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# A Sharp Estimate for the Weighted Hilbert Transform via Bellman Functions

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## 1. Introduction

It has long been of interest to find sharp estimates for the norm for the Hilbert transform and related operators in $L^p(\omega)$. In this paper we look at the estimates of the Hilbert transform in weighted spaces $L^2(\omega)$. Buckley [1] proved that the Hilbert transform is bounded by the square of the classical $A_2$ constant of the weight. In [8], Petermichl and Pott improved this estimate to the $3/2$ power. Their result is the best currently known but is probably not sharp. Here we consider a different $A_2$ constant that seems more natural to the unit disk: the invariant $A_2$ constant. The invariant $A_2$ constant is defined using Poisson averages instead of box averages.

We obtain the sharp estimate for the weighted Hilbert transform in terms of invariant $A_2$ by the method of Bellman functions. We estimate the norm using duality and then split the quantity to be estimated into four integrals, each of which can be estimated using a Bellman function. Our proof follows the outline of the sharp uniform estimates for dyadic martingales in [6] and [10].

The Hilbert transform can be considered the continuous analog of the dyadic martingale transforms, and the $A_2$ invariant constant can be viewed as the continuous analog of the regular $A_2$ constant. We also include a bilinear embedding theorem of Sawyer type. A two-weighted version was used by Nazarov, Volberg, and Treil [7], who referred to it as a bilinear Carleson embedding theorem. We change the assumptions to fit the one-weighted situation; they are no longer necessary but do allow for a concise statement. This change of conditions leads to a different choice of variables in our Bellman function and a set of conditions that is much simpler to check. In [7], the key assumption was boundedness of a certain positive operator on test functions. In our version, this assumption has been replaced by estimates for three simple sums.

## 2. Definitions and Statements

We consider the space $L^2_{\mathbb{T}}(\omega)$, where $\omega$ is a positive $L^1$ function known as a weight. Let $m$ be normalized Lebesgue measure on $\mathbb{T}$. The norm of $f \in L^2_{\mathbb{T}}(\omega)$ is $\left(\int_{\mathbb{T}} |f|^2 \omega \, dm\right)^{1/2}$ and is denoted by $\|f\|_\omega$. We are concerned with a special class of weights called $A_2$. We say $\omega \in A_2$ if

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$$\sup_I \langle\omega\rangle_I \langle\omega^{-1}\rangle_I = Q_2(\omega) < \infty. \tag{2.1}$$

Here the supremum is taken over all subarcs $I \subset \mathbb{T}$. The notation $\langle\omega\rangle_I$ means the average of the function $\omega$ over $I$.

In this paper, we will mostly use a different constant associated to each $\omega \in A_2$. Define $Q_2^{\mathrm{inv}}(\omega)$ by

$$\sup_{z\in\mathbb{D}} \omega(z)\omega^{-1}(z) = Q_2^{\mathrm{inv}}(\omega), \tag{2.2}$$

where $\omega(z)$ denotes the harmonic extension of $\omega$; that is,

$$\omega(z) = \int \omega(t) P_z(t)\, dm(t),$$

where $P_z(t) = (1-|z|^2)/|1-\bar{z}t|^2$. Note that, in general, $\omega^{-1}(z)$ and $\omega(z)^{-1}$ have different meanings: $\omega^{-1}(z)$ denotes the extension of the reciprocal of $\omega$, whereas $\omega(z)^{-1}$ is the reciprocal of the extension. $Q_2^{\mathrm{inv}}(\omega)$ is finite iff $Q_2(\omega)$ is finite. In [4], the following exact sharp relationship between the two different $A_2$ constants was proven:

$$c_1 Q_2(\omega) \le Q_2^{\mathrm{inv}}(\omega) \le c_2 Q_2(\omega)^2.$$

Because it involves only harmonic extensions, the invariant $A_2$ constant $Q_2^{\mathrm{inv}}(\omega)$ considered in (2.2) is invariant under Möbius transforms.

In what follows, $H$ stands for Hilbert transform on the circle $\mathbb{T}$. (In some texts, the Hilbert transform is defined as the operator we denote by $H_0$.) This transform $H$ acts on trigonometric polynomials as follows:

$$H\Big(\sum a_k e^{i\theta k}\Big) = -i \sum_{k\ge 0} a_k e^{i\theta k} + i \sum_{k<0} a_k e^{i\theta k}.$$

Let $H_0$ be the operator $H + iP_0$, where $P_0 \colon f \mapsto f(0)$. Our main result is the following.

THEOREM 2.1. *$H : L^2_{\mathbb{T}}(\omega) \to L^2_{\mathbb{T}}(\omega)$ has operator norm $\|H\| \le cQ_2^{\mathrm{inv}}(\omega)$, where $c$ does not depend on $Q_2^{\mathrm{inv}}(\omega)$.*

Observe that, in our notation, $Q_2^{\mathrm{inv}}(\omega) = \sup_{z\in\mathbb{D}} \omega(z)\omega^{-1}(z)$. We will show sharpness of this result in Section 5.

## 3. Proof of Theorem 2.1

Recall that, for $f(t)$ a function on $\mathbb{T}$, we use $f(z)$ to denote its harmonic extension to the disk.

We have $\|P_0\|_{L^2(\omega)\to L^2(\omega)} \le \sqrt{Q_2^{\mathrm{inv}}(\omega)}$, since

$$\|P_0(f)\|_\omega^2 = |f(0)|^2\omega(0) \le (|f|^2\omega)(0)\omega^{-1}(0)\omega(0) \le Q_2^{\mathrm{inv}}(\omega)\|f\|_\omega^2.$$

Since $\|H\| \le \|H_0\| + \|P_0\|$ and $Q_2^{\mathrm{inv}}(\omega) \ge 1$, it suffices to show that $\|H_0\| \le cQ_2^{\mathrm{inv}}(\omega)$. We estimate $\|H_0\|_{L^2_{\mathbb{T}}(\omega)\to L^2_{\mathbb{T}}(\omega)}$ by duality. Since $(H_0 tf, g/t) = (H_0 f, g)$, it is enough to show that $|(H_0 f, g)| \le c Q_2^{\mathrm{inv}}(\omega)(\|f\|_\omega^2 + \|g\|_{\omega^{-1}}^2)$ for all $f \in L^2_{\mathbb{T}}(\omega)$ and $g \in L^2_{\mathbb{T}}(\omega^{-1})$ (just use $t = \sqrt{\|g\|_{\omega^{-1}}/\|f\|_\omega}$ if $f \neq 0$). It suffices to consider real-valued and positive functions $f$ and $g$. Assuming positivity will both abbreviate notation and allow for a smaller domain of our Bellman functions. We polarize the formula in [3, p. 236], which is a simple consequence of Green's formula, and obtain

$$\int_{\mathbb{T}} (H_0 f - H_0 f(0))(g - g(0))\, dm = \frac{1}{2\pi} \int_{\mathbb{D}} (\nabla H_0 f)(\nabla g) \log \frac{1}{|z|}\, dA(z).$$

