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Xiumin Du, Larry Guth, Xiaochun Li

Institutions: University of Illinois at Urbana–Champaign, Massachusetts Institute of Technology

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A SHARP SCHRÖDINGER MAXIMAL ESTIMATE IN $\mathbb{R}^2$

BY

XIUMIN DU

DISSERTATION

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Doctoral Committee:

Professor M. Burak Erdogan, Chair
Associate Professor Xiaochun Li, Director of Research
Professor Florin P. Boca
Professor Richard S. Laugesen

Abstract

We study the almost everywhere pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$. It is shown that $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ almost everywhere for all $f \in H^s(\mathbb{R}^2)$ provided that $s > 1/3$.

This result is sharp up to the endpoint. It comes from the following Schrödinger maximal estimate:

$$\left\| \sup_{0<t\leq 1} |e^{it\Delta} f| \right\|_{L^3(B(0,1))} \leq C_s \|f\|_{H^s(\mathbb{R}^2)} ,$$

for any $s > 1/3$ and any function $f \in H^s(\mathbb{R}^2)$. The proof uses polynomial partitioning and decoupling.

To my daughter Sharon.

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Chapter 1

Introduction

This dissertation is based on our joint work with Larry Guth and Xiaochun Li [8].

1.1 Background and main results of the Thesis

The formal solution to the free Schrödinger equation

$$\begin{cases} iu_t + \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = f(x), & x \in \mathbb{R}^n \end{cases}$$

is given by

$$e^{it\Delta} f(x) = (2\pi)^{-n} \int e^{i\left(x\cdot\xi - t|\xi|^2\right)} \widehat{f}(\xi)\, d\xi\,, \tag{1.1}$$

where $\widehat{f}$ is the Fourier transform of f defined by $\widehat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\, dx$.

We consider the following problem posed by Carleson in [5]: determine the optimal s for which

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \text{almost everywhere} \tag{1.2}$$

whenever $f \in H^s(\mathbb{R}^n)$. Recall that $H^s(\mathbb{R}^n)$ is the L^2 Sobolev space which is defined by

$$H^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}' \,:\, \|f\|_{H^s} = \left(\int (1+|\xi|^2)^s \left|\widehat{f}(\xi)\right|^2 d\xi \right)^{1/2} < \infty \right\}.$$

Our main result is the following:

Theorem 1.1 *For every $f \in H^s(\mathbb{R}^2)$ with $s > 1/3$, $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ almost everywhere.*

Recently, Bourgain [3] gave examples showing that such convergence can fail for any $s < 1/3$, and so Theorem 1.1 is sharp up to the endpoint.

This problem originates from Carleson [5], who proved convergence for $s \geq 1/4$ when $n = 1$. Dahlberg and Kenig [6] showed that the convergence does not hold for $s < 1/4$ in any dimension. Sjölin [19] and Vega [21] proved independently the convergence for $s > 1/2$ in all dimensions. The sufficient condition for pointwise convergence was improved by Bourgain [1], Moyua-Vargas-Vega [17], and Tao-Vargas [20]. The best known sufficient condition in dimension $n = 2$ was $s > 3/8$, due to Lee [14] using Tao-Wolff's bilinear restriction method. In general dimension $n \geq 2$, Bourgain [2] showed the convergence for $s > 1/2 - 1/(4n)$, using multilinear methods. When $n = 2$, this approach gives a different proof of Lee's result for $s > 3/8$.

For many years, it had seemed plausible that convergence actually holds for $s > 1/4$ in every dimension. Only in 2012, Bourgain [2] gave a counterexample showing that this is false in sufficiently high dimensions. Improved counterexamples were given by Lucá-Rogers [15] and Demeter-Guo [7]. Very recently, in [3], Bourgain gave counterexamples showing that convergence can fail if $s < \frac{n}{2(n+1)}$. In particular, for $n = 2$, convergence can fail if $s < 1/3$.

We will follow the standard method by bounding the associated maximal function. We call such bounds Schrödinger maximal estimates. In Section 2.3 we will review in details how to approach the pointwise convergence problem via Schrödinger maximal estimates.

We use $B^n(c, r)$ to represent the ball centered at c with radius r in $\mathbb{R}^n$, and use χ_E to denote the characteristic function of any measurable set E. For brevity, $B(c, r)$ represents $B^2(c, r)$, a ball in $\mathbb{R}^2$.

For any Schwartz function f, we have that $e^{it\Delta} f(x) \to f(x)$ uniformly on $\mathbb{R}^n$. Given any function $f \in H^s$, we approximate f by Schwartz functions, and the pointwise convergence result in Theorem 1.1 follows from the following Schödinger maximal estimate (see Lemma 2.3):

Theorem 1.2 *For any $s > 1/3$, the following bound holds: for any function $f \in H^s(\mathbb{R}^2)$,*

$$\left\| \sup_{0<t\leq 1} |e^{it\Delta} f| \right\|_{L^3(B(0,1))} \leq C_s \|f\|_{H^s(\mathbb{R}^2)}. \tag{1.3}$$

If the support of $\widehat{f}$ lies in $A(R) = \{\xi \in \mathbb{R}^2 : |\xi| \sim R\}$, then Theorem 1.2 boils down to the bound

$$\left\| \sup_{0<t\leq 1} |e^{it\Delta} f| \right\|_{L^3(B(0,1))} \leq C_\epsilon R^{1/3+\epsilon} \|f\|_{L^2}. \tag{1.4}$$

Via a Littlewood-Paley decomposition and parabolic rescaling, Theorem 1.2 reduces to the following Schrödinger maximal estimate for functions f with $\widehat{f}$ supported in $B(0, 1)$ (see Lemma 2.5):

Theorem 1.3 *For any $\epsilon > 0$, there exists a constant C_ϵ such that*

$$\big\| \sup_{0<t\le R} |e^{it\Delta} f| \big\|_{L^3(B(0,R))} \le C_\epsilon R^\epsilon \|f\|_2 \tag{1.5}$$

holds for all $R \ge 1$ and all $f \in L^2(\mathbb{R}^2)$ with $\mathrm{supp}\widehat{f} \subset B(0,1)$.

Remark 1.4 *By the local bound (1.5) from Theorem 1.3 and Hölder's inequality, for $2 \le p \le 3$,*

$$\big\| \sup_{0<t\le R} |e^{it\Delta} f| \big\|_{L^p(B(0,R))} \le C_\epsilon R^{\frac{2}{p}-\frac{2}{3}+\epsilon} \|f\|_2 \tag{1.6}$$

holds for all $R \ge 1$ and all $f \in L^2(\mathbb{R}^2)$ with $\mathrm{supp}\widehat{f} \subset B(0,1)$.

Note that for functions f with $\mathrm{supp}\widehat{f} \subset B(0,1)$, we have a trivial bound

$$|e^{it\Delta} f(x)| \lesssim \|f\|_2 \,.$$

Therefore, it follows from (1.5) that for $3 < p \le \infty$,

$$\big\| \sup_{0<t\le R} |e^{it\Delta} f| \big\|_{L^p(B(0,R))} \le C_\epsilon R^\epsilon \|f\|_2 \tag{1.7}$$

holds for all $R \ge 1$ and all $f \in L^2(\mathbb{R}^2)$ with $\mathrm{supp}\widehat{f} \subset B(0,1)$.

By Proposition 2.6, the local bounds (1.6) and (1.7) imply that the following estimate

$$\left\| \sup_{0<t\le 1} |e^{it\Delta} f| \right\|_{L^p(B(0,1))} \le C_s \|f\|_{H^s(\mathbb{R}^2)} \,, \quad \forall f \in H^s(\mathbb{R}^2) \tag{1.8}$$

holds for $s > 1/3$ when $2 \le p \le 3$, and for $s > 1 - \frac{2}{p}$ when $3 < p \le \infty$.

Given Bourgain's conterexample, for $2 \le p \le 3$, the exponent of R in (1.6) is sharp up to a factor R^ε, and the range of s in (1.8) is sharp up to the endpoint.

For $3 < p \le \infty$, the exponent of R in (1.7) is sharp up to a factor R^ε, and the range of s in (1.8) is sharp up to the endpoint. This sharpness follows from a simple example:
Consider the function $\widehat{f} = \eta$, where $\eta \ge 0$, $\mathrm{supp}\,\eta \subset B(0,1)$ and $\eta = 1$ on $B(0,1/2)$. Then for $0 < t < c$ and $|x| < c$, where $c < 1$ is chosen to be a small constant, we have

$$|e^{it\Delta} f(x)| \sim C \,.$$

We also remark that the local bound (1.5) from Theorem 1.3 can be used to derive immediately a global estimate (cf. Theorem 10 in [18]), we are indebted to K. Rogers for pointing this out to us.

Theorem 1.5 *For any $s > 1/3$, the following bound holds: for any function $f \in H^s(\mathbb{R}^2)$,*

$$\left\| \sup_{0<t\leq 1} |e^{it\Delta} f| \right\|_{L^3(\mathbb{R}^2)} \leq C_s \|f\|_{H^s(\mathbb{R}^2)}. \tag{1.9}$$

In Section 2.4, we will show that the global estimate (1.9) in Theorem 1.5 follows from the local bound (1.5) in Theorem 1.3, via a Littlewood-Paley decomposition, parabolic rescaling and wave packet decomposition.

In summary, we have

$$\text{Theorem 1.1} \;\Leftarrow \text{Theorem 1.2} \;\Leftarrow \text{Theorem 1.3} \Rightarrow \text{Theorem 1.5}\,.$$

Our main work is dedicated to a proof of Theorem 1.3, the proof uses inductions. In order to make all inductions work, we will state Theorem 1.3 in a more general setting (see the statement of our main inductive theorem - Theorem 1.7 in Section 1.3).

1.2 Key ingredients in the proof of our main result - Theorem 1.3

The proof uses polynomial partitioning. This technique was introduced by Nets Katz and Larry Guth in [9], where it was applied to incidence geometry. In [10] and [11], Guth applied this technique to restriction estimates in Fourier analysis. Polynomial partitioning is a divide and conquer technique. We begin by finding a polynomial whose zero set divides some object of interest into equal pieces. For instance, in [9], it was proven that for any finite volume set $E \subset \mathbb{R}^3$ and any degree $D \geq 1$, there is a polynomial P of degree at most D so that $\mathbb{R}^3 \setminus Z(P)$ is a union of $\sim D^3$ disjoint open sets O_i, and the volumes $|O_i \cap E|$ are all equal. Hence for any i, $|E| \lesssim D^3 |O_i \cap E|$. In our paper, we choose the polynomial P to behave well with respect to the $L^p_x L^q_t$ norm of $e^{it\Delta} f$. For any $R \geq 1$, any $f \in L^2$ with compact Fourier support, any $p \leq q < \infty$ and any degree $D \geq 1$, we show that there is a polynomial P of degree at most D so that $\mathbb{R}^3 \setminus Z(P)$ is a union of $\sim D^3$ disjoint open sets O_i, and for any i,

$$\|e^{it\Delta} f\|^p_{L^p_x L^q_t(B(0,R)\times[0,R])} \lesssim D^3 \|\chi_{O_i} e^{it\Delta} f\|^p_{L^p_x L^q_t(B(0,R)\times[0,R])}. \tag{1.10}$$

(To prove Theorem 1.3, we will use q finite but very large and p close to 3. The degree D will be a tiny power of R, so D is large compared to 1, but very small compared to R). We would like to point out that we are unable prove (1.10) for $q = \infty$. The polynomial partitioning involves a topological argument, which requires some continuity, and it does not work well with the L^∞-norm.

Breaking spacetime into cells O_i is useful because of the way it interacts with the wave packet decomposition of $e^{it\Delta}f$, which we now recall. We decompose f into pieces that are localized in both physical space and frequency space. We tile the physical space $B(0,R)$ with $R^{1/2}$-cubes ν, and we tile the frequency space $B(0,1)$ with $R^{-1/2}$-cubes θ. Then we decompose f as $f = \sum_{\theta,\nu} f_{\theta,\nu}$, where $f_{\theta,\nu}$ is essentially supported on ν in physical space and essentially supported on θ in frequency space. Each function $e^{it\Delta}f_{\theta,\nu}$ is called a wave packet. The restriction of $e^{it\Delta}f_{\theta,\nu}$ to the domain $B(0,R)\times[0,R]$ is essentially supported on a tube $T_{\theta,\nu}$ of radius $R^{1/2}$ and length R. This tube intersects the time slice $\{t=0\}$ at ν, and the direction of the tube depends on θ.

A key fact in the applications of polynomial partitioning in combinatorics is that a line can enter at most $D+1$ of the cells O_i. To see this, we note that the polynomial P can vanish at most D times along a line, unless it vanishes on the whole line, and so a line can cross $Z(P)$ at most D times. A wave packet $e^{it\Delta}f_{\theta,\nu}$ is supported on a tube $T_{\theta,\nu}$ of radius $R^{1/2}$. This tube can potentially enter many or even all the cells O_i, but it cannot penetrate deeply into very many cells. We define W to be the $R^{1/2}$-neighborhood of $Z(P)$ in $B(0,R)\times[0,R]$, and we define O_i' to be $O_i \setminus W$. Now the central line of $T_{\theta,\nu}$ can enter at most $D+1$ of the original cells O_i, and so the tube $T_{\theta,\nu}$ can enter at most $D+1$ of the smaller cells O_i'. In other words, each wave packet $e^{it\Delta}f_{\theta,\nu}$ is essentially supported on the union of W and $D+1$ cells O_i'.

We can use induction to study $e^{it\Delta}f$ on each smaller cell O_i'. To study $e^{it\Delta}f$ on a cell O_i', we only need to take account of those wave packets that intersect O_i'. Therefore, we define f_i to be the sum of $f_{\theta,\nu}$ over those pairs (θ,ν) for which $T_{\theta,\nu}$ enters O_i'. On the cell O_i', $e^{it\Delta}f$ is essentially equal to $e^{it\Delta}f_i$. We can control the L^2 norms of the f_i by using the fact that $f_{\theta,\nu}$ are (approximately) orthogonal and the fact that each tube $T_{\theta,\nu}$ enters $\lesssim D$ smaller cells O_i'. In particular, we will prove that

$$\sum_i \|f_i\|_2^2 \lesssim D\|f\|_2^2.$$

We can now use induction to control $e^{it\Delta}f$ on each cell O_i'. In this way, we get good control of the contribution to $\|e^{it\Delta}f\|_{L_x^pL_t^q(B(0,R)\times[0,R])}$ coming from the union of all smaller cells O_i'. It remains to control the contribution coming from W.

The most difficult scenario is the following: the tubes $T_{\theta,\nu}$ are all contained in W. The polynomial partitioning method allows us to reduce the original problem to this special scenario. This scenario indeed occurs in Bourgain's example in [3]. Let us take a moment to describe this example briefly (in Section 4.1, we will come back to this example and discuss about it in details):

In the example from [3], the zero set $Z(P)$ can be taken to be a plane $2t = x_1$. The set W is a planar slab of thickness $R^{1/2}$. The solution $e^{it\Delta}f$ is essentially supported in W. On the plane $2t = x_1$, $e^{it\Delta}f$ is a solution of the Schrodinger equation in 1 + 1 dimensions. In other words, we can choose coordinates (y, s) on this plane and an initial data g so that $e^{is\Delta}g$ is essentially equal to $e^{it\Delta}f$ on the plane. Also, $|e^{it\Delta}f(x_1, x_2)|$ is approximately constant as we vary x_1 within the slab W. The initial data is chosen so that $|e^{is\Delta}g(y)|$ is large on a set X of $\sim R^{3/2}$ unit squares in $[0, R] \times [0, R]$. It follows that $|e^{it\Delta}f(x)|$ is large on a set of $\sim R^{3/2}$ 3-dimensional rectangles of dimensions $R^{1/2} \times 1 \times 1$ in $B(0, R) \times [0, R]$. Moreoever, the projections of these rectangles are roughly disjoint, and so they cover a positive proportion of $B(0, R)$. Therefore $\sup_{0<t\leq R} |e^{it\Delta}f(x)|$ is large on a positive proportion of $B(0, R)$.

