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A SHARP LOWER BOUND OF THE SPECTRAL RADIUS OF SIMPLE GRAPHS

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Let G be a simple connected graph with n vertices and let $\rho(G)$ be its spectral radius. The 2-degree of vertex i is denoted by t_i, which is the sum of degrees of the vertices adjacent to i. Let $N_i = \sum_{j\sim i} t_j$ and $M_i = \sum_{j\sim i} N_j$. We find a sharp lower bound of $\rho(G)$, which only contains two parameter N_i and M_i. Our result extends recent known results.

1. INTRODUCTION

Let G be a simple connected graph with vertex set $V = \{1, 2, \dots, n\}$. Let $d(i, j)$ denote the distance between vertices i and j. For $i \in V$, the degree of i and the average of the degree of the vertices adjacent to i are denoted by d_i and m_i, respectively. The 2-*degree* of vertex i is denoted by t_i, which is the sum of degrees of the vertices adjacent to i, that is $t_i = m_i d_i$. Let N_i be the sum of the 2-degree of vertices adjacent to i.

Let $A(G)$ be the adjacency matrix of G. By the Perron-Frobenius theorem [**1**, **2**], the spectral radius $\rho(G)$ is simple and there is a unique positive unit eigenvector.

Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The sequence of n eigenvalues is called the spectrum of G, the largest eigenvalue $\lambda_1(G)$ is often called the spectral radius of G, denoted by $\rho(G) = \lambda_1(G)$.

In this paper, we give a sharp lower bound on the spectral radius of simple graphs. For some recent surveys of the known results about this problem and related topics, we refer the reader to [**3**, **4**, **7**] and references therein.

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2. MAIN RESULTS

Lemma 1. *Let G be a bipartite graph with $V = V_1 \cup V_2$, $V_1 = \{1, 2, \ldots, s\}$ and $V_2 = \{s+1, s+2, \ldots, n\}$. Let $Y_1 = (y_1, y_2, \ldots, y_s)^T$ and $Y_2 = (y_{s+1}, y_{s+2}, \ldots, y_n)^T$. If $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is an eigenvector of $A(G)$ corresponding to $\rho(G)$, then $\| Y_1 \| = \| Y_2 \|$.*

Proof. Let $A(G) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$, where B is an $s \times (n-s)$ matrix. We have

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \rho\big(A(G)\big) \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix},$$

$$BY_2 = \rho\big(A(G)\big) Y_1 \Rightarrow Y_1^T B Y_2 = \rho\big(A(G)\big) Y_1^T Y_1,$$

and

$$B^T Y_1 = \rho\big(A(G)\big) Y_2 \Rightarrow Y_2^T B^T Y_1 = \rho\big(A(G)\big) Y_2^T Y_2.$$

Since $(Y_1^T B Y_2)^T = Y_2^T B^T Y_1$, we have that $Y_1^T Y_1 = Y_2^T Y_2$, that is

$$\| Y_1 \| = \| Y_2 \| .$$

□

Theorem 2. *Let G be a simple connected graph of order n and $\rho(G)$ be the spectral radius of G. Then*

(1)
$$\rho(G) \geq \sqrt{\Big(\sum_{i=1}^{n} {M_i}^2\Big) \Big/ \Big(\sum_{i=1}^{n} {N_i}^2\Big)},$$

where $N_i = \sum_{j \sim i} t_j$ and $M_i = \sum_{j \sim i} N_j$. The equality in (1) holds if and only if

$$\frac{M_1}{N_1} = \frac{M_2}{N_2} = \cdots = \frac{M_n}{N_n}$$

or G is a bipartite graph with $V = V_1 \cup V_2$, $V_1 = \{1, 2, \ldots, s\}$ and $V_2 = \{s+1, s+2, \ldots, n\}$, such that

$$\frac{M_1}{N_1} = \frac{M_2}{N_2} = \cdots = \frac{M_s}{N_s} \quad \text{and} \quad \frac{M_{s+1}}{N_{s+1}} = \frac{M_{s+2}}{N_{s+2}} = \cdots = \frac{M_n}{N_n}.$$

Proof. By Rayleigh quotient, we have

$$\rho(G)^2 = \rho\big(A(G)^2\big) = \max_{x \neq 0} \frac{x^T A(G)^2 x}{x^T x}.$$

Let $A(G)^2 = \big(a_{ij}^{(2)}\big)$, where $a_{ij}^{(2)}$ is the number of (i,j)-*walks* of length 2 in G. Clearly, $a_{ii}^{(2)} = d_i$ and $a_{ij}^{(2)} = a_{ji}^{(2)}$.