Because $H_0 f(0) = 0$, the left-hand side equals $(H_0 f, g)$. We have $|\nabla H_0 f| = |\nabla f|$ by the Cauchy–Riemann equations, since $f + i H_0 f$ is holomorphic. Therefore,

$$|(H_0 f, g)| \le \frac{1}{2\pi} \int_{\mathbb{D}} |\nabla f||\nabla g| \log \frac{1}{|z|}\, dA(z). \tag{3.3}$$

Note that, for $f$ real-valued, $|\nabla f| = 2|\partial f/\partial z|$, where the latter denotes the derivative of the harmonic extension of $f$ defined by $\partial f/\partial z = 1/2(\partial f/\partial x - i\partial f/\partial y)$. We will write $f(z)'$ for the holomorphic function $\partial f/\partial z$.

We can now add and subtract terms to the integrand and then use the triangle inequality to split the integral into the following four parts:

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z)'||g(z)'| \log \frac{1}{|z|}\, dA(z) \\
&\quad \le \int_{\mathbb{D}} |f(z)||g(z)| \left| \frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right| \log \frac{1}{|z|}\, dA(z) \\
&\qquad + \int_{\mathbb{D}} |f(z)||g(z)| \left| \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right| \log \frac{1}{|z|}\, dA(z) \\
&\qquad + \int_{\mathbb{D}} |f(z)||g(z)| \left| \frac{\omega(z)'}{\omega(z)} \right| \left| \frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \log \frac{1}{|z|}\, dA(z) \\
&\qquad + \int_{\mathbb{D}} |f(z)||g(z)| \left| \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{\omega(z)'}{\omega(z)} \right| \log \frac{1}{|z|}\, dA(z).
\end{aligned}$$

Readers who are familiar with [7] will immediately see the resemblance to the four sums in the dyadic setting.

Equation (3.3) gives us some intuition as to why we can prove our result in a manner very similar to [7]: the Hilbert transform can be estimated via an expression involving absolute values of derivatives of the harmonic extensions. These are an obvious analog of dyadic martingale differences. The rest of the proof follows Bellman function methods. The first integral can be controlled in the same way as done in [7]; in fact, the proofs are identical. For the second and third integral we need to proceed in two steps. Again, we want to use the same proof as in [7], but in order to do so we must have an estimate for a certain Green's potential function involving the weight. But there is a dyadic analog for this estimate, which showed up in the proof for sharp bounds for the dyadic square function in $L^2(\omega)$ and was proven by a Bellman function technique (see [5]). We will use the same Bellman function to obtain estimates for the Green's potential. The fourth integral requires what is known as a bilinear harmonic embedding theorem. It is closely related to Sawyer's embedding theorems with weights. But it is a bilinear version of weighted embedding theorems and the most important ingredient of our proof. It gives an efficient and relatively simple sufficient condition for our bilinear estimate to hold. The use of this simpler sufficient condition for the bilinear embedding theorem is quite different from the approach in [7]. We observed that one can omit the most difficult part of [7] by inventing the aforementioned sufficient conditions. We formulate a version with relatively simple embedding conditions, again Green's potentials. The appropriate Bellman function is constructed, and we give an explicit expression for this function. We find the appropriate bounds for the embedding conditions using Bellman functions found in [7] and [10].

Before we start to estimate the four integrals, we need the following lemma to relate Laplacians to second differentials. This lemma is an elementary but crucial key to translating dyadic Bellman function methods to the continuous setting.

LEMMA 3.1. *If $b(z) = B(h(z))$, where $h = (f_i)_i : \mathbb{C} \to \mathbb{R}^n$ and $B : \mathbb{R}^n \to \mathbb{R}$ with $B$ and $h$ sufficiently smooth, then*

$$\Delta b(z) = 4\left(d^2B(h(z))\left(\frac{\partial f_i}{\partial z}\right)_i, \left(\frac{\partial f_i}{\partial z}\right)_i\right) + 4(\nabla B)(h(z))\left(\frac{\partial^2 f_i}{\partial z \partial \bar{z}}\right)_i. \tag{3.4}$$

*In particular, if all $f_i$ are harmonic, then*

$$\Delta b(z) = 4\left(d^2B(h(z))\left(\frac{\partial f_i}{\partial z}\right)_i, \left(\frac{\partial f_i}{\partial z}\right)_i\right). \tag{3.5}$$

*Proof*. The proof is by elementary computation, using harmonicity for (3.5). Also note that $\Delta = 4\partial^2/\partial z \partial \bar{z}$. □

We will bound all integrals using Bellman functions. Each variable carries meaning, usually harmonic extensions of functions or Green's potentials for some fixed $z$. The following variables show up frequently:

$$X = f^2\omega(z), \quad Y = g^2\omega^{-1}(z),$$
$$x = f(z), \qquad y = g(z),$$
$$r = \omega^{-1}(z), \quad s = \omega(z).$$

Recall that $f^2\omega(z)$ denotes the harmonic extension of the product $f^2(t)w(t)$. If we assume $f$, $g$ to be real and positive, then all variables will be positive. Furthermore, using $\int h(t)P_z(t)\,dm(t) = \int h(t)w(t)^{1/2}w(t)^{-1/2}P_z(t)\,dm(t)$ together with the Cauchy–Schwarz inequality yields the following natural estimates:

$$1 \le rs \le Q \quad \text{(if we write } Q \text{ for } Q_2^{\text{inv}}(\omega)); \tag{3.6}$$

$$x^2 \le Xr \quad \text{and} \quad y^2 \le Ys. \tag{3.7}$$

These restrictions give a natural domain for our Bellman functions.