In this construction, the set X needs to be fairly sparse because the projections of the $R^{1/2} \times 1 \times 1$ rectangles need to be disjoint in $B(0, R)$. In particular, there can be at most $R^{1/2}$ unit squares of X in any $R^{1/2}$-ball in $[0, R] \times [0, R]$. In the example of [3], $|e^{is\Delta}g| \sim R^{-5/12}\|g\|_{L^2}$ on the set X. During our proof, we will need to show that this quantity $R^{-5/12}\|g\|_{L^2}$ could not be any larger. In rough terms, we need to show that a solution $e^{is\Delta}g$ cannot focus too much on a set X which is sparse and spread out.

We will prove such bounds using the l^2 decoupling theorem of Bourgain and Demeter [4]. We think of these bounds as refinements of the Strichartz inequality. Here is one such estimate:

Theorem 1.6 *Let $g \in L^2(\mathbb{R})$ with $\operatorname{supp} \widehat{g} \subset B^1(0, 1)$. Suppose that $Q_1, Q_2, ...$ are lattice $R^{1/2}$-cubes in $[0, R]^2$, so that*

$$\|e^{it\Delta}g\|_{L^6(Q_j)} \text{ is essentially constant in } j.$$

Suppose that these cubes are arranged in horizontal strips of the form $\mathbb{R} \times \{t_0, t_0 + R^{1/2}\}$, and that each strip contains $\sim \sigma$ cubes Q_j. Let Y denote $\bigcup_j Q_j$. Then for any $\epsilon > 0$,

$$\|e^{it\Delta}g\|_{L^6(Y)} \leq C_\epsilon R^\epsilon \sigma^{-1/3}\|g\|_{L^2}.$$

The Strichartz inequality says that $\|e^{it\Delta}g\|_{L^6([0,R]^2)} \lesssim \|g\|_{L^2}$. Theorem 1.6 says that we get a stronger estimate when the solution $e^{it\Delta}g$ is spread out in space. To get a sense of what the theorem says, consider the following example. Suppose that $e^{it\Delta}g$ is a sum of σ wave packets supported on disjoint $R^{1/2} \times R$ rectangles. We can take Y to be the union of these rectangles. By scaling, we can suppose that $|e^{it\Delta}g| \sim 1$ on these σ rectangles and negligibly small elsewhere, and then a direct calculation shows that

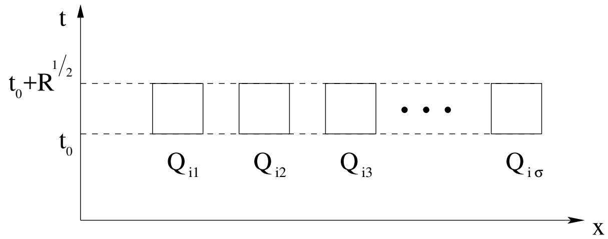

Figure 1.1: $\sim \sigma$ many cubes in a horizontal strip

$$\|e^{it\Delta}g\|_{L^6([0,R]^2)} \sim \|e^{it\Delta}g\|_{L^6(Y)} \sim (\sigma R^{3/2})^{1/6-1/2}\|e^{it\Delta}g\|_{L^2(Y)}$$

$$\sim (\sigma R^{3/2})^{1/6-1/2}\|e^{it\Delta}g\|_{L^2([0,R]^2)} \sim (\sigma R^{3/2})^{1/6-1/2}R^{1/2}\|g\|_2 \sim \sigma^{-1/3}\|g\|_2 \,.$$

So Theorem 1.6 roughly says that if $e^{it\Delta}g$ is "as spread out as" σ disjoint wave packets, then its L^6 norm cannot be much bigger than the L^6 norm of σ disjoint wave packets.

This theorem helps us to control the size of $e^{it\Delta}g$ on a sparse, spread out set X as above. Suppose that the function $e^{it\Delta}g$ is evenly spread out on $[0,R]^2$ in the sense that $\|e^{it\Delta}g\|_{L^6(Q)}$ is roughly constant among all $R^{1/2}$-boxes $Q \subset [0,R]^2$. In this case, we can take $\sigma = R^{1/2}$ in Theorem 1.6, which gives

$$\|e^{it\Delta}g\|_{L^6([0,R]^2)} \lesssim R^{-1/6+\epsilon}\|g\|_{L^2}.$$

In the example from [3], X contains $\sim R^{1/2}$ unit squares in each $R^{1/2}$-box of $[0,R]^2$, and each of these boxes indeed has a roughly equal value of $\|e^{it\Delta}g\|_{L^6(Q)}$. If $|e^{it\Delta}g| \sim H$ on the set X, then Theorem 1.6 gives

$$H|X|^{1/6} \lesssim \|e^{it\Delta}g\|_{L^6([0,R]^2)} \lesssim R^{-1/6+\epsilon}\|g\|_{L^2}.$$

Since $|X| \sim R^{3/2}$, we get the bound $H \lesssim R^{-5/12+\epsilon}\|g\|_{L^2}$. This upper bound matches the behavior of the example from [3] up to a factor R^ϵ.

Theorem 1.6 lets us deal with the case that $Z(P)$ is a plane. We need to deal with the more general case that $Z(P)$ is a possibly curved surface of degree at most D. We prove a more general version of Theorem 1.6, Theorem 3.5, which covers the case of wave packets concentrated into a curved surface.

1.3 Main inductive theorem

Our main goal is to prove Theorem 1.3. Here we state it in a slightly more complicated setting. Our proof uses inductions, and we need the slightly more complicated formulation to make all the inductions work. First of all, the polynomial partitioning involves a topological argument, and the topological argument does not work well with the sup appearing in our maximal function. Therefore, we replace the norm $L^p_x L^\infty_t$ with the norm $L^p_x L^q_t$ for q very large. Another technical issue has to do with parabolic rescaling. Suppose that $\widehat{f}$ is supported in a smaller ball $B(\xi_0, M^{-1}) \subset B(0,1)$. In this situation, one can often apply parabolic rescaling to reduce the problem at hand to a problem on a smaller ball in physical space. However, the change of coordinates in such a parabolic rescaling does not interact well with mixed norms of the form $L^p_x L^q_t$ (see Section 2.2 for details). Therefore, we instead do induction on the size of the ball $B(\xi_0, M^{-1})$, proving slightly stronger bounds when the ball is small. Taking account of these small issues, we formulate our result in the following way:

Theorem 1.7 *For $p > 3$, for any $\epsilon > 0$, there exists a constant $C_{p,\epsilon}$ such that for any $q > 1/\epsilon^4$,*

$$\left\|e^{it\Delta} f\right\|_{L^p_x L^q_t(B(0,R)\times[0,R])} \leq C_{p,\epsilon} M^{-\epsilon^2} R^\epsilon \|f\|_2 \tag{1.11}$$

holds for all $R \geq 1$, any $\xi_0 \in B^2(0,1)$, any $M \geq 1$ and all $f \in L^2(\mathbb{R}^2)$ with $\text{supp}\widehat{f} \subset B^2(\xi_0, M^{-1})$.

Let us explain how Theorem 1.7 implies our main result - Theorem 1.3 in Section 1.1. We note that by the dominated convergence theorem we have

$$\Big\| \sup_{0<t\leq R} |e^{it\Delta} f| \Big\|_{L^p(B(0,R))} = \lim_{q\to\infty} \left\|e^{it\Delta} f\right\|_{L^p_x L^q_t(B(0,R)\times[0,R])},$$

for any L^2-function f with compact Fourier support or any Schwartz function f. Therefore, Theorem 1.7 implies that for any $R \geq 1$ and any f with the support of $\widehat{f} \subset B(0,1)$, and for any $p > 3$, we have

$$\left\|e^{it\Delta} f\right\|_{L^p_x L^\infty_t(B(0,R)\times[0,R])} \leq C_{p,\epsilon} R^\epsilon \|f\|_2 \,. \tag{1.12}$$

So far we assume $p > 3$. In the meanwhile, since $\text{supp}\widehat{f} \subset B(0,1)$, we have

$$\|e^{it\Delta} f\|_\infty \lesssim \|f\|_2 \,,$$

and it is straightforward to prove a bound of the form

$$\|e^{it\Delta}f\|_{L^2_xL^\infty_t(B(0,R)\times[0,R])} \le R^{O(1)}\|f\|_2.$$

Combining these bounds using Hölder's inequality, we see that Equation (1.12) holds for $p=3$ as well. This establishes Theorem 1.3.

1.4 Outline of the Thesis

In the rest of the paper, we aim to prove the main inductive theorem - Theorem 1.7.

In Chapter 2, we first review two preliminary tools: wave packet decomposition and parabolic rescaling. Then using these tools, we show that the almost everywhere pointwise convergence problem (1.2) can be approached by bounding the associated maximal function, i.e. by Schrödinger maximal estimates. Moreover, using wave packet decomposition, we prove the global Schrödinger maximal estimate (1.9) in Theorem 1.5, as a corollary of Theorem 1.3. Finally, we show a polynomial partitioning theorem for mixed norm, which is one key ingredient in our proof of Theorem 1.7.

In Chapter 3, we prove the main inductive theorem - Theorem 1.7. Utilizing the way how tubes coming from wave packet decomposition interact with the variety arising from polynomial partitioning, we are able to reduce the original problem in $\mathbb{R}^3$ to a special scenario, in which case all wave packets are contained in a small neighborhood of a 2-dimensional variety. We call it the tangent-to-variety case. To deal with this special scenario, in section 3.3.1 we obtain both linear and bilinear local refinements of the Strichartz inequality, via the Bourgain-Demeter l^2-decoupling theorem. And in section 3.3.4 we use the bilinear refinement of Strichartz to prove a Schrödinger maximal estimate for a bilinear tangent term, which completes the proof of our main inductive theorem.

In Chapter 4, we give a few remarks about the Schrödinger maximal estimates in higher dimension. We first discuss in details about Bourgain's counterexample from [3], which shows that pointwise convergence (1.2) can fail for $s<\frac{n}{2(n+1)}$. We will see that Bourgain's example fits into a scenario where all wave packets are contained in a small neighborhood of a hyperplane. In this special scenario, or more generally, in tangent-to-variety case, where all wave packets are contained in a small neighborhood of a variety, we prove that the exponent $\frac{n}{2(n+1)}$ is optimal.

Chapter 2

Preliminaries

2.1 Wave packet decomposition

A (dyadic) rectangle in $\mathbb{R}^n$ is a product of (dyadic) intervals with respect to given coordinate axes of $\mathbb{R}^n$. A rectangle $\theta = \prod_{j=1}^n \theta_j$ in frequency space and a rectangle $\nu = \prod_{j=1}^n \nu_j$ in physical space are said to be dual if $|\theta_j||\nu_j| = 1$ for $j = 1, \cdots, n$. We say that (θ, ν) is a tile if it is a pair of *dual* (dyadic) rectangles. The dyadic condition is not essential in our decomposition.

Let φ be a Schwartz function from $\mathbb{R}$ to $\mathbb{R}$ whose Fourier transform is non-negative, supported in a small interval, of radius κ (κ is a fixed small constant), about the origin in $\mathbb{R}$, and identically 1 on another smaller interval around the origin. For a (dyadic) rectangular box $\theta = \prod_{j=1}^n \theta_j$, set

$$\widehat{\varphi_\theta}(\xi_1, \cdots, \xi_n) = \prod_{j=1}^n \frac{1}{|\theta_j|^{1/2}} \widehat{\varphi}\left(\frac{\xi_j - c(\theta_j)}{|\theta_j|}\right). \tag{2.1}$$

Here $c(\theta_j)$ is the center of the interval θ_j and hence $c(\theta) = (c(\theta_1), \cdots, c(\theta_n))$ is the center of the rectangle θ. We also note that $\|\varphi_\theta\|_{L^2} \sim 1$. We let $c(\nu)$ denote the center of ν. For a tile (θ, ν) and $\xi \in \mathbb{R}^n$, we define

$$\widehat{\varphi_{\theta,\nu}}(\xi) = e^{-ic(\nu)\cdot\xi}\widehat{\varphi_\theta}(\xi). \tag{2.2}$$

We say that two tiles (θ, ν) and (θ', ν') have the same dimensions if $|\theta_j| = |\theta'_j|$ for all j, which then implies that $|\nu_j| = |\nu'_j|$ for all j. Let $\mathbf{T}$ be a collection of all tiles with fixed dimensions and coordinate axes. Then for any Schwartz function f on $\mathbb{R}^n$, we have the following representation

$$f(x) = c_\kappa \sum_{(\theta,\nu)\in\mathbf{T}} f_{\theta,\nu} := c_\kappa \sum_{(\theta,\nu)\in\mathbf{T}} \langle f, \varphi_{\theta,\nu}\rangle \varphi_{\theta,\nu}(x), \tag{2.3}$$

where c_κ is an absolute constant. This representation can be proved directly (see [12]) or by employing inductively the one-dimensional result in [13].

When the time variable t is restricted to $[0,\lambda]$, we use **wave packet decomposition at scale** λ, that is, we use tiles (θ,ν) where θ is a $\lambda^{-\frac{1}{2}}$-cube in frequency space and ν is a $\lambda^{\frac{1}{2}}$-cube in physical space. Indeed, let θ be a $\lambda^{-\frac{1}{2}}$-cube (or ball) in $B^n(0,1)\subset\mathbb{R}^n$. Let $\mathbf{T}_\theta$ be a collection of all tiles (θ',ν) such that ν's are $\lambda^{\frac{1}{2}}$-cubes and $\theta'=\theta$. Then for any Schwartz function f with supp$\widehat{f}\subset B^n(0,1)$, we have

$$f(x)=c_\kappa\sum_\theta\sum_{(\theta',\nu)\in\mathbf{T}_\theta}\langle f,\varphi_{\theta',\nu}\rangle\varphi_{\theta',\nu}(x)\,. \tag{2.4}$$

Here θ's range over all possible cubes in supp$\widehat{f}$. We use $\mathbf{T}$ to denote $\bigcup_\theta\mathbf{T}_\theta$. It is clear that

$$\sum_{(\theta,\nu)\in\mathbf{T}}\big|\langle f,\varphi_{\theta,\nu}\rangle\big|^2\sim\|f\|_2^2\,. \tag{2.5}$$

We set

$$\psi_{\theta,\nu}(x,t)=e^{it\Delta}\varphi_{\theta,\nu}(x)\,. \tag{2.6}$$

From (2.4), we end up with the following representation for $e^{it\Delta}f$:

$$e^{it\Delta}f(x)=c_\kappa\sum_{(\theta,\nu)\in\mathbf{T}}e^{it\Delta}f_{\theta,\nu}(x)=c_\kappa\sum_{(\theta,\nu)\in\mathbf{T}}\langle f,\varphi_{\theta,\nu}\rangle\psi_{\theta,\nu}(x,t)\,. \tag{2.7}$$

We shall analyze the localization of $\psi_{\theta,\nu}$ in the physical and frequency space.