For a fixed (i,j)-walk in G, denote by $w(i,j)$ the length of this walk. Then $a_{ij}^{(2)} \neq 0$ if $w(i,j)=2$, and $a_{ij}^{(2)}=0$ otherwise. If $X=(N_1,N_2,\dots,N_n)^T$, we have

$$\begin{aligned} X^TA(G)^2X &= \sum_{i=1}^n N_i\Big(\sum_{j=1}^n a_{ij}^{(2)}N_j\Big) = \sum_{i=1}^n N_i \sum_{w(j,i)=2} a_{ij}^{(2)}N_j = \sum_{w(i,j)=2} a_{ij}^{(2)}N_iN_j \\ &= \sum_{i=1}^n d_i{N_i}^2 + \sum_{w(i,j)=2,i\neq j} a_{ij}^{(2)}N_iN_j = \sum_{i=1}^n\Big(\sum_{j\sim i}N_j\Big)^2 = \sum_{i=1}^n {M_i}^2 \end{aligned}$$

and $X^TX=\sum_{i=1}^n {N_i}^2$. So

$$\rho(G)=\sqrt{\max_{x\neq 0}\frac{x^TA(G)^2x}{x^Tx}} \geq \sqrt{\Big(\sum_{i=1}^n M_i^2\Big)\Big/\Big(\sum_{i=1}^n N_i^2\Big)}.$$

If the equality holds, then X is a positive eigenvector of $A(G)^2$ corresponding to $\rho(G)^2$. Moreover, if the eigenvalue $\rho\big(A(G)^2\big)$ of $A(G)^2$ has the multiplicity one, then by the Perron-Frobenius theorem, X is an eigenvector of $A(G)$ corresponding to $\rho(A(G))$, therefore $A(G)X=\rho(G)X$. For all $i=1,2,\dots,n$, we have $\big(A(G)X\big)_i=\big(\rho(G)X\big)_i$, that is $\sum_{j\sim i}N_j=\rho(G)N_i$. Since $\sum_{j\sim i}N_j=M_i$, we get

$$\frac{M_i}{N_i}=\rho(G) \qquad i=1,2,\dots,n,$$

and therefore

$$\frac{M_1}{N_1}=\frac{M_2}{N_2}=\cdots=\frac{M_n}{N_n}=\rho(G).$$

If the eigenvalue $\rho\big(A(G)^2\big)$ of $A(G)^2$ has the multiplicity two, it is well known that $-\rho\big(A(G)\big)$ is an eigenvalue of $A(G)$. Hence G is a bipartite graph. Without loss of generality, we assume that

$$A=\begin{pmatrix} 0 & B \\ B^T & 0\end{pmatrix},$$

hence

$$A^2=\begin{pmatrix} BB^T & 0 \\ 0 & B^TB\end{pmatrix}.$$

Let $X_1=(N_1,N_2,\dots,N_s)^T$ and $X_2=(N_{s+1},N_{s+2},\dots,N_n)^T$, we have

$$\begin{pmatrix} BB^T & 0 \\ 0 & B^TB\end{pmatrix}\begin{pmatrix} X_1 \\ X_2\end{pmatrix}=\rho\big(A(G)^2\big)\begin{pmatrix} X_1 \\ X_2\end{pmatrix},$$

$$BB^TX_1=\rho(A(G)^2)X_1 \ \text{ and } \ B^TBX_2=\rho\big(A(G)^2\big)X_2.$$

Let $Y=(y_1,y_2,\dots,y_n)^T$ be a positive eigenvector of $A(G)$ corresponding to $\rho(G)$. Let $Y_1=(y_1,y_2,\dots,y_s)^T$ and $Y_2=(y_{s+1},y_{s+2},\dots,y_n)^T$. Thus

$$BB^TY_1=\rho\big(A(G)^2\big)Y_1 \ \text{ and } \ B^TBY_2=\rho\big(A(G)^2\big)Y_2.$$