### *3.1. The First Integral*

Consider the following function of six real variables:

$$B(X, x, r, Y, y, s) = X - \frac{x^2}{r} + Y - \frac{y^2}{s}.$$

We get the following size estimate within the natural domain of $B$:

$$0 \le B \le X + Y.$$

Direct computation of the second differential yields

$$-d^2 B = \frac{2x^2}{r}\left|\frac{dx}{x} - \frac{dr}{r}\right|^2 + \frac{2y^2}{s}\left|\frac{dy}{y} - \frac{ds}{s}\right|^2. \tag{3.8}$$

Also consider the function $b\colon \mathbb{C} \to \mathbb{R}$, where

$$b(z) = B(h(z)) = B(f^2\omega(z), f(z), \omega^{-1}(z), g^2\omega^{-1}(z), g(z), \omega(z)).$$

Then we obtain the following estimate for $-\Delta b(z)$ using (3.5) and (3.8):

$$\begin{aligned} -\Delta b(z) &= 8\frac{|f(z)|^2}{\omega^{-1}(z)}\left|\frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)}\right|^2 + 8\frac{|g(z)|^2}{\omega(z)}\left|\frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)}\right|^2 \\ &\ge 16\frac{|f(z)g(z)|}{\sqrt{\omega^{-1}(z)\omega(z)}}\left|\frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)}\right|\left|\frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)}\right| \\ &\ge 16\frac{|f(z)g(z)|}{\sqrt{Q}}\left|\frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)}\right|\left|\frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)}\right|. \end{aligned}$$

To estimate the first integral, we use the previous estimate for $-\Delta b(z)$ and Green's second identity

$$\int_O (u\Delta v - v\Delta u)\, dA = \int_{\partial O}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$$

for the annulus $\mathbb{D} \setminus \varepsilon\mathbb{D}$ with $\varepsilon$ arbitrarily small:

$$\begin{aligned} &\int_{\mathbb{D}} |f(z)g(z)|\left|\frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)}\right|\left|\frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)}\right| \log\frac{1}{|z|}\, dA(z) \\ &\le c Q_2^{\text{inv}}(\omega)^{1/2}\int_{\mathbb{D}} -\Delta b(z)\log\frac{1}{|z|}\, dA(z) \\ &= c Q_2^{\text{inv}}(\omega)^{1/2}\left(b(0) - \int_{\mathbb{T}} b\, dm\right) \\ &\le c Q_2^{\text{inv}}(\omega)^{1/2}(\|f\|_\omega^2 + \|g\|_{\omega^{-1}}^2). \end{aligned}$$

The last step uses that $b \equiv 0$ on $\mathbb{T}$ and that the size estimate $B \le X + Y$ means $b(0) \le \|f\|_\omega^2 + \|g\|_{\omega^{-1}}^2$.

### 3.2. *The Second and Third Integral*

The second and the third integral are analogous, so let us only prove the estimate for the second one.

We consider the function

$$B(r,s) = r\left(-\frac{4Q^2}{rs} - rs + 4Q^2 + 1\right)$$

from [5]. This function has the following properties:

$$1 \le rs \le Q \implies 0 \le B(r,s) \le cQ^2 r,$$

$$1 \le rs \le Q \implies -d^2B \ge Cs(dr)^2.$$

Let us also consider the function $b\colon \mathbb{C} \to \mathbb{R}$, where

$$b(z) = B(h(z)) = B(\omega^{-1}(z), \omega(z)),$$

$$0 \le b(z) \le cQ^2\omega^{-1}(z),$$

$$-\Delta b(z) \ge c\omega(z)|\omega^{-1}(z)'|^2.$$

This function will help us to estimate the following Green's potential:

$$\begin{aligned} G(|\omega^{-1\prime}|^2\omega)(z) &= \int_{\mathbb{D}} \log\frac{1}{|S_z(\xi)|}|\omega^{-1}(\xi)'|^2\omega(\xi)\,dA(\xi) \\ &\le c\int_{\mathbb{D}} -\Delta b(\xi)\log\frac{1}{|S_z(\xi)|}\,dA(\xi) \\ &\overset{(\star)}{=} c\int_{\mathbb{D}} -\Delta b(S_{-z}(\xi))\log\frac{1}{|\xi|}\,dA(\xi) \\ &= c\left(b(z) - \int_{\mathbb{T}} b\,dm\right) \\ &\le cQ^2\omega^{-1}(z), \end{aligned}$$

where $S_z(\xi) = \frac{\xi - z}{1 - \bar{z}\xi}$, a Möbius transform. The equality $(\star)$ follows from a change of variables $\xi \mapsto S_{-z}(\xi)$. Hence we have proved that

$$G(|\omega^{-1\prime}|^2\omega)(z) \le cQ_2^{\mathrm{inv}}(\omega)^2\omega^{-1}(z)$$

and, analogously,

$$G(|\omega'|^2\omega^{-1})(z) \le cQ_2^{\mathrm{inv}}(\omega)^2\omega(z).$$

The reader should note the similarity between the estimate for the Green's potential and its dyadic analog found in [5]:

$$\frac{1}{|J|}\sum_{I\subset J}|\langle\omega\rangle_{I_+} - \langle\omega\rangle_{I_-}|^2\langle\omega^{-1}\rangle_I|I| \le cQ_2(\omega)^2\langle\omega\rangle_J.$$

Let us introduce a new variable,

$$G = G(|\omega^{-1\prime}|^2\omega)(z).$$

Now we are ready to steal the Bellman function used to prove weighted dyadic embedding theorem from [7]. We let

$$B(X, x, r, G, Y, y, s) = X - \frac{x^2}{r + G/Q^2} + Y - \frac{y^2}{s}.$$

PROPOSITION 3.2. *The function*

$$f(w, x, y, z) = w - \frac{x^2}{y + z} \tag{3.9}$$

*is concave in the domain* $\{y > 0$ *and* $z \geq 0\}$.