When t is restricted to the interval $[0,\lambda]$, the function $\psi_{\theta,\nu}$ is essentially supported on a tube $T_{\theta,\nu}$ defined as follows. Let

$$T_{\theta,\nu}:=\{(x,t)\in\mathbb{R}^n\times\mathbb{R}\,:\,0\le t\le\lambda,|x-c(\nu)-2tc(\theta)|\le\lambda^{1/2+\delta}\}\,, \tag{2.8}$$

where $\delta=\epsilon^2$ is a small positive parameter. We see that $T_{\theta,\nu}$ is a tube of length λ, of radius $\lambda^{1/2+\delta}$, in the direction $G_0(\theta)=(2c(\theta),1)$, and intersecting $\{t=0\}$ at an $\lambda^{1/2+\delta}$-ball centered at $c(\nu)$. In order to see this, let ψ be a Schwartz function with Fourier transform supported in $[-1,1]$ and $2\psi(t)\ge\chi_{[0,1]}(t)$. Here $\chi_{[0,1]}$ is the characteristic function on $[0,1]$. With t restricted to $[0,\lambda]$, we have $|\psi_{\theta,\nu}|\le2|\psi^*_{\theta,\nu}|$, where

$$\psi^*_{\theta,\nu}(x,t)=\psi_{\theta,\nu}(x,t)\psi\big(\frac{t}{\lambda}\big)\,. \tag{2.9}$$

From the definitions of $e^{it\Delta}$ and $\psi_{\theta,\nu}$, we get

$$|\psi^*_{\theta,\nu}(x,t)| = (2\pi)^{-n}\lambda^{-\frac{n}{4}}\psi(\frac{t}{\lambda})\left|\int e^{i\left[\lambda^{-1/2}(x-c(\nu)-2tc(\theta))\cdot\xi-\lambda^{-1}t|\xi|^2\right]}\prod_{j=1}^{n}\widehat{\varphi}(\xi_j)\,d\xi\right|$$

since $|t|\leq\lambda$, by integration by parts, $\psi^*_{\theta,\nu}$ is essentially supported in the tube $T_{\theta,\nu}$. More precisely, we have

$$|\psi^*_{\theta,\nu}(x,t)| \leq \lambda^{-n/4}\,\chi^*_{T_{\theta,\nu}}(x,t)\,, \tag{2.10}$$

where $\chi^*_{T_{\theta,\nu}}$ denotes a bump function satisfying that $\chi^*_{T_{\theta,\nu}}=1$ on $\{(x,t)\in\mathbb{R}^n\times\mathbb{R}\,:\,0\leq t\leq\lambda, |x-c(\nu)-2tc(\theta)|\leq\sqrt{\lambda}\}$, and $\chi^*_{T_{\theta,\nu}}=O(\lambda^{-1000})$ outside $T_{\theta,\nu}$. We can essentially treat $\chi^*_{T_{\theta,\nu}}$ as $\chi_{T_{\theta,\nu}}$, the indicator function on the tube $T_{\theta,\nu}$.

On the other hand, the Fourier transform of $\psi^*_{\theta,\nu}$ enjoys

$$\widehat{\psi^*_{\theta,\nu}}(\xi_1,\cdots,\xi_n,\xi_{n+1}) = \lambda\widehat{\varphi_{\theta,\nu}}(\xi_1,\cdots,\xi_n)\widehat{\psi}\left(\frac{\xi_{n+1}+(\xi_1^2+\cdots+\xi_n^2)}{1/\lambda}\right)\,. \tag{2.11}$$

Hence $\widehat{\psi^*_{\theta,\nu}}$ is supported in a $\frac{1}{\lambda}$-neighborhood of the parabolic cap over θ, that is,

$$\operatorname{supp}\widehat{\psi^*_{\theta,\nu}} \subseteq \left\{(\xi_1,\cdots,\xi_n,\xi_{n+1})\,:\,(\xi_1,\cdots,\xi_n)\in\theta, |\xi_{n+1}+(\xi_1^2+\cdots+\xi_n^2)|\leq\frac{1}{\lambda}\right\}. \tag{2.12}$$

We denote this $\frac{1}{\lambda}$-neighborhood of the parabolic cap over θ by θ^*. In the rest of the paper, we can assume that the function $\psi_{\theta,\nu}$ is essentially localized in $T_{\theta,\nu}$ in physical space, and localized in θ^* in frequency space.

2.2 Parabolic rescaling

For any function $f\in L^2(\mathbb{R}^n)$ with $\operatorname{supp}\widehat{f}\subset B^n(\xi_0,\rho)$, we write $\xi=\xi_0+\rho\zeta\in B^n(\xi_0,\rho)$, where $\zeta\in B^n(0,1)$, then

$$\left|e^{it\Delta}f(x)\right| = \rho^n(2\pi)^{-n}\left|\int e^{i\left[x\cdot(\xi_0+\rho\zeta)-t|\xi_0+\rho\zeta|^2\right]}\widehat{f}(\xi_0+\rho\zeta)\,d\zeta\right|$$

$$= \rho^{n/2}(2\pi)^{-n}\left|\int e^{i\left[(\rho x-2t\rho\xi_0)\cdot\zeta-\rho^2t|\zeta|^2\right]}\widehat{g}(\zeta)\,d\zeta\right| = \rho^{n/2}\left|e^{ir\Delta}g(y-2\rho^{-1}r\xi_0)\right|\,,$$

where the function g is given by

$$\widehat{g}(\zeta) = \rho^{n/2}\widehat{f}(\xi_0+\rho\zeta)\,,$$

and $\|g\|_2=\|f\|_2$, and the new coordinates (y,r) and old coordinates (x,t) are related by

$$\begin{cases} y = \rho x\,, \\ r = \rho^2 t\,. \end{cases}$$

Therefore, we have

$$\left\| e^{it\Delta} f(x) \right\|_{L^p_x L^q_t (B^n(0,L)\times[0,M])} = \rho^{n/2-2/q-n/p} \left\| e^{ir\Delta} g(y - 2\rho^{-1} r \xi_0) \right\|_{L^p_y L^q_r (B^n(0,\rho L)\times[0,\rho^2 M])}\,. \tag{2.13}$$

From the expression (2.13), we see that, in the case $\xi_0 \neq 0$, parabolic rescaling does not work well for mixed norm $L^p L^q$, where $p \neq q$. More precisely, in the case $\xi_0 \neq 0$ and $p \neq q$, after parabolic rescaling, the L^q_r-norm is taken along the inclined direction $(y - 2\rho^{-1} r \xi_0, r)$ instead of the vertical direction (y, r). Therefore the operator is changed by parabolic rescaling and to the new operator we can not apply induction on scales. When $p = q$, one can get around the difficulty as in the next lemma.

Here we state two results from parabolic rescaling that are used most commonly:

Lemma 2.1 *(a) Let $p = q$. For any function $f \in L^2(\mathbb{R}^n)$ with $\operatorname{supp} \widehat{f} \subset B^n(\xi_0, \rho)$, where $\xi_0 \in B^n(0,1)$, it follows from parabolic rescaling that*

$$\left\| e^{it\Delta} f(x) \right\|_{L^p_x L^p_t (B^n(0,L)\times[0,L])} \leq \rho^{n/2-(n+2)/p} \left\| e^{ir\Delta} g(y) \right\|_{L^p_y L^p_r (B^n(0,3\rho L)\times[0,\rho^2 L])}\,, \tag{2.14}$$

for some function $g \in L^2$ with $\operatorname{supp} \widehat{g} \subset B^n(0,1)$ and $\|g\|_2 = \|f\|_2$.

(b) For any function $f \in L^2(\mathbb{R}^n)$ with $\operatorname{supp} \widehat{f} \subset B^n(0, \rho)$, it follows from parabolic rescaling that

$$\left\| e^{it\Delta} f(x) \right\|_{L^p_x L^q_t (B^n(0,L)\times[0,M])} = \rho^{n/2-2/q-n/p} \left\| e^{ir\Delta} g(y) \right\|_{L^p_y L^q_r (B^n(0,\rho L)\times[0,\rho^2 M])}\,, \tag{2.15}$$

for some function $g \in L^2$ with $\operatorname{supp} \widehat{g} \subset B^n(0,1)$ and $\|g\|_2 = \|f\|_2$.

2.3 Approach to a.e. pointwise convergence: Schrödinger maximal estimates

To approach the almost everywhere pointwise convergence problem of the solutions to Schödinger equations (1.2), we note that we have uniform convergence $e^{it\Delta} f(x) \to f(x)$ for Schwartz functions f. First we recall that Schwartz functions are dense in H^s. Then for any $f \in H^s$, we approximate f by Schwartz functions, and we control the error terms by bounding the associated maximal function. We call such bounds Schrödinger maximal estimates.

Lemma 2.2 *The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.*

Proof: For any function $f \in H^s(\mathbb{R}^n)$, we know that $(1+|\xi|^2)^{s/2}\widehat{f} \in L^2$. Since $C_c^\infty(\mathbb{R}^n)$, the space of compactly supported smooth functions, is dense in L^2, we can find a sequence of functions $g_n \in C_c^\infty(\mathbb{R}^n)$ satisfying that

$$g_n \to (1+|\xi|^2)^{s/2}\widehat{f} \quad \text{in L}^2 .$$

Note that $(1+|\xi|^2)^{-s/2}g_n$ is in C_c^∞, hence its inverse Fourier transform f_n is a Schwartz function,

$$\widehat{f_n} := (1+|\xi|^2)^{-s/2}g_n .$$

Therefore, the sequence of functions $f_n \in \mathcal{S}(\mathbb{R}^n)$ converges to f in H^s:

$$\|f_n - f\|_{H^s} = \left(\int (1+|\xi|^2)^s \left|\widehat{f_n} - \widehat{f}\right|^2 d\xi\right)^{1/2}$$

$$= \left(\int \left|g_n - (1+|\xi|^2)^{s/2}\widehat{f}\right|^2\right)^{1/2} \to 0 .$$

■

The next lemma says that a.e. pointwise convergence result follows from certain Schrödinger maximal estimate. In particular, Theorem 1.2 implies Theorem 1.1.

Lemma 2.3 *Suppose that for some $s > 0$, and some $p \geq 1$,*

$$\left\|\sup_{0<t\leq 1} |e^{it\Delta} f|\right\|_{L^p(B^n(0,1))} \leq C_s \|f\|_{H^s(\mathbb{R}^n)} \tag{2.16}$$

holds for any function $f \in H^s(\mathbb{R}^n)$. Then

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \textit{almost everywhere} \tag{2.17}$$

whenever $f \in H^s(\mathbb{R}^n)$.

Proof: First we show that if f is Schwartz, then $e^{it\Delta}f(x) \to f(x)$ uniformly on $\mathbb{R}^n$, as $t \to 0$. Note that

$$\left|e^{it\Delta}f(x)-f(x)\right| = (2\pi)^{-n}\left|\int e^{ix\cdot\xi}\widehat{f}(\xi)\left(e^{-it|\xi|^2}-1\right)\,d\xi\right|$$

$$\lesssim |t|\cdot\int|\widehat{f}(\xi)||\xi|^2\,d\xi \le |t|\cdot\left\|\widehat{f}(\xi)|\xi|^2(1+|\xi|^2)^n\right\|_\infty\cdot\left\|(1+|\xi|^2)^{-n}\right\|_{L^1(\mathbb{R}^n)}\,.$$

For a Schwartz function f, its Fourier transform $\widehat{f}$ is also a Schwartz function, hence

$$\left\|\widehat{f}(\xi)|\xi|^2(1+|\xi|^2)^n\right\|_\infty < \infty\,,$$

and it follows that $\left|e^{it\Delta}f(x)-f(x)\right| \le C\,|t|$. Therefore, the uniform convergence is justified for Schwartz functions.

Given any function $f\in H^s(\mathbb{R}^n)$, since Schwartz functions are dense in H^s, for any $\varepsilon>0$, we write $f=g+h$, where g is Schwartz and $\|h\|_{H^s}<\varepsilon$. Then we get

$$\overline{\lim_{t\to 0}}\left|e^{it\Delta}f(x)-f(x)\right| \le \overline{\lim_{t\to 0}}\left|e^{it\Delta}g(x)-g(x)\right| + \overline{\lim_{t\to 0}}\left|e^{it\Delta}h(x)-h(x)\right|$$

$$\le 0+\sup_{0<t\le 1}\left|e^{it\Delta}h(x)\right|+|h(x)|\,.$$

For any $\alpha>0$, denote

$$E_\alpha := \left\{x\in B^n(0,1)\,:\,\overline{\lim_{t\to 0}}\left|e^{it\Delta}f(x)-f(x)\right|>\alpha\right\}\,,$$

and we bound $|E_\alpha|$, the measure of E_α, by

$$\left|\left\{x\in B^n(0,1)\,:\,\sup_{0<t\le 1}\left|e^{it\Delta}h(x)\right|>\alpha/2\right\}\right|+\left|\,\{x\in B^n(0,1)\,:\,|h(x)|>\alpha/2\}\,\right|.$$

Moreover by assumption (2.16) we have

$$\left|\left\{x\in B^n(0,1)\,:\,\sup_{0<t\le 1}\left|e^{it\Delta}h(x)\right|>\alpha/2\right\}\right|$$

$$\le \frac{\left\|\sup_{0<t\le 1}|e^{it\Delta}h|\right\|^p_{L^p(B^n(0,1))}}{(\alpha/2)^p} \le \frac{C_s^p\|h\|^p_{H^s(\mathbb{R}^n)}}{(\alpha/2)^p} \lesssim \frac{\varepsilon^p}{\alpha^p}\,,$$

and

$$\left|\{x \in B^n(0,1) \,:\, |h(x)| > \alpha/2\}\right| \le \frac{\|h\|_2^2}{(\alpha/2)^2} \le \frac{\|h\|_{H^s}^2}{(\alpha/2)^2} \lesssim \frac{\varepsilon^2}{\alpha^2}\,.$$

Therefore, for any $\alpha > 0$ the bound

$$|E_\alpha| \lesssim \frac{\varepsilon^p}{\alpha^p} + \frac{\varepsilon^2}{\alpha^2}$$

holds for any $\varepsilon > 0$. We conclude that $|E_\alpha| = 0$ for any $\alpha > 0$. We take the set $\bigcup_{k=1}^{\infty} E_{1/k}$, which has measure zero and

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \text{for any}\, x \in B^n(0,1)\setminus \bigcup_{k=1}^{\infty} E_{1/k}\,.$$

We have proved the almost everywhere pointwise convergence in $B^n(0,1)$. The same argument applies to any other ball $B^n(x_0,1)$, via the following observation: we write $x = x_0 + y \in B^n(x_0,1)$, where $y \in B^n(0,1)$, then

$$e^{it\Delta} f(x) = (2\pi)^{-n} \int e^{i\left(y\cdot\xi - t|\xi|^2\right)} e^{ix_0\cdot\xi} \widehat{f}(\xi)\, d\xi =: e^{it\Delta} g(y)\,,$$

where $\widehat{g}(\xi) = e^{ix_0\cdot\xi}\widehat{f}(\xi)$, so we have

$$\left\| \sup_{0<t\le 1} |e^{it\Delta} f| \right\|_{L^p(B^n(x_0,1))} = \left\| \sup_{0<t\le 1} |e^{it\Delta} g| \right\|_{L^p(B^n(0,1))} \le C_s \|g\|_{H^s(\mathbb{R}^n)} = C_s \|f\|_{H^s(\mathbb{R}^n)}\,.$$

Therefore, $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ for almost every $x \in \mathbb{R}^n$.