Since BB^T and B^TB have the same nonzero eigenvalues, BB^T and B^TB have eigenvalues $\rho(A(G)^2)$ with multiplicity one, respectively. Hence by the Perron-Frobenius theorem, we have $Y_1 = aX_1 (a \neq 0)$ and $Y_2 = bX_2$ $(b \neq 0)$. Now, it follows from $A(G)Y = \rho(G)Y$ that

$$\sum_{j\sim i} bN_j = \rho(G)aN_i, \quad i = 1, 2, \ldots, s$$

and

$$\sum_{j\sim i} aN_j = \rho(G)bN_i, \quad i = s+1, s+2, \ldots, n.$$

Since

$$\sum_{j\sim i} N_j = M_i,$$

thus we have

$$\frac{M_i}{N_i} = \frac{a}{b}\,\rho(G) \qquad i = 1, 2, \ldots, s$$

and

$$\frac{M_i}{N_i} = \frac{b}{a}\,\rho(G) \qquad i = s+1, s+2, \ldots, n.$$

Therefore,

$$\frac{M_1}{N_1} = \frac{M_2}{N_2} = \cdots = \frac{M_s}{N_s} = \frac{a}{b}\,\rho(G) \qquad i = 1, 2, \ldots s$$

and

$$\frac{M_{s+1}}{N_{s+1}} = \frac{M_{s+2}}{N_{s+2}} = \cdots = \frac{M_n}{N_n} = \frac{b}{a}\,\rho(G) \qquad i = s+1, s+2, \ldots, n.$$

In addition, by Lemma 1 we have

(2) $$a^2({N_1}^2 + \cdots + {N_s}^2) = b^2(N_{s+1}^2 + \cdots + N_n^2).$$

Conversely, we have:

(i) If $\frac{M_1}{N_1} = \frac{M_2}{N_2} = \cdots = \frac{M_n}{N_n}$, then

$$\sqrt{\Big(\sum_{i=1}^{n} {M_i}^2\Big)\Big/\Big(\sum_{i=1}^{n} {N_i}^2\Big)} = \rho(G).$$

(ii) If G is a bipartite graph with $\frac{M_1}{N_1} = \frac{M_2}{N_2} = \cdots = \frac{M_s}{N_s} = \frac{a}{b}\,\rho(G)$ $\quad i = 1, 2, \ldots, s$ and $\frac{M_{s+1}}{N_{s+1}} = \frac{M_{s+2}}{N_{s+2}} = \cdots = \frac{M_n}{N_n} = \frac{b}{a}\,\rho(G)$ $\quad i = s+1, s+2, \ldots, n$. Then by (2) we have

$$\sqrt{\frac{\sum_{i=1}^{n} {M_i}^2}{\sum_{i=1}^{n} {N_i}^2}} = \sqrt{\frac{\rho(G)^2\Big(\frac{a^2}{b^2}\left({N_1}^2 + \ldots + {N_s}^2\right) + \frac{b^2}{a^2}\left(N_{s+1}^2 + \cdots + {N_n}^2\right)\Big)}{{N_1}^2 + {N_2}^2 + \cdots + N_n^2}} = \rho(G),$$

and the proof follows. □

We now show that our bound improves the bound of Hong and Zhang [**6**].

Corollary 3 (Hong and Zhang [**6**]). *Let G be a simple connected graph of order n, then*

$$\rho(G) \geq \sqrt{\Big(\sum_{i=1}^{n} {N_i}^2\Big) \Big/ \Big(\sum_{i=1}^{n} {t_i}^2\Big)}, \tag{3}$$

with equality if and only if

$$\frac{N_1}{t_1} = \frac{N_2}{t_2} = \cdots = \frac{N_n}{t_n}$$

or G a bipartite graph with $V = V_1 \cup V_2$, $V_1 = \{1, 2, \ldots, s\}$ and $V_2 = \{s+1, s+2, \ldots, n\}$ such that

$$\frac{N_1}{t_1} = \frac{N_2}{t_2} = \cdots = \frac{N_s}{t_s} \quad and \quad \frac{N_{s+1}}{t_{s+1}} = \frac{N_{s+2}}{t_{s+2}} = \cdots = \frac{N_n}{t_n}\,.$$