*Proof*. The matrix

$$-d^2 f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{y+z} & \frac{-2x}{(y+z)^2} & \frac{-2x}{(y+z)^2} \\ 0 & \frac{-2x}{(y+z)^2} & \frac{2x^2}{(y+z)^3} & \frac{2x^2}{(y+z)^3} \\ 0 & \frac{-2x}{(y+z)^2} & \frac{2x^2}{(y+z)^3} & \frac{2x^2}{(y+z)^3} \end{pmatrix}$$

is positive semidefinite. □

Hence $B$, as a sum of concave functions, is concave. Consider

$$b(z) = B(h(z)) = B(f^2\omega(z), f(z), \omega^{-1}(z), G(|\omega^{-1\prime}|^2\omega)(z), g^2\omega^{-1}(z), \omega(z)).$$

We use equation (3.4) to estimate the part involving $X, x, r, G$, where the concavity of $B$ allows us to drop the part involving the second differential. We need only consider partial derivatives in the "nonharmonic variable" $G$. Note that $-\Delta G(|\omega^{-1\prime}|^2\omega) = |\omega^{-1\prime}|^2\omega$. We use (3.5) and (3.8) for the part involving $Y, y, s$. This yields

$$\begin{aligned} -\Delta b(z) &\geq Q^{-2}\frac{f(z)^2(-\Delta G(|\omega^{-1\prime}|^2\omega)(z))}{(\omega^{-1}(z) + Q^{-2}G(|\omega^{-1\prime}|^2\omega)(z))^2} + 8\frac{g(z)^2}{\omega(z)}\left|\frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)}\right|^2 \\ &\geq cQ^{-2}\frac{f(z)^2|\omega^{-1}(z)'|^2\omega(z)}{(\omega^{-1}(z) + Q^{-2}G(|\omega^{-1\prime}|^2\omega)(z))^2} + c\frac{g(z)^2}{\omega(z)}\left|\frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)}\right|^2 \\ &\geq cQ^{-2}\frac{f(z)^2|\omega^{-1}(z)'|^2\omega(z)}{\omega^{-1}(z)^2} + c\frac{g(z)^2}{\omega(z)}\left|\frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)}\right|^2 \\ &\geq cQ^{-1}|f(z)g(z)|\left|\frac{\omega^{-1}(z)'}{\omega^{-1}(z)}\right|\left|\frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)}\right|. \end{aligned}$$

Now we use Green's second identity, the fact that $b \geq 0$ on $\mathbb{T}$, and $b(0) \leq \|f\|_\omega^2 + \|g\|_{\omega^{-1}}^2$ to estimate the second integral:

$$\begin{aligned}\int_{\mathbb{D}} |f(z)||g(z)|\left|\frac{\omega^{-1}(z)'}{\omega^{-1}(z)}\right|\left|\frac{g(z)'}{g(z)}-\frac{\omega(z)'}{\omega(z)}\right|\log\frac{1}{|z|}\,dA(z)\\ \le cQ_2^{\text{inv}}(\omega)\int_{\mathbb{D}} -\Delta b(z)\log\frac{1}{|z|}\,dA(z)\\ \le cQ_2^{\text{inv}}(\omega)(\|f\|_{\omega}^2+\|g\|_{\omega^{-1}}^2).\end{aligned}$$

### *3.3. The Fourth Integral*

We will apply the following harmonic bilinear embedding theorem. Its proof can be found in Section 4.

LEMMA 3.3. *Let $\alpha(z)\ge 0$ and let $\omega,\upsilon$ be two weights such that $1\le\omega(z)\upsilon(z)\le Q$ for all $z\in\mathbb{D}$ and*

$$\int_{\mathbb{D}}\alpha(\xi)\omega(\xi)\log\frac{1}{|S_z(\xi)|}\,dA(\xi)\le Q\omega(z),$$

$$\int_{\mathbb{D}}\alpha(\xi)\upsilon(\xi)\log\frac{1}{|S_z(\xi)|}\,dA(\xi)\le Q\upsilon(z),$$

$$\int_{\mathbb{D}}\alpha(\eta)\omega(\eta)\upsilon(\eta)\log\frac{1}{|S_\xi(\eta)|}\,dA(\eta)\le Q.$$

*Then, for $f,g\ge 0\in L^2(\mathbb{T})$, we have*

$$\int_{\mathbb{D}}\alpha(z)f(z)g(z)\log\frac{1}{|z|}\,dA(z)\le cQ\|f\|_{\upsilon^{-1}}\|g\|_{\omega^{-1}}.$$

We will apply Lemma 3.3 to the weights $\omega$ and $\upsilon=\omega^{-1}$ with $Q_2^{\text{inv}}(\omega)=Q$ and (up to a normalization constant not depending on $Q$)

$$\alpha(z)=\frac{|\omega(z)'||\omega^{-1}(z)'|}{\omega(z)\omega^{-1}(z)}.$$

We need to find the estimates for $M$, $N$, and $K$. Consider the function

$$B(s,r)=s\left(-\frac{4Q}{rs}-\frac{rs}{4Q}+4Q+1\right)$$

from [10]. This function has the following properties:

$$1\le rs\le Q\implies 0\le B(r,s)\le cQs,$$

$$1\le rs\le Q\implies -d^2B\ge Cs\left|\frac{dsdr}{sr}\right|.$$

Let us also consider the function $b\colon\mathbb{C}\to\mathbb{R}$, where

$$b(z)=B(h(z))=B(\omega(z),\omega^{-1}(z)).$$

Then $0\le b(z)\le cQ\omega(z)$ and

$$-\Delta b(z) \geq 4C\omega(z)\frac{|\omega(z)'||\omega^{-1}(z)'|}{\omega(z)\omega^{-1}(z)} = 4C\omega(z)\alpha(z).$$

This function will help us to estimate the following integral:

$$\begin{aligned}\int_{\mathbb{D}} \log\frac{1}{|S_z(\xi)|}\alpha(\xi)\omega(\xi)\,dA(\xi) &\leq c\int_{\mathbb{D}} -\Delta b(\xi)\log\frac{1}{|S_z(\xi)|}\,dA(\xi)\\ &= c\int_{\mathbb{D}} -\Delta b(S_{-z}(\xi))\log\frac{1}{|\xi|}\,dA(\xi)\\ &= c\left(b(z) - \int_{\mathbb{T}} b\,dm\right)\\ &\leq cQ\omega(z).\end{aligned}$$