■

We have seen that the Schrödinger maximal bounds (2.16) imply a.e. pointwise convergence results. Next we make a few standard reductions of (2.16). Recall the following Lemma from [14]:

Lemma 2.4 (Lee) *Let $p, q \ge 2$, $\lambda \ge 1$, and $\alpha \in \mathbb{R}$. Suppose that*

$$\left\|e^{it\Delta} f\right\|_{L_x^p L_t^q(B^n(0,1)\times[0,1/\lambda])} \le C\lambda^\alpha \|f\|_2$$

holds for any $f \in L^2(\mathbb{R}^n)$ with $\operatorname{supp}\widehat{f} \subset A(\lambda)$. *Then*

$$\left\|e^{it\Delta} f\right\|_{L_x^p L_t^q(B^n(0,1)\times[0,1])} \le C_\epsilon \lambda^{\alpha+\varepsilon} \|f\|_2$$

holds for any $\varepsilon > 0$ and any $f \in L^2(\mathbb{R}^n)$ with $\operatorname{supp}\widehat{f} \subset A(\lambda)$.

Proof: By parabolic rescaling (2.15), it suffices to show that for any $\beta \in \mathbb{R}$, the estimate

$$\left\|e^{it\Delta}f\right\|_{L^p_xL^q_t(B^n(0,\lambda)\times[0,\lambda])} \le C\lambda^{\beta}\|f\|_2 \quad \text{for any } f\in L^2 \text{ with } \mathrm{supp}\widehat{f}\subset A(1) \tag{2.18}$$

implies

$$\left\|e^{it\Delta}f\right\|_{L^p_xL^q_t(B^n(0,\lambda)\times[0,\lambda^2])} \le C_\varepsilon\lambda^{\beta+\varepsilon}\|f\|_2 \quad \text{for any } f\in L^2 \text{ with } \mathrm{supp}\widehat{f}\subset A(1)\,. \tag{2.19}$$

To prove (2.19), since t is restricted to $[0,\lambda^2]$, we use wave packet decomposition at scale λ^2. Recall that for each tile (θ,ν), the corresponding tube $T_{\theta,\nu}$ is of radius $\lambda^{1+2\delta}$, of length λ^2, and in the direction given by $(2c(\theta),1)$. Note that the angle between each tube and the t-axis is nonzero (roughly 45 degree), since $c(\theta)\in A(1)$. To utilize the assumption (2.18), we partition the interval $[0,\lambda^2]$ into intervals I_j of length λ, and denote $\Omega_j := B^n(0,\lambda)\times I_j$. For each j, define

$$f_j := \sum_{(\theta,\nu):T_{\theta,\nu}\cap\Omega_j\neq\emptyset} f_{\theta,\nu}\,.$$

Then on each Ω_j, we have $e^{it\Delta}f(x)\sim e^{it\Delta}f_j(x)$, moreover

$$\sum_j\|f_j\|_2^2 \sim \sum_j\sum_{(\theta,\nu):T_{\theta,\nu}\cap\Omega_j\neq\emptyset}\|f_{\theta,\nu}\|_2^2 \lesssim \lambda^{O(\delta)}\|f\|_2^2\,, \tag{2.20}$$

where the last inequality follows from the fact that each tube $T_{\theta,\nu}$ has nonzero angle to the t-axis and it can only intersect at most $\lambda^{O(\delta)}$ many Ω_j's.

Now we write the right hand side of (2.19) as follows:

$$\left\|e^{it\Delta}f\right\|^p_{L^p_xL^q_t(B^n(0,\lambda)\times[0,\lambda^2])} = \int_{B^n(0,\lambda)}\left(\sum_j\int_{I_j}\left|e^{it\Delta}f(x)\right|^q\,dt\right)^{p/q}dx\,.$$

We consider the cases $p\ge q$ and $p<q$ separately. First, when $p<q$,

$$\left\|e^{it\Delta}f\right\|^p_{L^p_xL^q_t(B^n(0,\lambda)\times[0,\lambda^2])} \le \sum_j\left\|e^{it\Delta}f\right\|^p_{L^p_xL^q_t(\Omega_j)} \sim \sum_j\left\|e^{it\Delta}f_j\right\|^p_{L^p_xL^q_t(\Omega_j)}\,,$$

by applying the assumption (2.18) to each f_j, we get

$$\left\|e^{it\Delta}f\right\|^p_{L^p_xL^q_t(B^n(0,\lambda)\times[0,\lambda^2])} \lesssim \sum_j\lambda^{\beta p}\|f_j\|_2^p\,,$$

now because of $p\ge 2$ and (2.20),

$$\|e^{it\Delta}f\|^p_{L^p_xL^q_t(B^n(0,\lambda)\times[0,\lambda^2])}\lesssim\lambda^{\beta p}\left(\sum_j\|f_j\|_2^2\right)^{p/2}\lesssim\lambda^{\beta p+O(\delta)}\|f\|_2^p\,.$$

Next, when $p\geq q$, by Minkowski's inequality, the assumption (2.18), $q\geq 2$ and (2.20),

$$\|e^{it\Delta}f\|^p_{L^p_xL^q_t(B^n(0,\lambda)\times[0,\lambda^2])}\leq\left[\sum_j\|e^{it\Delta}f\|^q_{L^p_xL^q_t(\Omega_j)}\right]^{p/q}$$

$$\sim\left[\sum_j\|e^{it\Delta}f_j\|^q_{L^p_xL^q_t(\Omega_j)}\right]^{p/q}\lesssim\lambda^{\beta p}\left(\sum_j\|f_j\|_2^q\right)^{p/q}\lesssim\lambda^{\beta p+O(\delta)}\|f\|_2^p\,,$$

and this completes the proof.

■

By a Littlewood-Paley decomposition, parabolic rescaling and Lemma 2.4, we have the following reduction of Schrödinger maximal estimates (in particular, Theorem 1.3 implies Theorem 1.2 by taking $n=2,p=3$ and $s_0=1/3$):

Lemma 2.5 *Let $p\geq 2$ and $s_0\in\mathbb{R}$. Suppose that for any $\epsilon>0$, there is a constant C_ε such that*

$$\left\|\sup_{0<t\leq R}|e^{it\Delta}f|\right\|_{L^p(B^n(0,R))}\leq C_\varepsilon R^{s_0-n/2+n/p+\varepsilon}\|f\|_2 \tag{2.21}$$

holds for any $R\geq 1$, and any function $f\in L^2(\mathbb{R}^n)$ with $\mathrm{supp}\widehat{f}\subset B^n(0,1)$. Then

$$\left\|\sup_{0<t\leq 1}|e^{it\Delta}f|\right\|_{L^p(B^n(0,1))}\leq C_s\|f\|_{H^s(\mathbb{R}^n)} \tag{2.22}$$

holds for any function $f\in H^s(\mathbb{R}^n)$ with $s>s_0$.

Proof: By parabolic rescaling (2.15) and our assumption (2.21), for any $\varepsilon>0$, there is a constant C_ε such that

$$\left\|\sup_{0<t\leq 1/R}|e^{it\Delta}f|\right\|_{L^p(B^n(0,1))}\leq C_\varepsilon R^{s_0+\varepsilon}\|f\|_2 \tag{2.23}$$

holds for any $R\geq 1$, and any function $f\in L^2$ with $\mathrm{supp}\widehat{f}\subset A(R)$. Then by Lemma 2.4, for any $\varepsilon>0$, there is a constant C'_ε such that

$$\left\| \sup_{0<t\leq 1} |e^{it\Delta} f| \right\|_{L^p(B^n(0,1))} \leq C'_\varepsilon R^{s_0+\varepsilon} \|f\|_2 \,, \tag{2.24}$$

holds for any $R \geq 1$, and any function $f \in L^2$ with $\mathrm{supp}\widehat{f} \subset A(R)$.

Now given $s > s_0$ and $f \in H^s(\mathbb{R}^n)$, we choose $\varepsilon > 0$ such that $s > s_0 + \varepsilon$. We decompose f in a Littlewood-Paley decomposition:

$$f = \sum_{k\geq 0} f_k \,,$$

where $\widehat{f_0}$ is supported in $B^n(0,1)$ and $\widehat{f_k}$ is supported in $A(2^k)$ for $k \geq 1$. We have

$$\|f_k\|_{L^2} \lesssim 2^{-ks} \|f\|_{H^s} \,.$$

Applying (2.24) to each f_k, (2.21) to f_0 and using the triangle inequality, we get

$$\left\| \sup_{0<t\leq 1} |e^{it\Delta} f| \right\|_{L^p(B^n(0,1))} \lesssim \sum_{k\geq 1} 2^{k(s_0+\varepsilon-s)} \|f\|_{H^s} + \|f_0\|_2 \lesssim \|f\|_{H^s} \,,$$

as desired.

■

Combining Lemma 2.3 and Lemma 2.5, we get:

Proposition 2.6 *Let $p \geq 2$ and $s_0 \geq 0$. Suppose that for any $\epsilon > 0$, there is a constant C_ε such that*

$$\left\| \sup_{0<t\leq R} |e^{it\Delta} f| \right\|_{L^p(B^n(0,R))} \leq C_\varepsilon R^{s_0 - n/2 + n/p + \varepsilon} \|f\|_2 \tag{2.25}$$

holds for any $R \geq 1$, and any function $f \in L^2(\mathbb{R}^n)$ with $\mathrm{supp}\widehat{f} \subset B^n(0,1)$. Then

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \textit{almost everywhere} \tag{2.26}$$

whenever $f \in H^s(\mathbb{R}^n)$ with $s > s_0$.

2.4 A sharp global Schrödinger maximal estimate

In Section 2.3, we have seen that Theorem 1.3 implies the local Schrödinger maximal estimate (1.3) in Theorem 1.2. In this section, we show a sharp global estimate, as a corollary of Theorem 1.3.

Proposition 2.7 *Theorem 1.3 implies Theorem 1.5. In other words, suppose that for any $\varepsilon > 0$,*

$$\big\| \sup_{0<t\leq R} |e^{it\Delta} f| \big\|_{L^3(B(0,R))} \leq C_\epsilon R^\epsilon \|f\|_2 \tag{2.27}$$

for all $R \geq 1$ and all $f \in L^2(\mathbb{R}^2)$ with $\mathrm{supp}\widehat{f} \subset B(0,1)$. Then

$$\Big\| \sup_{0<t\leq 1} |e^{it\Delta} f| \Big\|_{L^3(\mathbb{R}^2)} \leq C_s \|f\|_{H^s(\mathbb{R}^2)} \tag{2.28}$$

holds for any function $f \in H^s(\mathbb{R}^2)$ provided that $s > 1/3$.

Proof: First, we show

$$\big\| \sup_{0<t\leq R^2} |e^{it\Delta} f| \big\|_{L^3(\mathbb{R}^2)} \leq C_\epsilon R^\epsilon \|f\|_2 \tag{2.29}$$

for all $R \geq 1$ and all $f \in L^2$ with $\mathrm{supp}\widehat{f} \subset B(0,1)$. By assumption, we have

$$\big\| \sup_{0<t\leq R^2} |e^{it\Delta} f| \big\|_{L^3(Q)} \leq C_\epsilon R^\epsilon \|f\|_2 \tag{2.30}$$

for any cube $Q \subset \mathbb{R}^2$ with side length R^2. Since t is restricted to $[0, R^2]$, we use wave packet decomposition at scale R^2. Recall that for each tile (θ, ν), the corresponding tube $T_{\theta,\nu}$ is of radius $R^{1+2\delta}$, of length R^2, and in the direction given by $(2c(\theta), 1)$. We partition $\mathbb{R}^2$ into disjoint cubes Q_l of side length R^2, and denote $\Omega_l = Q_l \times [0, R^2]$. For each l, we define

$$f_l := \sum_{(\theta,\nu): T_{\theta,\nu}\cap\Omega_l \neq \emptyset} f_{\theta,\nu}\,.$$

Then one each Ω_l, we have $e^{it\Delta} f(x) \sim e^{it\Delta} f_l(x)$, moreover

$$\sum_l \|f_l\|_2^2 \sim \sum_l \sum_{(\theta,\nu): T_{\theta,\nu}\cap\Omega_l \neq \emptyset} \|f_{\theta,\nu}\|_2^2 \lesssim \|f\|_2^2\,, \tag{2.31}$$

where the last inequality follows from the fact that each tube $T_{\theta,\nu}$ can only intersect $O(1)$ many Ω_l's. Now we have

$$\big\| \sup_{0<t\leq R^2} |e^{it\Delta} f| \big\|_{L^3(\mathbb{R}^2)}^3 = \sum_l \big\| \sup_{0<t\leq R^2} |e^{it\Delta} f| \big\|_{L^3(Q_l)}^3 \sim \sum_l \big\| \sup_{0<t\leq R^2} |e^{it\Delta} f_l| \big\|_{L^3(Q_l)}^3$$

by applying (2.30) to each f_l and utilizing (2.31) we get

$$\Big\|\sup_{0<t\le R^2}|e^{it\Delta}f|\Big\|^3_{L^3(\mathbb{R}^2)} \le \sum_l C_\varepsilon R^\varepsilon \|f_l\|_2^3 \le C_\varepsilon R^\varepsilon \|f\|_2^3\,,$$

which establishes (2.29).

By parabolic rescaling (2.15), (2.29) implies that

$$\Big\|\sup_{0<t\le 1}|e^{it\Delta}f|\Big\|_{L^3(\mathbb{R}^2)} \le C_\epsilon R^{1/3+\epsilon}\|f\|_2 \tag{2.32}$$

holds for all $R \ge 1$ and all $f \in L^2$ with $\mathrm{supp}\widehat{f} \subset A(R) = \{\xi \in \mathbb{R}^2 : |\xi| \sim R\}$.

Now given any function $f \in H^s(\mathbb{R}^2)$ with some $s > 1/3$, we choose $\varepsilon > 0$ such that $1/3 + \varepsilon < s$, and decompose f in a Littlewood-Paley decomposition:

$$f = \sum_{k\ge 0} f_k\,,$$

where $\widehat{f_0}$ is supported in $B^n(0,1)$ and $\widehat{f_k}$ is supported in $A(2^k)$ for $k \ge 1$. We have

$$\|f_k\|_{L^2} \lesssim 2^{-ks}\|f\|_{H^s}\,.$$

Applying (2.32) to each f_k, (2.29) to f_0 and using the triangle inequality, we get

$$\left\|\sup_{0<t\le 1}|e^{it\Delta}f|\right\|_{L^3(\mathbb{R}^2)} \le C_\varepsilon \sum_{k\ge 1} 2^{k(1/3+\varepsilon-s)}\|f\|_{H^s} + C\|f_0\|_2 \le C_s\|f\|_{H^s}\,,$$

the last inequality follows from the choice of ε: $1/3+\varepsilon < s$, and this completes the proof.

