + Lu(T-t) = f(T-t), \\ u(T-t)\mid_{t=0} = u(T). \end{cases}$$

Both L^* and $L(T-t)$ satisfy $(A1)$–$(A3)$.

Corollary 3.11. *Lemma* 3.9, *Remark* 3.10 *and Proposition* 3.1 *imply Proposition* 3.7.

This completes the proof of Theorem 2.1 by Corollary 3.8.

4. Ill-posedness

Ill-posedness is proved by justifying the formal geometrical optics argument (cf. [4]) for a special choice of initial data. It is instructive to first consider the case of constant dispersion $a_3 \equiv 1$:

$$\partial_t v + \partial_x^3 v + \sum_{j=0}^{2} c_2(x,t)\partial_x^j v = g. \tag{4.1}$$

Then condition $(A3N')$ is equivalent to

$$\sup_{N>0,x_0} \int_{x_0-N}^{x_0} c_2(x',0)dx' = \infty.$$

The general case of (1.1) is later reduced to ill-posedness for (4.1). For this reduction it is desirable to relax condition $(A2)$ to smooth, but not necessarily bounded, coefficients:

(A2′): Assume that $c_2 \in C_t^1 C_x^0 \cap C_t^0 C_x^2$ and $c_1,\ c_0 \in C_{t,x}^0$.

From now on, the notation $C = C(\alpha)$ means that there exists a constant $C \geq 1$ that depends continuously α and may depend on the norms of coefficients c_j evaluated on some compact set, whose size also depends on α. The constants required to be small are reciprocal to the large constants.

The proof of the ill-posedness for (4.1) rests on the following explicit construction that violates estimate (2.1) for $s = 0$. Let $\psi(x) = \eta^{-\frac{1}{2}}\psi_0(\frac{x-x_0}{\eta})$, where $\psi_0 \in C_0^\infty([-1,1])$, $\|\psi_0\|_{L^2} = 1$ and the small parameter $0 < \eta \leq 1$ is to be chosen. Then

$$\operatorname{supp}\psi \subset [x_0-\eta, x_0+\eta],\ \|\psi\|_{L^2} = 1 \text{ and } \|\psi\|_{H^k} \leq C\eta^{-k} \text{ for } k \geq 0. \tag{4.2}$$

Define

(4.3)
$$v(x,t) := e^{iS}w, \text{ with } S = x\xi + t\xi^3, \quad \text{and} \quad w = e^{\frac{1}{3}\int_x^{x_0} c_2(x',t)dx'}\psi(x+3\xi^2 t),$$

with parameters $\xi \geq 1$, x_0, $0 < \eta \leq 1$ to be chosen. It is immediate from $(A2')$ that $w \in C_t^1 C_x^0 \cap C_t^0 C_x^3$. A substitution of the ansatz $v = e^{iS}w$ into (4.1) gives

$$g = \frac{1}{3}\int_x^{x_0} \partial_t c_2(x',t)dx' \cdot v + e^{iS}\left\{(3i\xi\partial_x^2 w + \partial_x^3 w) + 2c_2 i\xi\partial_x w\right\} + e^{iS}\left\{c_2\partial_x^2 w + c_1(i\xi\cdot w + \partial_x w) + c_0 w\right\}.$$

Taking absolute values gives

$$|g(x,t)| \leq \xi \sum_{j=0}^{3} g_j(x,t)|\partial_x^j w(x,t)|, \tag{4.4}$$

where $g_j(x,t)$ are continuous non-negative functions independent of ξ.

Observe from (4.2) and (4.3) that $\operatorname{supp}_x w(x,t) \subset [x_0 - 3\xi^2 t - \eta, x_0 - 3\xi^2 t + \eta]$. Therefore,

$$\int_x^{x_0} c_2(x',t)dx' = \int_{x_0-3\xi^2 t}^{x_0} c_2(x',t)dx' + I(x,t),$$

where on the support of $w(x,t)$, $|I(x,t)| \leq C(x_0 - 3\xi^2 t)\eta$.

Using (4.3), (4.2) and the estimate above implies that $0 < \eta \leq \frac{1}{C(x_0 - 3\xi^2 t)}$ gives

$$\|v_n(t)\|_{L^2_x} \approx_2 e^{\frac{1}{3}\int_{x_0-3\xi^2 t}^{x_0} c_2(x',t)dx'} \tag{4.5}$$

with comparability constant chosen to be 2. The estimates (4.4) and (4.5) are the main ingredients for the proof of the following theorem.