Similarly, we obtain

$$\int_{\mathbb{D}} \log\frac{1}{|S_z(\xi)|}\alpha(\xi)\omega^{-1}(\xi)\,dA(\xi) \leq cQ\omega^{-1}(z),$$

which gives the desired estimates for the outer integrals of $M$ and $N$. We are left to show the bound for $K$—namely, that

$$\int_{\mathbb{D}} \log\frac{1}{|S_z(\xi)|}\alpha(\xi)\omega(\xi)\omega^{-1}(\xi)\,dA(\xi) \leq cQ.$$

Consider the function

$$B(s,r) = 4\sqrt{Q}\sqrt{sr} - sr$$

from [7]. This function has the following properties:

$$1 \leq rs \leq Q \implies 0 \leq B(r,s) \leq 4Q,$$

$$1 \leq rs \leq Q \implies -d^2B \geq c|dsdr|.$$

Let us also consider the function $b\colon \mathbb{C} \to \mathbb{R}$, where

$$b(z) = B(h(z)) = B(\omega(z), \omega^{-1}(z)).$$

Then $0 \leq b(z) \leq cQ$ and

$$-\Delta b(z) \geq 4C\omega(z)\omega^{-1}(z)\frac{|\omega(z)'||\omega^{-1}(z)'|}{\omega(z)\omega^{-1}(z)} = 4C\omega(z)\omega^{-1}(z)\alpha(z).$$

This function will take care of the following integral:

$$\begin{aligned}\int_{\mathbb{D}} \log\frac{1}{|S_z(\xi)|}\alpha(\xi)\omega(\xi)\omega^{-1}(\xi)\,dA(\xi) &\leq c\int_{\mathbb{D}} -\Delta b(\xi)\log\frac{1}{|S_z(\xi)|}\,dA(\xi)\\ &= c\int_{\mathbb{D}} -\Delta b(S_{-z}(\xi))\log\frac{1}{|\xi|}\,dA(\xi)\\ &= c\left(b(z) - \int_{\mathbb{T}} b\,dm\right)\\ &\leq cQ.\end{aligned}$$

## 4. Bilinear Carleson Embedding Theorem

Recall that the Möbius transform is given by $S_z(\xi) = \frac{\xi - z}{1-\bar{z}\xi}$.

LEMMA 4.1. *Let $\alpha(z) \geq 0$ and let $\omega, \upsilon$ be two weights such that $1 \leq \omega(z)\upsilon(z) \leq Q$ for all $z \in \mathbb{D}$ and*

$$\int_{\mathbb{D}} \alpha(\xi)\omega(\xi) \log \frac{1}{|S_z(\xi)|}\, dA(\xi) \leq Q\omega(z),$$
$$\int_{\mathbb{D}} \alpha(\xi)\upsilon(\xi) \log \frac{1}{|S_z(\xi)|}\, dA(\xi) \leq Q\upsilon(z),$$
$$\int_{\mathbb{D}} \alpha(\eta)\omega(\eta)\upsilon(\eta) \log \frac{1}{|S_\xi(\eta)|}\, dA(\eta) \leq Q.$$

*Then, for $f, g \geq 0 \in L^2(\mathbb{T})$, we have*

$$\int_{\mathbb{D}} \alpha(z) f(z) g(z) \log \frac{1}{|z|}\, dA(z) \leq cQ\|f\|_{\upsilon^{-1}}\|g\|_{\omega^{-1}}.$$

*Proof.* As before, it is more convenient to switch to Young's inequality. It suffices to show that

$$\int_{\mathbb{D}} \alpha(z) f(z) g(z) \log \frac{1}{|z|}\, dA(z) \leq cQ(\|f\|^2_{\upsilon^{-1}} + \|g\|^2_{\omega^{-1}}).$$

Let us consider the variables

$$X = f^2\upsilon^{-1}(z), \quad x = f(z), \quad r = \upsilon(z),$$
$$Y = g^2\omega^{-1}(z), \quad y = g(z), \quad s = \omega(z),$$

as well as the nonharmonic variables

$$M = \int_{\mathbb{D}} \alpha(\xi)\upsilon(\xi) \log \frac{1}{|S_z(\xi)|} \int_{\mathbb{D}} \alpha(\eta)\upsilon(\eta)\omega(\eta) \log \frac{1}{|S_\xi(\eta)|}\, dA(\eta)\, dA(\xi),$$
$$N = \int_{\mathbb{D}} \alpha(\xi)\omega(\xi) \log \frac{1}{|S_z(\xi)|} \int_{\mathbb{D}} \alpha(\eta)\omega(\eta)\upsilon(\eta) \log \frac{1}{|S_\xi(\eta)|}\, dA(\eta)\, dA(\xi),$$
$$K = \int_{\mathbb{D}} \alpha(\eta)\upsilon(\eta)\omega(\eta) \log \frac{1}{|S_z(\eta)|}\, dA(\eta).$$

We then have the following natural estimates:

$$1 \leq rs \leq Q,$$
$$x^2 \leq Xr \quad \text{and} \quad y^2 \leq Ys \quad \text{(by Jensen's inequality)},$$
$$M \leq Q^2 r \quad \text{and} \quad N \leq Q^2 s \quad \text{(by combining assumptions)},$$
$$K \leq Q \quad \text{(by assumption)}.$$

Furthermore, we may assume that all variables are strictly positive: $X, Y, x, y, r, s$ are harmonic extensions of positive functions, and in $K, M, N$ the integrands are strictly positive (for $\alpha = 0$ there is nothing to prove). This suggests the domain

$$\mathcal{K} = \{(X, x, r, Y, y, s, M, N, K) : X, x, r, Y, y, s, M, N, K > 0; \\ 1 \le rs \le Q;\ x^2 \le Xr;\ y^2 \le Ys;\ M \le Q^2 r;\ N \le Q^2 s;\ K \le Q\}.$$

Let us consider the following function of nine (!) real variables:

$$B(X, x, r, Y, y, s, M, N, K) \\ = B_1(X, x, r, M) + B_2(Y, y, s, N) + B_3(X, x, r, Y, y, s, K),$$

where

$$B_1(X, x, r, M) = X - \frac{x^2}{r + M/Q^2},$$

$$B_2(Y, y, s, N) = Y - \frac{y^2}{s + N/Q^2},$$

$$B_3(X, x, r, Y, y, s, K) \\ = \begin{cases} X + Y - \frac{x^2 s - 2xy(K/Q) + y^2 r}{rs - K^2/Q^2} & \text{if } yr - x\frac{K}{Q} > 0 \\ & \text{and } xs - y\frac{K}{Q} > 0, \\ X + Y - y^2/s & \text{otherwise, and } \frac{x^2}{r} \ge \frac{y^2}{s}, \\ X + Y - x^2/r & \text{otherwise, and } \frac{x^2}{r} \le \frac{y^2}{s}. \end{cases}$$

As before, $b(z), b_1(z), b_2(z), b_3(z)$ are the corresponding functions on $\mathbb{D}$. We now discuss the various properties of $B$.