■

2.5 Polynomial partitioning

First we state a variation of the ham-sandwich theorem, which introduces a polynomial P in the polynomial ring $\mathbb{R}[x,t]$ such that the variety $Z(P) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : P(x,t) = 0\}$ bisects every member in a collection of some quantities. It relies on the Borsuk-Ulam Theorem, which asserts that *if $F : \mathbb{S}^N \to \mathbb{R}^N$ is a continuous function, where $\mathbb{S}^N$ is the N-dimensional unit sphere, then there exists a point $v \in \mathbb{S}^N$ with $F(v) = F(-v)$.*

Lemma 2.8 *Suppose that $W_1, W_2, \cdots, W_N \in L^1_x L^r_t(\mathbb{R}^n \times \mathbb{R}), 1 \le r < \infty$. Then there exists a non-zero polynomial P on $\mathbb{R}^n \times \mathbb{R}$ of degree $\le c_n N^{1/(n+1)}$ such that for each W_j,*

$$\|\chi_{\{P>0\}}W_j\|_{L_x^1L_t^r(\mathbb{R}^n\times\mathbb{R})}=\|\chi_{\{P<0\}}W_j\|_{L_x^1L_t^r(\mathbb{R}^n\times\mathbb{R})}.$$

Proof: Let V be the vector space of polynomials on $\mathbb{R}^n\times\mathbb{R}$ of degree at most D, then

$$\mathrm{Dim}V=\binom{D+n+1}{n+1}\sim_n D^{n+1}.$$

So we can choose $D\sim N^{1/(n+1)}$ such that $\mathrm{Dim}V\geq N+1$, and without loss of generality we can assume $\mathrm{Dim}V=N+1$ and identify V with $\mathbb{R}^{N+1}$. We define a function G as follows:

$$\mathbb{S}^N\subseteq V\backslash\{0\}\xrightarrow{G}\mathbb{R}^N$$
$$P\mapsto\{G_j(P)\}_{j=1}^N,$$

where

$$G_j(P):=\|\chi_{\{P>0\}}W_j\|_{L_x^1L_t^r(\mathbb{R}^n\times\mathbb{R})}-\|\chi_{\{P<0\}}W_j\|_{L_x^1L_t^r(\mathbb{R}^n\times\mathbb{R})},$$

it is obvious that $G(-P)=-G(P)$. Assume that the function G is continuous, then the Borsuk-Ulam Theorem tells us that there exists a polynomial $P\in\mathbb{S}^N\subseteq V\backslash\{0\}$ with $G(P)=G(-P)$, hence $G(P)=0$, and P obeys the conclusion of Lemma 2.8. It remains to check the continuity of the functions G_j on $V\backslash\{0\}$.

Suppose that $P_k\to P$ in $V\backslash\{0\}$. Note that

$$|G_j(P_k)-G_j(P)|\leq 2\|\chi_{\{P_kP\leq 0\}}W_j\|_{L_x^1L_t^r(\mathbb{R}^n\times\mathbb{R})},$$

while $P_k\to P$ implies that

$$\bigcap_{k_0}\bigcup_{k\geq k_0}\{(x,t):P_k(x,t)\cdot P(x,t)\leq 0\}\subseteq P^{-1}(0).$$

By the dominated convergence theorem,

$$\lim_{k_0\to\infty}\|\chi_{\cup_{k\geq k_0}\{P_kP\leq 0\}}W_j\|_{L_x^1L_t^r(\mathbb{R}^n\times\mathbb{R})}=\|\chi_{\{P^{-1}(0)\}}W_j\|_{L_x^1L_t^r(\mathbb{R}^n\times\mathbb{R})}=0.$$

This proves that $\lim_{k\to\infty}|G_j(P_k)-G_j(P)|=0$, showing that G_j is continuous on $V\backslash\{0\}$.

∎

By applying Lemma 2.8 repeatedly, we get the following polynomial partitioning result:

Theorem 2.9 *Suppose that $W \in L^1_x L^r_t(\mathbb{R}^n \times \mathbb{R})\backslash\{0\}, 1 \le r < \infty$. Then for each degree $D \ge 1$ there exists a non-zero polynomial P on $\mathbb{R}^n \times \mathbb{R}$ of degree at most D such that $(\mathbb{R}^n \times \mathbb{R})\backslash Z(P)$ is a union of $\sim_n D^{n+1}$ disjoint open sets O_i and for each i we have*

$$\|W\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})} \le c_n D^{n+1} \|\chi_{O_i} W\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})}.$$

Proof: By Lemma 2.8, we obtain a polynomial P_1 of degree $\lesssim 1$ such that

$$\|\chi_{\{P_1>0\}} W\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})} = \|\chi_{\{P_1<0\}} W\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})}.$$

Next, we let $W_+ := \chi_{\{P_1>0\}} W$ and $W_- := \chi_{\{P_1<0\}} W$, and by Lemma 2.8 again we obtain a polynomial P_2 of degree $\lesssim 2^{1/(n+1)}$ such that

$$\|\chi_{\{P_2>0\}} W_j\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})} = \|\chi_{\{P_2<0\}} W_j\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})},$$

for $j = +, -$. Continuing inductively, we construct polynomials $P_1, P_2, \cdots, P_s$. Let $P := \prod_{k=1}^s P_k$. The sign conditions of the polynomials cut $(\mathbb{R}^n \times \mathbb{R})\backslash Z(P)$ into 2^s cells O_i, and by construction and triangle inequality we have that, for each i,

$$\|W\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})} \le 2^s \|\chi_{O_i} W\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})}.$$

By construction, $\deg P_k \lesssim 2^{(k-1)/(n+1)}$, therefore $\deg P \le c_n 2^{s/(n+1)}$. We can choose s such that $c_n 2^{s/(n+1)} \in [D/2, D]$, then $\deg P \le D$ and the number of cells is $2^s \sim_n D^{n+1}$.

■

Definition 2.10 *We say that a polynomial P is non-singular if $\nabla P(z) \ne 0$ for each point z in $Z(P)$.*

It is well-known that non-singular polynomials are dense in the space of all polynomials, cf. Lemma 1.5 in [10].

Lemma 2.11 (Guth) *Let $\mathrm{Poly}_D(\mathbb{R}^n)$ be the vector space of polynomials on $\mathbb{R}^n$ of degree at most D. Then non-singular polynomials are dense in $\mathrm{Poly}_D(\mathbb{R}^n)$ for any D and n. Moreover, the singular polynomials have measure zero.*

Proof: Consider the map

$$\mathbb{R}^n \times \mathrm{Poly}_D(\mathbb{R}^n) \xrightarrow{E} \mathbb{R} \times \mathrm{Poly}_D(\mathbb{R}^n)$$
$$(x, Q) \mapsto (Q(x), Q)\,.$$

The map E is C^∞ smooth, and so by Sard's theorem, the critical values of E have measure zero.

Suppose that (h, Q) is a regular value of E, then we claim that $Q - h$ is a non-singular polynomial. Given $x \in \mathbb{R}^n$ satisfying that $(Q-h)(x) = 0$, we want to show that $\nabla(Q-h)(x) = \nabla Q(x) \neq 0$. Note that $(Q-h)(x) = 0$ if and only if $(x, Q) \in E^{-1}(h, Q)$. Since (h, Q) is a regular value of E, we know that the differential map

$$dE_{x,Q} = \nabla Q(x) \times id : \mathbb{R}^n \times \mathrm{Poly}_D(\mathbb{R}^n) \to \mathbb{R} \times \mathrm{Poly}_D(\mathbb{R}^n)$$

is surjective, therefore, $\nabla Q(x)$ is nonzero.

We have seen that for almost every $(h, Q) \in \mathbb{R} \times \mathrm{Poly}_D(\mathbb{R}^n)$, $Q - h$ is non-singular. In other words, the set $\{(h, P+h) \in \mathbb{R} \times \mathrm{Poly}_D(\mathbb{R}^n) \,:\, P \text{ is singular}\}$ has measure zero. By Fubini's theorem it follows that the set of singular polynomials has measure zero in $\mathrm{Poly}_D(\mathbb{R}^n)$, and so the non-singular polynomials are dense.

■

Following from the density of non-singular polynomials and the proof of Theorem 2.9, we can assume that the polynomial in the partitioning theorem enjoys nice geometric properties.

Lemma 2.12 *Suppose that $W_1, W_2, \cdots, W_N \in L^1_x L^r_t(\mathbb{R}^n \times \mathbb{R}), 1 \leq r < \infty$. Then for any $\epsilon > 0$, there exists a non-singular polynomial P on $\mathbb{R}^n \times \mathbb{R}$ of degree $\leq c_n N^{1/(n+1)}$ such that for each W_j,*

$$(1-\varepsilon)\big\|\chi_{\{P<0\}} W_j\big\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})} \leq \big\|\chi_{\{P>0\}} W_j\big\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})} \leq (1+\varepsilon)\big\|\chi_{\{P<0\}} W_j\big\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})}.$$

Proof: Let P_0 be a non-zero polynomial on $\mathbb{R}^n \times \mathbb{R}$ of degree $\leq c_n N^{1/(n+1)}$ such that for each W_j,

$$\big\|\chi_{\{P>0\}} W_j\big\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})} = \big\|\chi_{\{P<0\}} W_j\big\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})}.$$

Then let P_k be a sequence of non-singular polynomials of degree $\leq c_n N^{1/(n+1)}$, approaching P_0. By the same continuity argument as in the proof of Lemma 2.8, we have

$$\lim_{k\to\infty} \big\|\chi_{\{P_k>0\}} W_j\big\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})} = \big\|\chi_{\{P>0\}} W_j\big\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})}$$

and

$$\lim_{k\to\infty} \left\|\chi_{\{P_k<0\}} W_j\right\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})} = \left\|\chi_{\{P<0\}} W_j\right\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})},$$

therefore, for large k, P_k obeys the desired inequality.

■

Using Lemma 2.12 in place of Lemma 2.8, we get a partitioning result involving non-singular polynomials.

Theorem 2.13 *Suppose that $W \in L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})\backslash\{0\}, 1 \le r < \infty$. Then for each D there exists a non-zero polynomial P on $\mathbb{R}^n\times\mathbb{R}$ of degree at most D such that $(\mathbb{R}^n\times\mathbb{R})\backslash Z(P)$ is a union of $\sim_n D^{n+1}$ disjoint open sets O_i and for each i we have*

$$\left\|W\right\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})} \le c_n D^{n+1} \left\|\chi_{O_i} W\right\|_{L^1_x L^r_t(\mathbb{R}^n\times\mathbb{R})}.$$

Moreover, the polynomial P is a product of distinct non-singular polynomials.

Chapter 3

Proof of main inductive theorem - Theorem 1.7

This chapter is devoted to a proof of Theorem 1.7, using polynomial partitioning.

3.1 Cell contributions

Recall that the functions f in Theorem 1.7 are Fourier supported in $B(\xi_0, M^{-1}) \subset \mathbb{R}^2$ with arbitrary $\xi_0 \in B(0,1)$ and $M \geq 1$. Also $p > 3$ and $q > \epsilon^{-4}$. The functions f can be assumed to be Schwartz functions since the collection of all Schwartz functions is dense in L^2. We need to prove the bound (1.11):

$$\left\|e^{it\Delta} f\right\|_{L_x^p L_t^q(B(0,R)\times[0,R])} \leq C_{p,\epsilon} M^{-\epsilon^2} R^{\epsilon} \|f\|_2 .$$

The proof of Theorem 1.7 is by induction on the radius R in physical space and the radius $1/M$ in frequency space. First we cover the bases of the induction. Suppose that $M \geq R^{10}$, then we bound $|e^{it\Delta} f(x)|$ by $M^{-1}\|f\|_2$ and Theorem 1.7 is trivial. Suppose that $R^{1/2-O(\delta)} < M < R^{10}$, then all associated wave packets are in the same direction, and by a direct computation we can bound the left-hand side of (1.11) by $R^{(3-p)/(2p)+O(\delta)}\|f\|_2$, from which Theorem 1.7 follows immediately. Therefore we can assume that $M \ll \sqrt{R}$. We can assume that R is sufficiently large, otherwise Theorem 1.7 is trivial. This covers the base of the induction. Now we turn to the inductive step. By induction, we can assume that Theorem 1.7 holds for physical radii less than $R/2$ or for physical radius R and frequency radius less than $\frac{1}{2M}$.

Let $\mathbf{B}_R^*$ denote the set $B(0,R) \times [0,R]$. We pick a degree $D = R^{\epsilon^4}$, and apply polynomial partitioning with this degree to the function $\chi_{\mathbf{B}_R^*}|e^{it\Delta} f(x)|^p$. By Theorem 2.13 with $r = q/p$, there exists a non-zero polynomial P on $\mathbb{R}^2 \times \mathbb{R}$ of degree at most D such that $(\mathbb{R}^2 \times \mathbb{R})\backslash Z(P)$ is a union of $\sim D^3$ disjoint open sets O_i and for each i we have

$$\left\|e^{it\Delta} f(x)\right\|_{L_x^p L_t^q(\mathbf{B}_R^*)}^p \leq cD^3 \left\|\chi_{O_i} e^{it\Delta} f(x)\right\|_{L_x^p L_t^q(\mathbf{B}_R^*)}^p. \tag{3.1}$$

Moreover, the polynomial P is a product of distinct non-singular polynomials.

We define

$$W := N_{R^{1/2+\delta}} Z(P) \cap \mathbf{B}_R^* \,, \tag{3.2}$$

where $\delta = \epsilon^2$ and $N_{R^{1/2+\delta}} Z(P)$ stands for the $R^{1/2+\delta}$-neighborhood of the variety $Z(P)$ in $\mathbb{R}^3$. We have the wave packet decomposition for $e^{it\Delta} f$ as in (2.7). For each cell O_i, we set

$$O_i' = [O_i \cap \mathbf{B}_R^*] \setminus W \;\text{ and }\; \mathbf{T}_i = \{(\theta,\nu) \in \mathbf{T} \,:\, T_{\theta,\nu} \cap O_i' \neq \emptyset\} \,. \tag{3.3}$$

Here $T_{\theta,\nu}$ is the tube associated to each tile (θ, ν), as defined in (2.8). For a function f we define

$$f_i = \sum_{(\theta,\nu)\in \mathbf{T}_i} f_{\theta,\nu} \,. \tag{3.4}$$

From (2.10), it follows that on each cell O_i',

$$e^{it\Delta} f(x) \sim e^{it\Delta} f_i(x) \,. \tag{3.5}$$

By the fundamental theorem of algebra, we have a simple yet important geometric observation:

Lemma 3.1 *For each tile $(\theta,\nu) \in \mathbf{T}$, the number of cells O_i' that intersect the tube $T_{\theta,\nu}$ is $\leq D+1$.*

Proof: If $T_{\theta,\nu}$ intersects O_i', then the central line of $T_{\theta,\nu}$ must cross O_i. On the other hand, a line can cross the variety $Z(P)$ at most D times, hence can cross at most $D+1$ cells O_i.

■

By triangle inequality, we dominate $\left\| e^{it\Delta} f(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}_R^*)}$ by

$$\sum_i \left\| \chi_{O_i'} e^{it\Delta} f(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}_R^*)} + \left\| \chi_W e^{it\Delta} f(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}_R^*)} \,. \tag{3.6}$$

We call the first term in (3.6) the cellular term, and the second the wall term. Using induction we will see that the desired bound (1.11) holds unless the wall term makes a significant contribution. In particular, we will show that (1.11) holds unless

$$\left\| e^{it\Delta} f \right\|_{L^p_x L^q_t(\mathbf{B}_R^*)} \lesssim R^{O(\epsilon^4)} \left\| \chi_W e^{it\Delta} f \right\|_{L^p_x L^q_t(\mathbf{B}_R^*)}. \tag{3.7}$$

Define

$$\mathcal{I} = \left\{ i \,:\, \left\| e^{it\Delta} f(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)} \leq 10cD^3 \left\| \chi_{O'_i} e^{it\Delta} f(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)} \right\}, \tag{3.8}$$

where c is the constant from (3.1). By triangle inequality and (3.1), for each $i \in \mathcal{I}^c$, we have

$$\left\| e^{it\Delta} f(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)} \leq \frac{10}{9} cD^3 \left\| \chi_{O_i \cap W} e^{it\Delta} f(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)}$$

$$\lesssim R^{3\epsilon^4} \left\| \chi_W e^{it\Delta} f(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)}.$$

So if $\mathcal{I}^c$ is non-empty, then (3.7) holds. For the moment, we are considering the case where (3.7) does not hold, and so every index i is in $\mathcal{I}$, and hence $|\mathcal{I}| \sim D^3$.