Theorem 4.1. *Suppose*

$$\sup_{N>0,x_0} \int_{x_0-N}^{x_0} c_2(x',0)dx' = \infty. \tag{4.6}$$

Then there exists a sequence $t_n \to 0$ *and sequences* x_0^n *and* ξ_n, η_n *such that* v_n *defined by* (4.3) *is in* $C_t^1 L_x^2 \cap C_t^0 H_x^3$ *and* g_n *from* (4.1) *satisfies*

$$\|v_n(t_n)\|_{L^2_x} \geq n(\|v_n(0)\|_{L^2_x} + \int_0^{t_n} \|g_n(t)\|_{L^2_x} dt) > 0, \tag{4.7}$$

$$\int_0^{t_n} \|v_n(t)\|_{L^2_x} \leq \frac{1}{n}\|v_n(0)\|_{L^2}. \tag{4.8}$$

Proof. By (4.6), there exist $x_0 \in \mathbb{R}$ and $N > 0$ such that

$$e^{\frac{1}{3}\int_{x_0-N}^{x_0} c_2(x',0)dx'} \geq 16n. \tag{4.9}$$

Let $t_n = \dfrac{N}{3\xi^2}$ with $\xi = \xi(x_0, N)$ to be chosen below. From now on only t such that $0 \leq t \leq t_n$ will be considered. For this range of t, the small parameter $\eta = \eta(x_0 - 3\xi^2 t) > 0$ can be chosen to depend only on (x_0, N). As the choice of x_0 and η completely determines ψ, ψ is independent of ξ.

To estimate the right hand side of (4.7), observe from (4.5) that

$$\frac{1}{2} \leq \|v_n(0)\|_{L^2_x} \leq 2.$$

Furthermore, (4.2) and (4.3) imply that $\operatorname{supp}_x w(x,t) \subset [x_0 - N - 1, x_0 + 1]$ for $0 \leq t \leq t_n$. Hence, w, v and g have compact supports independent of ξ and are bounded. More precisely, (4.2), (4.3) and (4.4) imply

$$\|g(t)\|_{L^2} \leq C(N, x_0)\xi.$$

Integrating this inequality in time gives

$$\int_0^{t_n} \|g(t)\|_{L^2} dt \leq \frac{C(N, x_0)}{\xi}. \tag{4.10}$$

Therefore, for $\xi \geq C(N, x_0)$, $\int_0^{t_n} \|g(t)\|_{L^2} \leq 1$. This finishes the analysis of the right hand side of (4.7).

Similarly, (4.5) implies that for $0 \leq t \leq t_n$, $\|v(t)\|_{L^2_x} \leq C(x_0, N)$. Hence, for $\xi \geq C(x_0, N)$,

$$\int_0^{t_n} \|v_n(t)\|_{L^2} dt \leq \frac{1}{2n} \leq \frac{1}{n}\|v_n(0)\|_{L^2_x}.$$

To finish the proof it suffices to show that there exists $\xi = \xi_n(x_0, N) \geq C(x_0, N)$ such that

$$\|v_n(t_n)\|_{L^2_x} \geq 4n. \tag{4.11}$$

This estimate requires a comparison of (4.5) and (4.9). To this end, by (4.5) and the Fundamental Theorem of Calculus,

$$\|v_n(t_n)\|_{L^2_x} \geq \frac{1}{2}e^{\frac{1}{3}\int_{x_0-3\xi^2 t_n}^{x_0} c_2(x',0)dx'}e^{\frac{1}{3}\int_{x_0-3\xi^2 t_n}^{x_0}\int_0^{t_n}\partial_t c_2(x',t)dx'dt},$$

whereas, using $t_n = \frac{N}{3\xi^2}$,

$$|\int_{x_0-N}^{x_0}\int_0^{\frac{N}{3\xi^2}} \partial_t c_2(x',t)dtdx'| \leq \frac{C(x_0,N)}{\xi^2} \leq \log 2$$

for $\xi \geq C(x_0,N)+1$. Combining the last two estimates and (4.9) implies (4.11). □

4.1. **Reduction to constant dispersion.** Ill-posedness for (1.1) relies on a change of variables to reduce to (4.1).