Proof. By Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\Big(\sum_{i=1}^{n} {M_i}^2\Big)\Big(\sum_{i=1}^{n} {t_i}^2\Big) \geq \Big(\sum_{i=1}^{n} t_i M_i\Big)^2 &= \Big(\sum_{i=1}^{n} t_i \sum_{j\sim i} N_j\Big)^2 = \Big(\sum_{i=1}^{n} t_i \sum_{j\sim i}\Big(\sum_{k\sim j} t_k\Big)\Big)^2 \\
&= \Big(\sum_{i=1}^{n} t_i \sum_{w(k,i)=2} t_k\Big)^2 = \Big(\sum_{w(i,j)=2} t_i t_j\Big)^2 \\
&= \Big(\sum_{i=1}^{n} d_i {t_i}^2 + \sum_{w(i,j)=2, i\neq j} t_i t_j\Big)^2 \\
&= \Big(\sum_{i=1}^{n}\Big(\sum_{j\sim i} t_j\Big)^2\Big)^2 = \Big(\sum_{i=1}^{n} {N_i}^2\Big)^2.
\end{aligned}$$

The equality holds if and only if

$$\frac{M_1}{N_1} = \frac{M_2}{N_2} = \cdots = \frac{M_n}{N_n}.$$

Hence

$$\frac{\sum_{i=1}^{n} {M_i}^2}{\sum_{i=1}^{n} {N_i}^2} \geq \frac{\sum_{i=1}^{n} {N_i}^2}{\sum_{i=1}^{n} {t_i}^2}.$$

Therefore it follows from Theorem 1 that the result holds. □

A graph is called *pseudo-semiregular* if its vertices have the same average degree. A bipartite graph is called *pseudo-semiregular* if all vertices in the same part of a bipartition have the same average degree.

Corollary 4 (Yu et al. [**8**]). *Let G be a simple connected graph of order n. Then*

$$\rho(G) \geq \sqrt{\Big(\sum_{i=1}^{n} {t_i}^2\Big)\Big/\Big(\sum_{i=1}^{n} {d_i}^2\Big)}, \tag{4}$$

with equality if and only if G is either a pseudo-regular graph or a pseudo-semiregular graph.

Proof. This corollary follows from Corollary 3 (See [**3**]). □

Corollary 5 (Hofmeister [**5**]). *Let G be a simple connected graph with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\rho(G) \geq \sqrt{\frac{1}{n}\sum_{i=1}^{n} {d_i}^2}. \tag{5}$$

Proof. This corollary follows from Corollary 4 (See [**6**]. □

The lower bound (1) is complicated though it is better than (3). But if G is a tree, then the N_i and the M_i have a simple expression

$$N_i = \sum_{j\sim i} t_j = \sum_{j\sim i}\Big(\sum_{k\sim j} d_k\Big) = {d_i}^2 + \sum_{d(j,i)=2} d_j$$

and

$$M_i = \sum_{j\sim i} N_j = \sum_{j\sim i}\Big({d_j}^2 + \sum_{d(k,j)=2} d_k\Big) = \sum_{j\sim i} {d_j}^2 + (d_i - 1)\sum_{j\sim i} d_j + \sum_{d(j,i)=3} d_j.$$

Hence we have

Corollary 6. *Let T be a tree of order n and $\rho(T)$ be the spectral radius of T. Then*

$$\rho(T) \geq \sqrt{\frac{\sum_{i=1}^{n}\Big(\sum_{j\sim i} {d_j}^2 + (d_i - 1)\sum_{j\sim i} d_j + \sum_{d(j,i)=3} d_j\Big)^2}{\sum_{i=1}^{n}\Big({d_i}^2 + \sum_{d(j,i)=2} d_j\Big)^2}}. \tag{6}$$

The equality holds if and only if

$$\frac{M_1}{N_1} = \frac{M_2}{N_2} = \cdots = \frac{M_n}{N_n}$$

or for the bipartite graph T with $V = V_1 \cup V_2$, $V_1 = \{1, 2, \ldots, s\}$ and $V_2 = \{s + 1, s + 2, \ldots, n\}$ such that

$$\frac{M_1}{N_1} = \frac{M_2}{N_2} = \cdots = \frac{M_s}{N_s} \quad \textit{and} \quad \frac{M_{s+1}}{N_{s+1}} = \frac{M_{s+2}}{N_{s+2}} = \cdots = \frac{M_n}{N_n},$$

where

$$M_i = \sum_{j\sim i} d_j^{\,2} + (d_i - 1)\sum_{j\sim i} d_j + \sum_{d(j,i)=3} d_j$$

and

$$N_i = d_i^{\,2} + \sum_{d(j,i)=2} d_j .$$

Proof. This corollary follows from Theorem 2. □

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