In addition, by Lemma 3.1,

$$\sum_i \|f_i\|_2^2 \lesssim (D+1) \sum_{\theta,\nu} \|f_{\theta,\nu}\|_2^2 \lesssim D\|f\|_2^2 \,. \tag{3.9}$$

Henceforth, by pigeonhole principle, there exists $i \in \mathcal{I}$ such that

$$\|f_i\|_2^2 \lesssim D^{-2} \|f\|_2^2 \,. \tag{3.10}$$

Now we use induction: we apply (1.11) to this special f_i at radius $\frac{R}{2}$. We can cover $B(0,R) \times [0,R]$ by $O(1)$ cylinders with dimensions $B(0,R/2) \times [0,R/2]$. Therefore, we get the bound

$$\left\| e^{it\Delta} f(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)} \lesssim D^3 \left\| \chi_{O'_i} e^{it\Delta} f(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)} \lesssim D^3 \left\| e^{it\Delta} f_i(x) \right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)}$$

$$\lesssim D^3 \left[C_{p,\epsilon} M^{-\epsilon^2} R^\epsilon \|f_i\|_2 \right]^p \lesssim D^{3-p} \left[C_{p,\epsilon} M^{-\epsilon^2} R^\epsilon \|f\|_2 \right]^p \,.$$

Recall that $D = R^{\epsilon^4}$, and we can assume R is very large (compared to p). Since $p > 3$ we have $D^{3-p} \ll 1$. Therefore, we see that induction closes (unless (3.7) holds).

It remains to prove the desired bounds when (3.7) holds – when the wall term is almost as big as the whole.

3.2 Contribution from the wall: transverse term

From Section 3.1, it remains to estimate the wall contribution, the second term in (3.6). To deal with the contribution from the wall W, we break B_R^* into $\sim R^{3\delta}$ balls B_j of radius $R^{1-\delta}$. (Recall from the last chapter that δ is defined to be ϵ^2.)

For any tile $(\theta, \nu) \in \mathbf{T}$, we say that $T_{\theta,\nu}$ is *tangent* to the wall W in a given ball B_j if it satisfies that $T_{\theta,\nu} \cap B_j \cap W \neq \emptyset$ and

$$\text{Angle}(G_0(\theta), T_z[Z(P)]) \leq R^{-1/2+2\delta} \tag{3.11}$$

for any non-singular point $z \in 10T_{\theta,\nu} \cap 2B_j \cap Z(P)$. Recall that $G_0(\theta) = (2c(\theta), 1)$ is the direction of the tube $T_{\theta,\nu}$. Here $T_z[Z(P)]$ stands for the tangent space to the variety $Z(P)$ at the point z, and by a non-singular point we mean a point z in $Z(P)$ with $\nabla P(z) \neq 0$. Since P is a product of distinct non-singular polynomials, the non-singular points are dense in $Z(P)$. We note that if $T_{\theta,\nu}$ is tangent to W in B_j, then $T_{\theta,\nu} \cap B_j$ is contained in the $R^{1/2+\delta}$-neighborhood of $Z(P) \cap 2B_j$.

We say that $T_{\theta,\nu}$ is *transverse* to the wall W in the ball B_j if it enjoys that $T_{\theta,\nu} \cap B_j \cap W \neq \emptyset$ and

$$\text{Angle}(G_0(\theta), T_z[Z(P)]) > R^{-1/2+2\delta} \tag{3.12}$$

for some non-singular point $z \in 10T_{\theta,\nu} \cap 2B_j \cap Z(P)$.

Let $\mathbf{T}_{j,\text{tang}}$ represent the collection of all tiles $(\theta, \nu) \in \mathbf{T}$ such that $T_{\theta,\nu}$'s are tangent to the wall W in B_j, and $\mathbf{T}_{j,\text{trans}}$ denote the collection of all tiles $(\theta, \nu) \in \mathbf{T}$ such that $T_{\theta,\nu}$'s are transverse to the wall W in B_j.

We define $f_{j,\text{tang}} := \sum_{(\theta,\nu)\in\mathbf{T}_{j,\text{tang}}} f_{\theta,\nu}$ and $f_{j,\text{trans}} := \sum_{(\theta,\nu)\in\mathbf{T}_{j,\text{trans}}} f_{\theta,\nu}$. Then on $B_j \cap W$, we have

$$e^{it\Delta} f(x) \sim e^{it\Delta} f_{j,\text{tang}}(x) + e^{it\Delta} f_{j,\text{trans}}(x)\,. \tag{3.13}$$

The following Lemma is about how a tube crosses a variety transversely, which was proved by Guth in [10]. It says that $T_{\theta,\nu}$ crosses the wall W transversely in at most $R^{O(\epsilon^4)}$ many balls B_j.

Lemma 3.2 *(Lemma 3.5 in [10]) For each tile $(\theta, \nu) \in \mathbf{T}$, the number of $R^{1-\delta}$-balls B_j for which $(\theta, \nu) \in \mathbf{T}_{j,\text{trans}}$ is at most Poly$(D) = R^{O(\epsilon^4)}$.*

For points $(x, t) \in B_j \cap W$, we could break up $e^{it\Delta} f(x)$ into a transverse term and a tangent term. However, when we analyze the tangent contribution in subsequent sections, we will need to use a bilinear structure. So we do a more refined decomposition: we break $e^{it\Delta}f(x)$ into a linear transverse term and a bilinear tangent term.

We decompose $B(\xi_0, M^{-1}) \subset \mathbb{R}^2$, the Fourier support of function f, into balls τ of radius $1/(KM)$. Here $K = K(\epsilon)$ is a large parameter. We write $f = \sum_\tau f_\tau$, where supp $\widehat{f_\tau} \subseteq \tau$.

We let $B_\epsilon := \{(x,t) \in B(0,R)\times[0,R] \,:\, \exists\, \tau \text{ s.t. } |e^{it\Delta}f_\tau(x)| > K^{-\epsilon^4}|e^{it\Delta}f(x)|\}$. We will show by induction on the radius $(1/M)$ in frequency space that the contribution from B_ϵ is acceptable. In fact, by the definition of B_ϵ,

$$\left\|\chi_{B_\epsilon} e^{it\Delta}f(x)\right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)} \le K^{\epsilon^4 p}\sum_\tau \left\|e^{it\Delta}f_\tau(x)\right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)}.$$

By applying (1.11) in Theorem 1.7 the right-hand side is bounded by

$$\lesssim K^{\epsilon^4 p}\sum_\tau \left[C_\epsilon (KM)^{-\epsilon^2} R^\epsilon \|f_\tau\|_2\right]^p$$

$$\le K^{(\epsilon^4-\epsilon^2)p}\left[C_\epsilon M^{-\epsilon^2} R^\epsilon \|f\|_2\right]^p$$

We choose $K = K(\epsilon)$ large so that $K^{(\epsilon^4-\epsilon^2)} \ll 1$, and the term involving B_ϵ plays an unimportant role by induction. So from now on, we can assume that the contribution of B_ϵ is negligible.

For points (x,t) not in B_ϵ, we have the following decomposition into a transverse term and a bilinear tangent term.

Lemma 3.3 *For each point $(x,t) \in B_j \cap W$ satisfying $\max_\tau |e^{it\Delta}f_\tau(x)| \le K^{-\epsilon^4}|e^{it\Delta}f(x)|$, there exists a sub-collection I of the collection of all possible $1/(KM)$ balls τ, such that*

$$|e^{it\Delta}f(x)| \lesssim |e^{it\Delta}f_{I,j,\text{trans}}(x)| + K^{10}\text{Bil}(e^{it\Delta}f_{j,\text{tang}}(x)), \tag{3.14}$$

where

$$f_{I,j,\text{trans}}(x) := \sum_{\tau\in I} f_{\tau,j,\text{trans}}(x),$$

and the bilinear tangent term is given by

$$\text{Bil}(e^{it\Delta}f_{j,\text{tang}}(x)) := \max_{\substack{\tau_1,\tau_2 \\ dist(\tau_1,\tau_2)\ge 1/(KM)}} |e^{it\Delta}f_{\tau_1,j,\text{tang}}(x)|^{1/2}|e^{it\Delta}f_{\tau_2,j,\text{tang}}(x)|^{1/2}. \tag{3.15}$$

Proof: Let I be defined by $I := \{\tau : |e^{it\Delta} f_{\tau,j,\mathrm{tang}}(x)| \le K^{-10}|e^{it\Delta} f(x)|\}$. Then clearly

$$I^c = \{\tau : |e^{it\Delta} f_{\tau,j,\mathrm{tang}}(x)| > K^{-10}|e^{it\Delta} f(x)|\}.$$

If there exist $\tau_1, \tau_2 \in I^c$ with $\mathrm{dist}(\tau_1,\tau_2) \ge 1/(KM)$, then $|e^{it\Delta} f(x)| \lesssim K^{10}\mathrm{Bil}(e^{it\Delta} f_{j,\mathrm{tang}}(x))$. Otherwise, the number of balls τ in I^c is $O(1)$, and

$$\sum_{\tau \in I^c} |e^{it\Delta} f_\tau(x)| \le CK^{-\epsilon^4}|e^{it\Delta} f(x)| \le \frac{1}{10}|e^{it\Delta} f(x)|.$$

Hence, by the fact that $f = \sum_\tau f_\tau$ and the definition of I,

$$\frac{9}{10}|e^{it\Delta} f(x)| \le |\sum_{\tau \in I} e^{it\Delta} f_\tau(x)|$$

$$\lesssim |e^{it\Delta} f_{I,j,\mathrm{tang}}(x)| + |e^{it\Delta} f_{I,j,\mathrm{trans}}(x)|$$

$$\le CK^{-8}|e^{it\Delta} f(x)| + |e^{it\Delta} f_{I,j,\mathrm{trans}}(x)|,$$

which implies that $|e^{it\Delta} f(x)| \lesssim |e^{it\Delta} f_{I,j,\mathrm{trans}}(x)|$.

■

By Lemma 3.3 we can now estimate the wall contribution in (3.6) by

$$\left\|\chi_W \, e^{it\Delta} f(x)\right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)}$$

$$\lesssim \sum_j \left\| \max_I \chi_{B_j \cap W} \, |e^{it\Delta} f_{I,j,\mathrm{trans}}(x)| \right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)} \tag{3.16}$$

$$+K^{10p} \sum_j \left\|\chi_{B_j \cap W} \mathrm{Bil}(e^{it\Delta} f_{j,\mathrm{tang}}(x))\right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)} . \tag{3.17}$$

We now estimate the linear transverse term (3.16). The term (3.16) is dominated by

$$\sum_j \sum_{I\subseteq\mathcal{T}} \left\|\chi_{B_j\cap W}\, e^{it\Delta} f_{I,j,\mathrm{trans}}(x)\right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)}, \tag{3.18}$$

where $\mathcal{T}$ is the collection of all possible $1/(KM)$-balls in $B(\xi_0, 1/M)$, and the sum is taken over all subsets of $\mathcal{T}$. Since there are at most 2^{K^2} I's, we apply (1.11) in Theorem 1.7 with radius $R^{1-\delta}$ to obtain

$$(3.18) \le \sum_j 2^{K^2} \left[C_\epsilon M^{-\epsilon^2} R^{(1-\delta)\epsilon} \|f_{j,\mathrm{trans}}\|_2\right]^p, \tag{3.19}$$

which is bounded by, using Lemma 3.2,

$$2^{K^2} R^{O(\epsilon^4)-\delta\epsilon p} \left[C_\epsilon M^{-\epsilon^2} R^{\epsilon} \|f\|_2\right]^p. \tag{3.20}$$

Since $\delta = \epsilon^2$, it is clear that $2^{K^2} R^{O(\epsilon^4)-\delta\epsilon p} < 1/100$ and so the induction on the transverse term closes.

It remains to estimate the bilinear tangent term (3.17). We state the Schrödinger maximal estimate for the bilinear tangent term in this section, and prove it in Subsection 3.3.4.

Proposition 3.4 *For $p > 3$, the following maximal estimate of the bilinear tangent term* (3.15) *holds, uniformly in M:*

$$\left(\int_{B(0,R)} \sup_{t:(x,t)\in W\cap B_j} \left|\mathrm{Bil}(e^{it\Delta} f_{j,\mathrm{tang}}(x))\right|^p dx\right)^{1/p} \le C_\epsilon R^{\epsilon/2} \|f\|_2. \tag{3.21}$$

Given Proposition 3.4, we estimate the bilinear tangent term (3.17) as follows, for any $q > 1/\epsilon^4$,

$$\left\|\chi_{B_j\cap W} \mathrm{Bil}(e^{it\Delta} f_{j,\mathrm{tang}}(x))\right\|^p_{L^p_x L^q_t(\mathbf{B}^*_R)}$$

$$\le R^{p/q} \int_{B(0,R)} \sup_{t:(x,t)\in W\cap B_j} \left|\mathrm{Bil}(e^{it\Delta} f_{j,\mathrm{tang}}(x))\right|^p dx$$

$$\le R^{O(\delta)+\epsilon p/2} \|f\|_2^p.$$

Hence Theorem 1.7 follows from Proposition 3.4 and the inductions.

3.3 Contribution from the wall: bilinear tangent term

In this section we obtain both linear and bilinear local refinements of the Strichartz inequality, via the Bourgain-Demeter l^2-decoupling theorem [4]. In subsection 3.3.4 we will use the bilinear refinement of Strichartz to prove the Schrödinger maximal estimate for the bilinear tangent term in Proposition 3.4.

3.3.1 Variations on the Strichartz inequality using decoupling

For the bilinear tangent term in Proposition 3.4, all wave packets are tangent to a variety. Suppose that $Z = Z(P)$ where P is a product of non-singular polynomials. For any tile $(\theta, \nu) \in \mathbf{T}$, we say that $T_{\theta,\nu}$ is $ER^{-1/2}$*-tangent* to Z if

$$T_{\theta,\nu} \subset N_{ER^{1/2}} Z \cap \mathbf{B}^*_R \,,$$

and

$$\text{Angle}(G_0(\theta), T_z[Z(P)]) \leq ER^{-1/2} \tag{3.22}$$

for any non-singular point $z \in N_{2ER^{1/2}}(T_{\theta,\nu}) \cap 2\mathbf{B}^*_R \cap Z$.

Let

$$\mathbf{T}_Z(E) := \{(\theta, \nu) \in \mathbf{T} : T_{\theta,\nu} \text{ is } ER^{-1/2}\text{-tangent to } Z\} \,,$$

and we say that f is concentrated in wave packets from $\mathbf{T}_Z(E)$ if

$$\sum_{(\theta,\nu)\notin \mathbf{T}_Z(E)} \|f_{\theta,\nu}\|_2 \leq \text{RapDec}(R)\|f\|_2.$$

Since the radius of $T_{\theta,\nu}$ is $R^{1/2+\delta}$, R^δ is the smallest interesting value of E.

We write $A \lessapprox B$ if $A \leq C_\epsilon R^\epsilon B$ for any $\epsilon > 0$. In this section, we establish the following local refinements of the Strichartz estimates.