Definition 4.2. For a_3 satisfying $(A1)$ and $(A2)$ define

$$y(x,t) = \int_0^x a_3^{-\frac{1}{3}}(x',t)dx'. \tag{4.12}$$

This construction allows us to replace the roles of x and y as follows.

Lemma 4.3. *Consider*

$$y - y(x,t) = 0 \tag{4.13}$$

with $y(x,t)$ *from* (4.12). *Then there exists a unique smooth function* $x = x(y,t)$ *that satisfies* (4.13). *Moreover,*

$$\frac{\partial x}{\partial y} = \frac{1}{\frac{\partial y}{\partial x}} = a_3^{\frac{1}{3}}(x,t).$$

Proof. By $(A1)$ and the Fundamental Theorem of Calculus, $\frac{\partial y}{\partial x}(x,t) = a_3^{-\frac{1}{3}}(x,t) \neq 0$ for all (x,t). An application of the Implicit Function Theorem for (4.13) completes the proof. □

Define

$$v(y,t) = a_3^{-\frac{1}{3}}(x(y,t),t)u(x(y,t),t)$$

using the Lemma 4.3. Equivalently

$$u(x,t) = \frac{1}{\frac{\partial y}{\partial x}}v(y,t). \tag{4.14}$$

From this definition, L^2 norms of u and v are comparable by $(A1)$:

$$\|u(t)\|^2_{L^2_x} = \int a_3(x,t)|v(y,t)|^2 dy \approx_{\lambda,\Lambda} \|v(t)\|^2_{L^2_y}. \tag{4.15}$$

A computation shows that

$$\partial_t u = \frac{1}{\frac{\partial y}{\partial x}}(\partial_t v + \frac{\partial y}{\partial t}\partial_y v - \frac{\partial_t \frac{\partial y}{\partial x}}{\frac{\partial y}{\partial x}}v), \qquad \partial_x u = \partial_y v - v\frac{\partial_x^2 y}{(\frac{\partial y}{\partial x})^2},$$

$$\partial_x^2 u = \partial_y^2 v\frac{\partial y}{\partial x} + \sum_{j=0}^{1} b_j(\partial_x y, \partial_x^2 y)\partial_y^j v,$$

$$\partial_x^3 u = \partial_y^3 v(\frac{\partial y}{\partial x})^2 + \sum_{j=0}^{1} \tilde{b}_j(\partial_x y, \partial_x^2 y, \partial_x^3 y)\partial_y^j v$$

for smooth functions b_j and $\tilde{b}_j$. Using this computation and $a_3(x,t)(\frac{\partial y}{\partial x})^3 \equiv 1$, substitute (4.14) into (1.1) to get

$$\partial_t v + \partial_y^3 v + \sum_{j=0}^{2} c_j(y,t)\partial_y^j v = g, \tag{4.16}$$

where the coefficients c_j satisfy $(A2')$ and, in particular,

$$c_2(y,t) = a_2(x,t)a_3^{-\frac{2}{3}}(x,t), \quad g(y,t) = \frac{\partial y}{\partial x} f(x,t). \tag{4.17}$$

The relationship between f and g is identical to (4.14); thus (4.15) implies

$$\|f(t)\|_{L^2_x} \approx_{\lambda,\Lambda} \|g(t)\|_{L^2_y}. \tag{4.18}$$

Therefore, (1.1) can be reduced to (4.1), which was analyzed in Theorem 4.1.

Lemma 4.4. *Suppose $(A1)$, $(A2)$ and $(A3N')$ hold. Let $s \in \mathbb{R}$. Then there exists a sequence $w_n \in C^1 H^s \cap C_t^0 H^{s+3}$ and $t_n \to 0$ such that*

$$\|w_n(t_n)\|_{H^s} \gtrsim n(\|w_n(0)\|_{H^s_x} + \int_0^{t_n} \|(\partial_t + L)w_n(t)\|_{H^s_x} dt) > 0. \tag{4.19}$$

Note that (4.6) for $c_2(y,t)$ defined by (4.17) is equivalent to $(A3N')$. Therefore, Theorem 4.1 applies to (4.16). Define u_n by applying (4.14) to v_n from Theorem 4.1, which can be written explicitly as

$$u_n(x,t) = a_3^{\frac{1}{3}}(x,t)e^{iy\xi_n + it\xi_n^3} e^{\frac{1}{3}\int_x^{x_0} \frac{a_2}{a_3}(x',t)dx'} \psi_n(y + 3\xi_n^2 t). \tag{4.20}$$