*Derivative estimates:*

$$\frac{\partial B_1}{\partial M} \ge \frac{1}{4Q^2}\frac{x^2}{r^2},$$

$$\frac{\partial B_2}{\partial N} \ge \frac{1}{4Q^2}\frac{y^2}{s^2},$$

$$\frac{\partial B_3}{\partial K} \ge \begin{cases} \frac{c}{Q}\frac{xy}{rs} & \text{if } K \le Q\frac{yr}{4x} \text{ and } K \le Q\frac{xs}{4y}, \\ 0 & \text{otherwise.} \end{cases}$$

For the derivative estimate of $B_1$, note that $M \le Q^2 r$; use $N \le Q^2 s$ for $B_2$. The weaker derivative estimate for $B_3$ (nonnegativity) holds everywhere and is easy to check. Harder is the stronger estimate that we will only need for "small" $K$, that is, $\left\{K \le Q\frac{yr}{4x} \text{ and } K \le Q\frac{xs}{4y}\right\}$. By exchanging $x$ and $y$, we need only consider the case $x^2/r \ge y^2/s$. Let us point out that $B_3$ was taken from an early version of [7], where it was written (up to normalization) in the following form:

$$B_3(X, x, r, Y, y, s, K) = X + Y - \sup_{a>0} \beta(a, X, x, r, Y, y, s, K);$$

here

$$\beta(a, X, x, r, Y, y, s, K) = \frac{x^2}{r + a\dfrac{K}{Q}} + \frac{y^2}{s + a^{-1}\dfrac{K}{Q}}.$$

Note that $\beta$ is continuously differentiable in $a$ for $a > 0$. Testing for critical points yields

$$\frac{\partial\beta}{\partial a} = -\frac{x^2\tilde{K}}{(r + a\tilde{K})^2} + \frac{y^2\tilde{K}}{(as + \tilde{K})^2},$$

and

$$\frac{\partial\beta}{\partial a} = 0 \iff a = \frac{yr - x\tilde{K}}{xs - y\tilde{K}},$$

provided this fraction is finite and different from 0. We see that $\partial\beta/\partial a$ changes sign from positive to negative at $a = a_m := (yr - x\tilde{K})/(xs - y\tilde{K})$ if both numerator and denominator are positive, so in this case $\beta$ attains its maximum at this point. If $a_m$ is negative then the supremum is "attained" at 0 or $\infty$. One can also see that if both $yr - x\tilde{K}$ and $xs - y\tilde{K}$ are negative then the extremum is a minimum, so again the supremum is attained at 0 or $\infty$. We found our $B_3$ by letting $a = a_m = (yr - x\tilde{K})/(xs - y\tilde{K})$ when both $yr - x\tilde{K}$ and $xs - y\tilde{K}$ are positive and by setting $a = 0$ or $a = \infty$ in all other cases.

Recall that, for the derivative estimate of $B_3$, only the case where $\beta$ attains its maximum at $a_m = (yr - x\tilde{K})/(xs - y\tilde{K})$ is relevant because we only need the estimate for small $K$—namely, $K \le Q\frac{yr}{4x}$ and $K \le Q\frac{xs}{4y}$.

Let us write $\tilde{K}$ for $K/Q$. It was shown in an early version of [7] that, if $K$ is small, then

$$\beta(a_0, X, x, r, Y, y, s, K) \ge \frac{x^2}{r} + \frac{1}{2}\frac{y^2}{s} \quad \text{for } a_0 = \frac{yr}{xs}. \tag{4.10}$$

We will include the proof for the sake of completeness. Let us first observe that

$$\frac{x^2}{r + a\tilde{K}} \ge \frac{x^2}{r} - a\tilde{K}\frac{x^2}{r^2} \quad \text{and} \quad \frac{y^2}{s + a^{-1}\tilde{K}} \ge \frac{y^2}{s} - a^{-1}\tilde{K}\frac{y^2}{s^2};$$

hence

$$\beta(a, X, x, r, Y, y, s, K) \ge \frac{x^2}{r} + \frac{y^2}{s} - \left(a\tilde{K}\frac{x^2}{r^2} + a^{-1}\tilde{K}\frac{y^2}{s^2}\right). \tag{4.11}$$

The part in parentheses for $a = a_0 = yr/xs$ is

$$\frac{yr}{xs}\tilde{K}\frac{x^2}{r^2} + \frac{xs}{yr}\tilde{K}\frac{y^2}{s^2} = 2\frac{xy}{rs}\tilde{K} \le \frac{y^2}{2s}, \tag{4.12}$$

where we use the assumption $\tilde{K} \le yr/4x$. Now we obtain the required estimate from below for $\beta$ at $a_0$:

$$\beta(a_0, X, x, r, Y, y, s, K) \ge \frac{x^2}{r} + \frac{y^2}{s} - \frac{y^2}{2s} = \frac{x^2}{r} + \frac{1}{2}\frac{y^2}{s},$$

where we use (4.11) for $a_0$ together with (4.12). Thus, taking supremum in the first variable yields

$$\sup_{a>0} \beta(a, X, x, r, Y, y, s, K) \geq \frac{x^2}{r} + \frac{1}{2}\frac{y^2}{s}.$$

We now consider the parameter family of functions

$$B_3^a(X, x, r, Y, y, s, K) := X + Y - \beta(a, X, x, r, Y, y, s, K).$$

In an early version of [7], the following derivative estimate was proved for small $K$:

$$\left.\frac{\partial B_3^a}{\partial \tilde{K}}\right|_{a=a_m} \geq c\frac{xy}{rs} \quad \text{where } a_m = \frac{yr - x\tilde{K}}{xs - y\tilde{K}}. \tag{4.13}$$