Theorem 3.5 *Suppose that $f \in L^2(\mathbb{R}^2)$ has Fourier support in $B^2(0,1)$, and is concentrated in wave packets from $\mathbf{T}_Z(E)$, where $Z = Z(P)$ and P is a product of distinct non-singular polynomials. Suppose that $Q_1, Q_2, ...$ are lattice $R^{1/2}$-cubes in $B^3(R)$, so that*

$$\|e^{it\Delta}f\|_{L^6(Q_j)} \text{ is essentially constant in } j.$$

Suppose that these cubes are arranged in horizontal strips of the form $\mathbb{R}\times\mathbb{R}\times\{t_0, t_0+R^{1/2}\}$*, and that each strip contains* $\sim\sigma$ *cubes* Q_j*. Let* Y *denote* $\bigcup_j Q_j$*. Then*

$$\|e^{it\Delta}f\|_{L^6(Y)} \lessapprox E^{O(1)}R^{-1/6}\sigma^{-1/3}\|f\|_{L^2}. \tag{3.23}$$

To get some intuition, we consider a special case of Theorem 3.5, in which the variety Z is naturally replaced by a 2-plane V, and $E \approx 1$. In the planar case, the angle condition (3.22) restricts the support of $\widehat{f}$ to a $\approx R^{-1/2}$-strip and all wave packets are contained in the $\approx R^{1/2}$-neighborhood of V, so the absolute value $|e^{it\Delta}f(x)|$ is essentially constant along a transverse direction which is roughly normal to V. (It is not rigorous to treat $|e^{it\Delta}f(x)|$ essentially as constant along a transverse direction to V, but useful for intuition. In subsection 3.3.4 we will give a rigorous argument to deal with the issue of $|e^{it\Delta}f(x)|$ being morally roughly constant along a transverse direction.) By a direct computation, the absolute value of $e^{it\Delta}f(x)|_V$ is roughly equal to the absolute value of a Schrödinger solution in dimension 2, denoted by $e^{ir\Delta}h(y)$ for some function h with Fourier support in $B^1(0,1)$, where (y,r) are coordinates of V. Hence the conclusion in Theorem 3.5 can be rephrased in terms of h. Indeed, observe that

$$\|e^{it\Delta}f(x)\|_{L^6(Y)}^6 \sim R^{1/2}\|e^{ir\Delta}h(y)\|_{L^6(Y\cap V)}^6,$$

$$\|f\|_2^2 \sim R^{-1}\|e^{it\Delta}f\|_{L^2(B^3(R))}^2 \sim R^{-1}R^{1/2}\|e^{ir\Delta}h\|_{L^2(B^3_R\cap V)}^2 \sim R^{1/2}\|h\|_2^2.$$

Therefore the estimate (3.23) is equivalent to

$$\|e^{ir\Delta}h\|_{L^6(Y\cap V)} \lessapprox \sigma^{-1/3}\|h\|_{L^2}. \tag{3.24}$$

It follows from the Strichartz inequality that $\|e^{ir\Delta}h\|_{L^6(Y\cap V)} \lesssim \|h\|_{L^2}$. We get an improvement when σ is large. The condition that σ is large forces the solution $e^{it\Delta}f$ to be spread out in space, and we will exploit this spreading out to get our improvement.

Moreover, Theorem 3.5 has the following bilinear refinement.

Theorem 3.6 *For functions* f_1 *and* f_2 *in* $L^2(\mathbb{R}^2)$ *with separated Fourier supports in* $B^2(0,1)$*, separated by* ~ 1*, suppose that* f_1 *and* f_2 *are concentrated in wave packets from* $\mathbf{T}_Z(E)$*, where* $Z = Z(P)$ *and* P *is a* *product of distinct non-singular polynomials. Suppose that* $Q_1, Q_2, \cdots, Q_N$ *are lattice* $R^{1/2}$*-cubes in* $B^3(R)$*, so that for each* i*,*

$$\|e^{it\Delta} f_i\|_{L^6(Q_j)} \text{ is essentially constant in } j.$$

Let Y *denote* $\bigcup_{j=1}^N Q_j$*. Then*

$$\left\| |e^{it\Delta} f_1 e^{it\Delta} f_2|^{1/2} \right\|_{L^6(Y)} \lessapprox E^{O(1)} R^{-1/6} N^{-1/6} \|f_1\|_{L^2}^{1/2} \|f_2\|_{L^2}^{1/2}.$$

3.3.2 Proof of the linear refinement of Strichartz - Theorem 3.5

The proof uses the Bourgain-Demeter l^2-decoupling theorem, together with induction on the radius and parabolic rescaling. First we recall the decoupling result of Bourgain and Demeter in [4].

Theorem 3.7 (Bourgain-Demeter) *Suppose that the* R^{-1}*-neighborhood of the unit parabola in* $\mathbb{R}^2$ *is divided into* $R^{1/2}$ *disjoint rectangular boxes* τ*, each with dimensions* $R^{-1/2} \times R^{-1}$*. Suppose* $\widehat{F_\tau}$ *is supported in* τ *and* $F = \sum_\tau F_\tau$*. Then*

$$\|F\|_{L^6(\mathbb{R}^2)} \lessapprox \left(\sum_\tau \|F_\tau\|_{L^6(\mathbb{R}^2)}^2 \right)^{1/2}.$$

If $E \geq R^{1/4}$ (or any fixed power of R), then the estimate (3.23) is trivial because of the factor $E^{O(1)}$. So we assume that $E \leq R^{1/4}$.

To set up the argument, we decompose f as follows. We break the unit ball $B^2(1)$ in frequency space into small balls τ of radius $R^{-1/4}$, and divide the physical space ball $B^2(R)$ into balls B of radius $R^{3/4}$. For each pair (τ, B), we let $f_{\square_{\tau,B}}$ be the function formed by cutting off f on the ball B (with a Schwartz tail) in physical space and the ball τ in Fourier space. We note that $e^{it\Delta} f_{\square_{\tau,B}}$, restricted to $B^3(R)$, is essentially supported on an $R^{3/4} \times R^{3/4} \times R$-box, which we denote by $\square_{\tau,B}$ (compare the discussion in Section 2.1). The box $\square_{\tau,B}$ is in the direction given by $(2c(\tau), 1)$ and intersects $t = 0$ at a disk centered at $(c(B), 0)$, where $c(\tau)$ and $c(B)$ are the centers of τ and B respectively. For a fixed τ, the different boxes $\square_{\tau,B}$ tile $B^3(R)$. In particular, for each τ, a given cube Q_j lies in exactly one box $\square_{\tau,B}$.

Since f is concentrated in wave packets from $\mathbf{T}_Z(E)$, we only need to consider those $R^{1/2}$-cubes Q_j that are contained in the $ER^{1/2}$-neighborhood of Z. For each such $R^{1/2}$-cube Q_j, we will see that the wave packets that pass through Q_j are nearly coplanar. Because of this, we will be able to apply the 2-dimensional decoupling theorem to study $e^{it\Delta} f$ on Q_j:

Lemma 3.8 *Suppose that $f \in L^2(\mathbb{R}^2)$ has Fourier support in $B^2(0,1)$ and is concentrated in wave packets from $\mathbf{T}_Z(E)$, where $E \leq R^{1/4}$ and $Z = Z(P)$ is a finite union of non-singular varieties. Suppose that an $R^{1/2}$-cube Q is in $N_{ER^{1/2}}(Z)$. Then we have the decoupling bound*

$$\|e^{it\Delta}f\|_{L^6(Q)} \lessapprox \left(\sum_{\square} \|e^{it\Delta}f_\square\|^2_{L^6(10Q)}\right)^{1/2} + R^{-1000}\|f\|_{L^2}. \tag{3.25}$$

Remark: Recall that the parameter R is assumed to be sufficiently large. The $R^{-1000}\|f\|_{L^2}$ is a negligibly small term which covers minor contributions coming from the tails of the Fourier transforms of smooth functions. We will neglect this term in the sequel.

Proof: Observe that $Q \subset N_{ER^{1/2}}Z$ implies that there exists a non-singular point $z_0 \in Z \cap N_{ER^{1/2}}Q$. Thus for each wave packet $T_{\theta,\nu}$ that intersects Q, we have $z_0 \in Z \cap N_{2ER^{1/2}}(T_{\theta,\nu})$. By the definition of $\mathbf{T}_Z(E)$ we get the angle bound

$$\text{Angle}(G_0(\theta), T_{z_0}[Z(P)]) \leq ER^{-1/2}\,. \tag{3.26}$$

We recall from Section 2.1 that $G_0(\theta) = (2c(\theta), 1)$. Suppose that $T_{z_0}Z$ is the plane given by $a_1x_1 + a_2x_2 + bt = 0$, with $a_1^2 + a_2^2 + b^2 = 1$. The angle condition above restricts the location of θ as follows:

$$|2a \cdot c(\theta) + b| \lesssim ER^{-1/2}. \tag{3.27}$$

We note that each tube $T_{\theta,\nu}$ makes an angle $\gtrsim 1$ with the plane $t = 0$, because $\theta \subset B(0,1)$. We can assume that there are some tubes $T_{\theta,\nu}$ tangent to $T_{z_0}Z$, and so $|a| \gtrsim 1$. Therefore, (3.27) confines θ to a strip of width $\sim ER^{-1/2}$ inside of $B(0,1)$. We denote this strip by $S \subset B(0,1)$.

Let $\mathbf{T}_{Z,Q}(E)$ be the set of (θ,ν) in $\mathbf{T}_Z(E)$ for which each $T_{\theta,\nu}$ intersects Q. For each (θ,ν) in $\mathbf{T}_{Z,Q}(E)$, θ obeys (3.27), and so $\theta \subset S$. Let η be a smooth bump function which approximates χ_Q. We note that $\eta e^{it\Delta}f$ is essentially equal to

$$\sum_{(\theta,\nu)\in\mathbf{T}_{Z,Q}(E)} \eta e^{it\Delta}f_{\theta,\nu}.$$

Therefore, the Fourier transform of the localized solution $\eta e^{it\Delta}f$ is essentially supported in

$$S^* := \{(\xi_1,\xi_2,\xi_3) : (\xi_1,\xi_2) \in S \text{ and } |\xi_3 + \xi_1^2 + \xi_2^2| \lesssim R^{-1/2}\}. \tag{3.28}$$

(The contribution of the not essential parts is covered by the negligible term $R^{-1000}\|f\|_{L^2}$ in the statement of the Lemma.)

After a rotation in the (x_1, x_2)-plane we can suppose that the strip S is defined by

$$a_1 \leq \xi_1 \leq a_1 + ER^{-1/2},$$

for some $a_1 \in [-1, 1]$. We note that at each point $(\xi_1, \xi_2) \in S$,

$$\partial_1 \left(-\xi_1^2 - \xi_2^2\right) = -2a_1 + O(ER^{-1/2}). \tag{3.29}$$

Let v be the vector

$$v = (1, 0, -2a_1).$$

Let Π be a 2-plane perpendicular to v. Because $E \leq R^{1/4}$, we claim that the projection of S^* onto Π lies in the $\sim R^{-1/2}$-neighborhood of a parabola. We can see this as follows. Let

$$S^*_{core} := \{(\xi_1, \xi_2, \xi_3) : \xi_1 = a_1, |\xi_2| \leq 1, \xi_3 = -\xi_1^2 - \xi_2^2\}.$$

The set S^*_{core} is a parabola, and its projection onto Π is also a parabola. We claim that the projection of S^* to Π lies in the $\sim R^{-1/2}$-neighborhood of this parabola. If $(\xi_1, \xi_2, \xi_3) \in S^*$, then (3.29) tells us that

$$(-\xi_1^2 - \xi_2^2) = -a_1^2 - \xi_2^2 - 2a_1(\xi_1 - a_1) + O(ER^{-1/2} \cdot |\xi_1 - a_1|).$$

Therefore,

$$(\xi_1, \xi_2, \xi_3) = (a_1, \xi_2, -a_1^2 - \xi_2^2) + (\xi_1 - a_1)v + O(ER^{-1/2}|\xi_1 - a_1| + R^{-1/2}).$$

The first term on the right-hand side lies is S^*_{core}. Since Π is perpendicular to v, the projection to Π kills the second term on the right-hand side. So the distance from the projection of ξ to the projection of S^*_{core} is at most

$$ER^{-1/2}|\xi_1 - a_1| + R^{-1/2} \lesssim E^2 R^{-1} + R^{-1/2} \sim R^{-1/2}.$$

Therefore, if we restrict $\eta e^{it\Delta} f$ to Π, the resulting 2-dimensional function has Fourier support in the $\sim R^{-1/2}$-neighborhood of a parabola.

We consider the decomposition $f = \sum_{(\tau,B):\square_{\tau,B} \cap Q \neq \emptyset} f_{\square_{\tau,B}}$. If $e^{it\Delta} f_{\square_{\tau,B}}$ contributes to $\|e^{it\Delta} f\|_{L^6(Q)}$, there must be a wave packet $T_{\theta,\nu}$ that intersects the $R^{1/2}$-cube Q with $\theta \subset \tau$, and so $\tau \cap S$ must be non-empty. Also, for a given τ, there is only one B so that $\Box_{\tau,B} \cap Q$ is non-empty. Also, the Fourier support of $\eta e^{it\Delta} f_{\Box_{\tau,B}}$ lies in $S^* \cap (\tau \times \mathbb{R})$, by the same argument we used above for $\eta e^{it\Delta} f$. The projection onto Π of $S^* \cap (\tau \times \mathbb{R})$ is an $R^{-1/4} \times R^{-1/2}$ rectangular box. The union of these boxes over all τ intersecting S is the $R^{-1/2}$-neighborhood of a parabola. Therefore, we have the hypotheses to apply the 2-dimensional decoupling theorem, Theorem 3.7, which gives:

$$\|\eta e^{it\Delta} f\|_{L^6(\Pi)} \lessapprox \left(\sum_{\Box} \|\eta e^{it\Delta} f_\Box\|^2_{L^6(\Pi)} \right)^{1/2}.$$

Now we integrate in the direction perpendicular to Π and apply Fubini and Minkowski to get

$$\|\eta e^{it\Delta} f\|_{L^6(\mathbb{R}^3)} \lessapprox \left(\sum_{\Box} \|\eta e^{it\Delta} f_\Box\|^2_{L^6(\mathbb{R}^3)} \right)^{1/2}.$$

This implies the desired conclusion.

■

Next, by induction on the radius R, we will show that each function $f_\Box$ obeys a version of Theorem 3.5. Here is the statement. Suppose that $S_1, S_2, ...$ are $R^{1/2} \times R^{1/2} \times R^{3/4}$-tubes in $\Box$ (running parallel to the long axis of $\Box$), and that

$$\|e^{it\Delta} f_\Box\|_{L^6(S_j)} \text{ is essentially constant in } j.$$

Suppose that these tubes are arranged into $R^{3/4}$-strips running parallel to the short axes of $\Box$ and that each such strip contains $\sim \sigma_\Box$ tubes S_j. Let $Y_\Box$ denote $\cup_j S_j$. Then

$$\|e^{it\Delta} f_\Box\|_{L^6(Y_\Box)} \lessapprox E^{O(1)} R^{-1/12} R^{-1/12} \sigma_\Box^{-1/3} \|f_\Box\|_{L^2}. \tag{3.30}$$

This inequality follows by doing a parabolic rescaling and then using Theorem 3.5 at scale $R^{1/2}$, which we can assume holds by induction on R. We write down the details of this parabolic rescaling, and in particular we will check that the tangent-to-variety condition is preserved under parabolic rescaling. For each $R^{-1/4}$-ball τ in $B^2(1)$, we write $\xi = \xi_0 + R^{-1/4}\zeta \in \tau$, then

$$|e^{it\Delta} f_\tau(x)| = R^{-1/4} |e^{i\tilde{t}\Delta} g(\tilde{x})|$$

for some function g with Fourier support in $B^2(1)$ and $\|g\|_2 = \|f_\tau\|_2$, where the new coordinates $(\tilde{x}, \tilde{t})$ are related to the old coordinates (x, t) by

$$\begin{cases} \tilde{x} = R^{-1/4}x - 2tR^{-1/4}\xi_0\,, \\ \tilde{t} = R^{-1/2}t\,. \end{cases} \tag{3.31}$$

Therefore

$$\|e^{it\Delta} f_\Box(x)\|_{L^6(Y_\Box)} = R^{-1/12}\|e^{i\tilde{t}\Delta} g(\tilde{x})\|_{L^6(\tilde{Y})},$$

where $\tilde{Y}$ is the image of $Y_\Box$ under the new coordinates. Note that $\tilde{Y}$ is a union of $R^{1/4}$-cubes inside an $R^{1/2}$-cube. These $R^{1/4}$-cubes are arranged in $R^{1/4}$-horizontal strips, and each strip contains $\sim \sigma_\Box$ $R^{1/4}$-cubes. Moreover, by the relation (3.31), we see that each wave packet T, at scale R, of dimensions $R^{1/2+\delta} \times R^{1/2+\delta} \times R$ in the old coordinates is mapped to a corresponding wave packet $\tilde{T}$, at scale $R^{1/2}$, of dimensions $R^{1/4+\delta} \times R^{1/4+\delta} \times R^{1/2}$ in the new coordinates. The variety $Z(P)$ corresponds to a new variety $Z(Q)$, given by the relation $Q(\tilde{x}, \tilde{t}) = Q(R^{-1/4}x - 2tR^{-1/4}\xi_0, R^{-1/2}t) = P(x, t)$. We claim that, under the above correspondence, if the wave packet T at scale R is $ER^{-1/2}$-tangent to $Z(P)$, then the wave packet $\tilde{T}$ at scale $R^{1/2}$ is $ER^{-1/4}$-tangent to $Z(Q)$ in the new coordinates.