Let $f_n = \partial_t u_n + L u_n$ and g_n be defined by (4.17). Then (4.7), (4.15) and (4.18) imply up to a constant

$$\|u_n(t_n)\|_{L^2_x} \gtrsim n(\|u_n(0)\|_{L^2_x} + \int_0^{t_n} \|(\partial_t + L)u_n(t)\|_{L^2_x} dt) > 0. \tag{4.21}$$

Likewise, (4.8) holds for u_n instead of v_n. This completes the proof of (4.19) for $s = 0$ by taking $w_n := u_n$.

For the general $s \in \mathbb{R}$, commute J^s with L as in Lemma 3.9: $J^s(\partial_t + L) = (\partial_t + \tilde{L})J^s$, where $\tilde{L} = L + [J^s L]J^{-s}$. By Lemma 3.9, $\tilde{L} = P + A_s(x,t,\partial_x)$, where $A_s \in S^0$ and the differential operator P satisfies $(A1)$, $(A2)$ and $(A3N')$. Define $\tilde{u}_n$ via (4.20) with L replaced by P; i.e. $\tilde{u}_n$ differs from u_n by a factor of $(\frac{a_3(x_0,t)}{a_3(x,t)})^s$. Further, define

$$w_n(x,t) = J^{-s}\tilde{u}_n(x,t).$$

Hence $\|w_n(t)\|_{H^s} = \|\tilde{u}_n(t)\|_{L^2}$. Applying (4.21) to $\tilde{u}_n$ implies

$$\|w_n(t_n)\|_{H^s} \gtrsim n(\|w_n(0)\|_{H^s_x} + \int_0^{t_n} \|(\partial_t + P)\tilde{u}_n(t)\|_{L^2_x} dt) > 0.$$

By (4.8) for $n \gtrsim \|A_s\|_{L^2 \to L^2}$,

$$\int_0^{t_n} \|A_s \tilde{u}_n\|_{L^2} \lesssim \|w_n(0)\|_{H^s_x}.$$

As $J^s f = (\partial_t + P)\tilde{u}_n + A_s\tilde{u}_n$, combining the last two estimates implies that w_n satisfies (4.19) up to a constant.

4.2. **Proof of ill-posedness.**

Corollary 4.5. *Lemma* 4.4 *implies that* (2.1) *fails, or, more generally, for any* $T > 0$ *there is no non-decreasing function* $C(T') : [0, T] \to \mathbb{R}$ *such that*

$$\sup_{0 \le t \le T'} \|u(t)\|_{H^s} \le C(T')(\|u_0\|_{H^s} + \int_0^{T'} \|f(t)\|_{H^s} dt) \tag{4.22}$$

holds for all $u \in C^0_{[0,T]} H^s$ *solving* (1.1).

Proof. Assuming (4.22), for the sake of contradiction, and using (4.19) imply that $C(t_n) \ge n$ for all $n \in \mathbb{N}$. As $t_n \to 0$ and $C(t)$ is non-decreasing in t, $C(t) \ge n$ for all $0 < t \le T$ and $n \in \mathbb{N}$. This is a contradiction. □

Corollary 4.6. *Assuming* (*A*1), (*A*2) *and* (*A*3*N*) *implies that* (1.1) *is ill-posed in* H^s *the sense of Theorem* 2.4.

Proof. Suppose, for the sake of contradiction, that (2.3) holds for some $[0, T]$ and some continuous function $C(t_0, t)$ for $0 \le t_0 \le t \le T$. Define a non-decreasing function $C(T') = \sup_{0 \le t_0 \le t \le T'} C(t_0, t)$. Then by the Duhamel principle every solution of (1.1) satisfies

$$u(t) = S(t, 0)u_0 + \int_0^t S(t, t_0)f(t_0)dt_0,$$

where $u(t) = S(t, t_0)g$ solves (1.1) on $[t_0, T]$ with data $u(t_0) = g$ and $f \equiv 0$. Moreover,

$$\sup_{0 \le t_0 \le t \le T'} \|S(t, t_0)\| \le C(T').$$

Thus the Duhamel principle implies (4.22) for all $u \in C^0_{[0,T]} H^s$ solutions of (1.1), which contradicts Corollary 4.5. □

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