But $B_3(X, x, r, Y, y, s, K) = B_3^{a_m}(X, x, r, Y, y, s, K)$, so

$$\frac{\partial B_3}{\partial \tilde{K}} = \left.\frac{\partial B_3^a}{\partial a}\right|_{a=a_m} \cdot \frac{\partial a}{\partial \tilde{K}} + \left.\frac{\partial B_3^a}{\partial \tilde{K}}\right|_{a=a_m}.$$

Note that

$$\left.\frac{\partial B_3^a}{\partial a}\right|_{a=a_m} = -\left.\frac{\partial \beta}{\partial a}\right|_{a=a_m} = 0,$$

since $\beta$ attains its maximum in $a_m$. Provided that $K$ is small, we have the derivative estimate

$$\frac{\partial B_3}{\partial K} \geq \frac{c}{Q}\frac{xy}{rs}.$$

We include the proof of (4.13). First observe that, according to (4.10), we have

$$\frac{x^2}{r + a_m\tilde{K}} + \frac{y^2}{s + a_m^{-1}\tilde{K}} \geq \frac{x^2}{r} + \frac{1}{2}\frac{y^2}{s}.$$

This implies $y^2/(s + a_m^{-1}\tilde{K}) \geq y^2/2s$ and hence $s \geq a_m^{-1}\tilde{K}$. But since $x^2/r \geq y^2/s$, (4.10) implies also that

$$\frac{x^2}{r + a_m\tilde{K}} + \frac{y^2}{s + a_m^{-1}\tilde{K}} \geq \frac{y^2}{s} + \frac{1}{2}\frac{x^2}{r}.$$

Thus, similarly we obtain $r \geq a_m\tilde{K}$. Consequently,

$$\frac{\partial B_3^a}{\partial \tilde{K}} = \frac{ax^2}{(r + a\tilde{K})^2} + \frac{a^{-1}y^2}{(s + a^{-1}\tilde{K})^2} \geq 2\frac{xy}{(r + a\tilde{K})(s + a^{-1}\tilde{K})}.$$

Using $r \geq a_m\tilde{K}$ and $s \geq a_m^{-1}\tilde{K}$ yields (4.13).

*Size:* We have the following obvious size estimates for $B_i$:

$$0 \leq B_1 \leq X, \quad 0 \leq B_2 \leq Y, \quad 0 \leq B_3 \leq X + Y.$$

Here $0 \leq B_3$ follows from the fact that $X - x^2/(r + a\tilde{K}) \geq 0$ and $Y - y^2/(s + a^{-1}\tilde{K}) \geq 0$ for positive $a$.

*Concavity:* Both $B_1$ and $B_2$ are of the form (3.2), so

$$-d^2B_1 \geq 0 \quad \text{and} \quad -d^2B_2 \geq 0.$$

Since $\beta$ is convex for all parameters $a$, it follows that $B_3$, as the infimum of a family of concave functions, is also concave. To use $B_3$ in a Green's formula it needs to be sufficiently smooth. For $\delta < 1$ we consider the set $\mathcal{K}_\delta = \{v(z) : z \leq \delta\}$, where $v(z) = (X(z), x(z), r(z), Y(z), y(z), s(z), M(z), N(z), K(z))$. It is easy to see that the set $\mathcal{K}_\delta$ is a compact subset of the domain $\mathcal{K}$ of $B$. Let us pick $\varepsilon(\delta)$ to be the distance between $\mathcal{K}_\delta$ and the hyperplanes $X = 0$, $x = 0$, $r = 0$, $Y = 0$, $y = 0$, $s = 0$, $M = 0$, $N = 0$, and $K = 0$. We pick $\phi_{\varepsilon(\delta)}$ to be a $C^\infty$ approximate identity in $\mathbb{R}^9$ with radius $\varepsilon(\delta)/2$. By this we mean a smooth, radial, and nonnegative bump function supported by a disk around 0 of radius $\varepsilon(\delta)/2$ and normalized to $\int \phi_{\varepsilon(\delta)} = 1$. We consider the convolution $\tilde{B}_3^\delta = B_3 * \phi_{\varepsilon(\delta)}$. The resulting function is smooth in $\mathcal{K}_\delta$ and has size, derivative, and concavity properties similar to $B_3$. It is easy to see that only the constants change by a factor of $3/2$ for the size estimate and by $1/9$ in case of the derivative estimate for all choices of $\delta$. The derivative estimate of $\tilde{B}_3$ will only hold for smaller $K$, namely $\left\{K \leq Q\frac{yr}{18x} \text{ and } K \leq Q\frac{xs}{18y}\right\}$. In this sense,

$$-d^2\tilde{B}_3^\delta \geq 0 \text{ in } \mathcal{K}_\delta.$$

As before, we plug in our variables and call the resulting function $\tilde{b}_3^\delta(z)$. The sum $b_1 + b_2 + \tilde{b}_3^\delta$ is denoted by $\tilde{b}^\delta$. This function is defined on $\delta\mathbb{D}$ only, so we estimate our integral on a slightly smaller region first and then pass to the limit. We divide $\delta\mathbb{D}$ into three parts:

$$\begin{aligned} A_1 &= \left\{z \in \delta\mathbb{D} : K(z) \geq Q\frac{g(z)\upsilon(z)}{18f(z)}\right\}, \\ A_2 &= \left\{z \in \delta\mathbb{D} : K(z) \geq Q\frac{f(z)\omega(z)}{18g(z)}\right\}, \\ A_3 &= \delta\mathbb{D} \setminus (A_1 \cup A_2). \end{aligned}$$

If $z \in A_1$, then

$$\begin{aligned} -\Delta b_1(z) &\geq \frac{\partial B_1}{\partial M}(-\Delta M) \\ &\geq \frac{1}{4Q^2}\frac{f(z)^2}{\upsilon(z)^2}\alpha(z)\upsilon(z)K(z) \\ &\geq \frac{1}{4Q^2}\frac{f(z)^2}{\upsilon(z)^2}\alpha(z)\upsilon(z)Q\frac{g(z)\upsilon(z)}{18f(z)} \\ &= \frac{1}{72Q}\alpha(z)f(z)g(z). \end{aligned}$$

Similarly, if $z \in A_2$ then

$$-\Delta b_2(z) \geq \frac{1}{72Q}\alpha(z)f(z)g(z).$$

If $z \in A_3$, then

$$\begin{aligned}-\Delta \tilde{b}_3^{\delta}(z) &\geq \frac{\partial \tilde{B}_3^{\delta}}{\partial K}(-\Delta K)\\ &\geq \frac{c}{Q}\frac{f(z)g(z)}{\omega(z)\upsilon(z)}\alpha(z)\omega(z)\upsilon(z)\\ &= \frac{c}{Q}\alpha(z)f(z)g(z).\end{aligned}$$

Since $-\Delta b_{1,2}, -\Delta \tilde{b}_3^{\delta} \geq 0$ on all of $\delta\mathbb{D}$, we have all together

$$-cQ\Delta \tilde{b}^{\delta}(z) \geq \alpha(z) f(z) g(z),$$

with $c$ not depending on $\delta$. We are now ready to run the Green's formula trick:

$$\begin{aligned}\int_{\delta\mathbb{D}} \alpha(z) f(z) g(z) \log \frac{1}{|z|}\, dA(z) &\leq cQ \int_{r\mathbb{D}} -\Delta \tilde{b}^{\delta}(z) \log \frac{1}{|z|}\, dA(z)\\ &= cQ\left(\tilde{b}^{\delta}(0) - \int_{\delta\mathbb{T}} \tilde{b}^{\delta}(t)\, dm\right)\\ &\leq cQ(\|f\|^2_{\upsilon^{-1}} + \|g\|^2_{\omega^{-1}}).\end{aligned}$$

Passing to the limit $\delta \to 1^-$ delivers the desired estimate. □

## 5. Sharpness of the Result

In this section, we demonstrate that $Q_2^{\mathrm{inv}}(\omega)$ is indeed the best possible bound for the Hilbert transform. First, we will create an example on $\mathbb{R}$ (see [2]).

The definition of $A_2$ on $\mathbb{R}$ is very similar to that on $\mathbb{T}$: We say $\omega \in A_2$ (on $\mathbb{R}$) if

$$\sup_I \langle \omega \rangle_I \langle \omega^{-1} \rangle_I = Q_2^{\mathbb{R}}(\omega) < \infty, \tag{5.14}$$

where the supremum is taken over all intervals $I$ in $\mathbb{R}$.

Define $Q_2^{\mathbb{R},\mathrm{inv}}(\omega)$ by

$$\sup_{z \in \mathbb{R}^2_+} \omega(z)\omega^{-1}(z) = Q_2^{\mathbb{R},\mathrm{inv}}(\omega), \tag{5.15}$$

where $\omega(z)$ denotes the harmonic extension of $\omega$ onto the upper half-plane. (Note that $\omega(z) = w(x, y) = P_y * w(x)$, where $P_y(x) = cy/(|x|^2 + y^2)$ and $(x, y) \in \mathbb{R}^2_+$.)

In this section, we will use the symbol $\sim$ to denote comparable size; that is, $u \sim v$ if there exist positive constants $c, C$ such that $cu \leq v \leq Cu$.

Let $v = |x|^{\alpha}$. Such weights are called power weights, and a simple calculation shows that $v \in A_2$ (on $\mathbb{R}$) iff $\alpha \in (-1, 1)$ and that

$$Q_2^{\mathbb{R}}(v) \sim \frac{1}{1-\alpha^2}.$$

For power weights it is known that $Q_2^{\mathbb{R},\mathrm{inv}}(v) \sim 1/(1-\alpha^2)$ also, and thus $Q_2^{\mathbb{R},\mathrm{inv}}(v) \sim Q_2^{\mathbb{R}}(v)$ (see [4, Chap. 3]). We now use power weights to demonstrate that the main theorem is sharp. Let $s$ be a fixed number in $(0,1)$, and let $v_s(x) = |x|^{(1-s)}$ and $f_s(x) = |x|^{s-1}\chi_{[0,1]}$. Then $Q_2^{\mathbb{R},\mathrm{inv}}(v_s) \sim 1/s$ and $\|f_s\|^2_{L^2(v_s)} = 1/s$.

Let $\mathcal{H}$ denote the Hilbert transform on $\mathbb{R}$. Then, for $x > 2$,

$$\mathcal{H}f_s(x) = \int_0^1 \frac{y^{s-1}}{x-y}\,dy \sim \frac{1}{x}\int_0^1 y^{s-1}\,dy = \frac{1}{sx},$$

since $\frac{2}{x} \geq \frac{1}{x-y} \geq \frac{1}{x}$ for $x > 2$ and $y \in [0,1]$. Therefore,

$$\int_2^\infty |\mathcal{H}f_s|^2 v_s \sim s^{-3}$$

and so

$$\|\mathcal{H}f_s\|_{v_s} \geq cs^{-3/2} \sim s^{-1}\|f_s\|_{v_s} \sim Q_2^{\mathbb{R},\mathrm{inv}}(v_s)\|f_s\|_{v_s}.$$

Here we see that the first power of the $A_2$ constant is sharp if we let $s \to 0^+$.

To transform this example to one on $\mathbb{T}$, we use the Möbius transformation that maps the real line to the circle, $\frac{i-x}{i+x}$. Specifically, let

$$Jh(x) = \frac{1}{\sqrt{\pi}(i+x)}h\left(\frac{i-x}{i+x}\right).$$

Then

$$J^{-1}k(z) = \sqrt{\pi}\frac{2i}{z+1}k\left(\frac{z-1}{iz+i}\right).$$

Let $\tilde{v}_s(z) = v_s\left(\frac{z-1}{iz+i}\right)$. Recall that the $A_{2,\mathrm{inv}}$ constant is invariant under Möbius transforms and thus $Q_2^{\mathbb{R},\mathrm{inv}}(v_s) = Q_2^{\mathrm{inv}}(\tilde{v}_s)$. A calculation shows that $J$ is an isometry from $L^2_{\mathbb{T}}(\tilde{v}_s)$ to $L^2_{\mathbb{R}}(v_s)$. Furthermore (cf. [9]),

$$H = cJ^{-1}\mathcal{H}J.$$

Let $g(z) = J^{-1}f$. We have

$$\|Hg\|_{\tilde{v}_s} \sim \|J^{-1}\mathcal{H}Jg\|_{\tilde{v}_s} = \|\mathcal{H}f\|_{v_s} \leq cQ_2^{\mathbb{R},\mathrm{inv}}(v_s)\|f\|_{v_s} = cQ_2^{\mathrm{inv}}(\tilde{v}_s)\|g\|_{\tilde{v}_s}.$$

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