By the relation (3.31), the distance condition $T \subset N_{ER^{1/2}}Z(P)$ implies that $\tilde{T} \subset N_{ER^{1/4}}Z(Q)$. Given the direction $(2\xi, 1)$ of T, the angle condition

$$\mathrm{Angle}((2\xi, 1), T_{z_0}[Z(P)]) \leq ER^{-1/2}$$

is equivalent to

$$\frac{|(2\xi, 1) \cdot (P_x(x_0, t_0), P_t(x_0, t_0))|}{|(P_x(x_0, t_0), P_t(x_0, t_0))|} \lesssim ER^{-1/2}, \tag{3.32}$$

where $z_0 = (x_0, t_0)$. Note that the direction of the corresponding wave packet $\tilde{T}$ is given by $(2\zeta, 1)$, where ξ and ζ are related by $\xi = \xi_0 + R^{-1/4}\zeta$. Let $\tilde{z}_0 = (\tilde{x}_0, \tilde{t}_0)$ denote the point corresponding to z_0. Using the relations

$$P_x = R^{-1/4}Q_{\tilde{x}}, \quad P_t = -2R^{-1/4}\xi_0 \cdot Q_{\tilde{x}} + R^{-1/2}Q_{\tilde{t}}\,,$$

after some computation, (3.32) yields that

$$\frac{|(2\zeta, 1) \cdot (Q_{\tilde{x}}(\tilde{x}_0, \tilde{t}_0), Q_{\tilde{t}}(\tilde{x}_0, \tilde{t}_0))|}{|(Q_{\tilde{x}}(\tilde{x}_0, \tilde{t}_0), Q_{\tilde{t}}(\tilde{x}_0, \tilde{t}_0))|} \lesssim ER^{-1/4},$$

which implies that

$$\text{Angle}((2\zeta,1),\tilde{T}_{\tilde{z}_0}[Z(Q)])\le ER^{-1/4}.$$

Therefore the tangent-to-variety condition is preserved under parabolic rescaling and the induction on radius is justified.

We have now established inequality (3.30). To apply this inequality, we need to identify a good choice of $Y_{\square}$. We do this by some dyadic pigeonholing. For each $\square$, we apply the following algorithm to regroup tubes in $\square$.

1. We sort those $R^{1/2}\times R^{1/2}\times R^{3/4}$-tubes S's contained in the box $\square$ according to the order of magnitude of $\|e^{it\Delta}f_{\square}\|_{L^6(S)}$, which we denote λ. For each dyadic number λ, we use $\mathbb{S}_\lambda$ to stand for the collection of tubes $S\subset\square$ with $\|e^{it\Delta}f_{\square}\|_{L^6(S)}\sim\lambda$.

2. For each λ, we sort the tubes $S\in\mathbb{S}_\lambda$ by looking at the number of such tubes in an $R^{3/4}$-strip. For any dyadic number η, we let $\mathbb{S}_{\lambda,\eta}$ be the set of tubes $S\in\mathbb{S}_\lambda$ so that the number of tubes of $\mathbb{S}_\lambda$ in the $R^{3/4}$-strip containing S is $\sim\eta$.

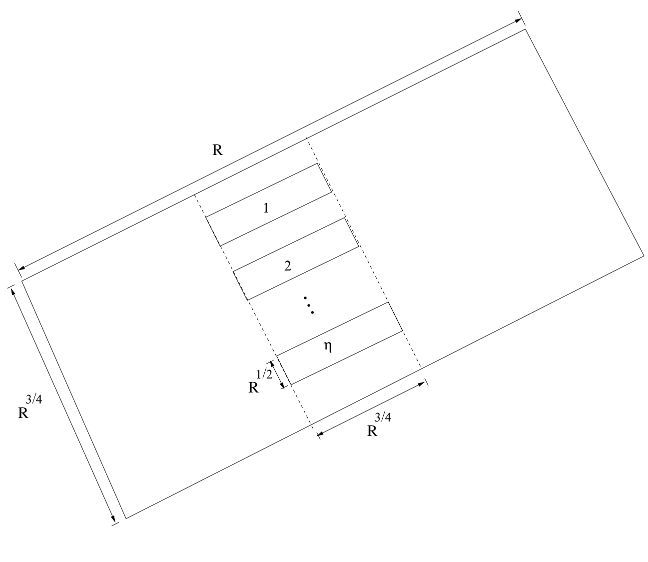

Figure 3.1: Tubes in a given strip in the $\square$

Let $Y_{\square,\lambda,\eta}$ be the union of the tubes in $\mathbb{S}_{\lambda,\eta}$. Then we represent

$$e^{it\Delta}f=\sum_{\lambda,\eta}\left(\sum_{\square}e^{it\Delta}f_{\square}\cdot\chi_{Y_{\square,\lambda,\eta}}\right).$$

Since there are $O(\log R)$ choices for each of λ,η, by pigeonholing, we can choose λ,η so that

$$\|e^{it\Delta}f\|_{L^6(Q_j)} \lesssim (\log R)^2 \Big\| \sum_{\Box} e^{it\Delta} f_\Box \cdot \chi_{Y_{\Box,\lambda,\eta}} \Big\|_{L^6(Q_j)} \tag{3.33}$$

holds for a fraction ≈ 1 of all cubes Q_j in Y. We need this uniform choice of (λ, η), which is independent of Q_j, because later we will sum over all Q_j and arrive at $\|e^{it\Delta}f_\Box\|_{L^6(Y_{\Box,\lambda,\eta})}$.

We fix λ and η for the rest of the proof. Let $Y_\Box$ stand for the abbreviation of $Y_{\Box,\lambda,\eta}$. We note that $Y_\Box$ obeys the hypotheses for our inductive estimate (3.30), with $\sigma_\Box$ being the value of η that we have fixed.

The following geometric estimate will play a crucial role in our proof. Each set $Y_\Box$ contains $\lesssim \sigma_\Box$ tubes in each strip parallel to the short axes of $\Box$. Since the angle between the short axes of $\Box$ and the x-axes is bounded away from $\pi/2$, it follows that $Y_\Box$ contains $\lesssim \sigma_\Box$ cubes Q_j in any $R^{1/2}$-horizontal row. Therefore,

$$|Y_\Box \cap Y| \lesssim \frac{\sigma_\Box}{\sigma}|Y|. \tag{3.34}$$

Next we sort the the boxes $\Box$ according to the dyadic size of $\|f_\Box\|_{L^2}$. We can restrict matters to $\lesssim \log R$ choices of this dyadic size, and so we can choose a set of $\Box$'s, $\mathbb{B}$, so that $\|f_\Box\|_{L^2}$ is essentially constant for $\Box \in \mathbb{B}$ and

$$\|e^{it\Delta}f\|_{L^6(Q_j)} \lessapprox \| \sum_{\Box\in\mathbb{B}} e^{it\Delta} f_\Box \cdot \chi_{Y_\Box} \|_{L^6(Q_j)} \tag{3.35}$$

for a fraction ≈ 1 of cubes Q_j in Y.

Finally we sort the cubes $Q_j \subset Y$ according to the number of $Y_\Box$ that contain them. We let $Y' \subset Y$ be a set of cubes Q_j which obey (3.35) and which each lies in $\sim \mu$ of the sets $\{Y_\Box\}_{\Box\in\mathbb{B}}$. Because (3.35) holds for a large fraction of cubes, and because there are only dyadically many choices of μ, $|Y'| \approx |Y|$. By the equation (3.34), we see that

$$|Y_\Box \cap Y'| \le |Y_\Box \cap Y| \lessapprox \frac{\sigma_\Box}{\sigma}|Y| \approx \frac{\sigma_\Box}{\sigma}|Y'|.$$

Therefore, the multiplicity μ is bounded by

$$\mu \lessapprox \frac{\sigma_\Box}{\sigma}|\mathbb{B}|. \tag{3.36}$$

We now are ready to combine all our ingredients and finish our proof. For each $Q_j \subset Y'$, we have

$$\|e^{it\Delta}f\|_{L^6(Q_j)} \lessapprox \left\| \sum_{\square\in\mathbb{B}} e^{it\Delta}f_\square \cdot \chi_{Y_\square} \right\|_{L^6(Q_j)} .$$

Now we apply Lemma 3.8 to the function $\sum_{\square\in\mathbb{B},Q_j\subset Y_\square} f_\square$ to bound the right hand side by

$$\lessapprox \left(\sum_{\square\in\mathbb{B},Q_j\subset Y_\square} \left\|e^{it\Delta}f_\square\right\|^2_{L^6(Q_j)} \right)^{1/2} .$$

Since the number of $Y_\square$ containing Q_j is $\sim \mu$, we can apply Hölder to get

$$\left\| \sum_{\square\in\mathbb{B}} e^{it\Delta}f_\square \cdot \chi_{Y_\square} \right\|_{L^6(Q_j)} \lessapprox \mu^{1/3} \left(\sum_{\square\in\mathbb{B},Q_j\subset Y_\square} \left\|e^{it\Delta}f_\square\right\|^6_{L^6(Q_j)} \right)^{1/6} .$$

Now we raise to the sixth power and sum over $Q_j \subset Y'$ to get

$$\left\|e^{it\Delta}f\right\|^6_{L^6(Y')} \lessapprox \mu^2 \sum_{\square\in\mathbb{B}} \left\|e^{it\Delta}f_\square\right\|^6_{L^6(Y_\square)} .$$

Since $|Y'| \gtrapprox |Y|$, and since each cube $Q_j \subset Y$ makes an equal contribution to $\|e^{it\Delta}f\|_{L^6(Y)}$, we see that $\|e^{it\Delta}f\|_{L^6(Y)} \approx \|e^{it\Delta}f\|_{L^6(Y')}$ and so

$$\left\|e^{it\Delta}f\right\|^6_{L^6(Y)} \lessapprox \mu^2 \sum_{\square\in\mathbb{B}} \left\|e^{it\Delta}f_\square\right\|^6_{L^6(Y_\square)} .$$

By a parabolic rescaling, Figure 3.1 becomes Figure 3.2. Henceforth, applying our inductive hypothesis

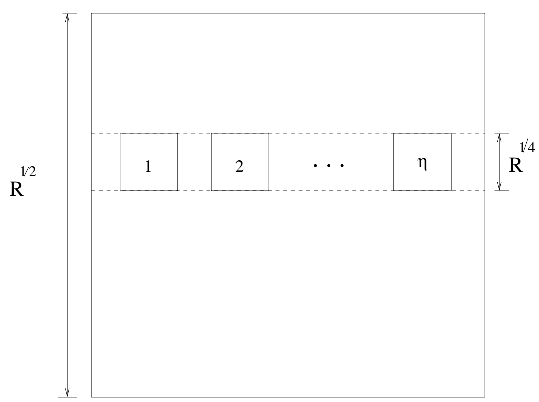

Figure 3.2: Cubes in a given strip in an $R^{1/2}$-cube

(3.30) at scale $R^{1/2}$ to the right-hand side, we see that

$$\left\|e^{it\Delta}f\right\|_{L^6(Y)}^6 \lessapprox E^{O(1)}R^{-1}\mu^2\sigma_\Box^{-2}\sum_{\Box\in\mathbb{B}}\|f_\Box\|_{L^2}^6. \tag{3.37}$$

Plugging in our bound for μ in (3.36), this is bounded by

$$\lesssim E^{O(1)}R^{-1}\sigma^{-2}|\mathbb{B}|^2\sum_{\Box\in\mathbb{B}}\|f_\Box\|_{L^2}^6.$$

Now since $\|f_\Box\|_{L^2}$ is essentially constant among all $\Box\in\mathbb{B}$, the last expression is

$$\sim E^{O(1)}R^{-1}\sigma^{-2}(\sum_{\Box\in\mathbb{B}}\|f_\Box\|_{L^2}^2)^3 \le E^{O(1)}R^{-1}\sigma^{-2}\|f\|_{L^2}^6.$$

Taking the sixth root, we obtain our desired bound:

$$\|e^{it\Delta}f\|_{L^6(Y)} \lessapprox E^{O(1)}R^{-1/6}\sigma^{-1/3}\|f\|_{L^2}.$$

This closes the induction on radius and completes the proof.

3.3.3 Proof of the bilinear refinement of Strichartz - Theorem 3.6

It can be proved by the method used in the proof of Theorem 3.5. By Hölder,

$$\left\|\left|e^{it\Delta}f_1e^{it\Delta}f_2\right|^{1/2}\right\|_{L^6(Y)} \le \prod_{i=1}^2\left\|e^{it\Delta}f_i\right\|_{L^6(Y)}^{1/2}.$$

For each i, we process $\|e^{it\Delta}f_i\|_{L^6(Y)}$ following the proof of Theorem 3.5. We decompose $f_i=\sum_\Box f_{i,\Box}$, and we follow the proof of Theorem 3.5. We define $Y_{i,\Box}$ by dyadic pigeonholing, so that $Y_{i,\Box}$ is arranged in several $R^{3/4}$-strips (running parallel to the short axes of $\Box$) with $\sim\sigma_{i,\Box}$ $R^{1/2}\times R^{1/2}\times R^{3/4}$-tubes in each strip. When we use dyadic pigeonholing to pick a subset of cubes $Q_j\subset Y$, we pigeonhole for f_1 and f_2 simultaneously, and so we pick out a set of cubes that works well for both functions. Following the argument up to Equation (3.35), we see that for a fraction ≈ 1 of cubes Q_j,

$$\|e^{it\Delta}f_i\|_{L^6(Q_j)} \lessapprox \|\sum_{\Box\in\mathbb{B}_i}e^{it\Delta}f_{i,\Box}\cdot\chi_{Y_{i,\Box}}\|_{L^6(Q_j)} \text{ for } i=1,2. \tag{3.38}$$

Similarly, we sort the cubes $Q_j\subset Y$ according to the number of $Y_{i,\Box}$ that contain them. We let $Y'\subset Y$ be a set of cubes Q_j which obey (3.38) and which each lies in $\sim\mu_1$ of the sets $\{Y_{1,\Box}\}_{\Box\in\mathbb{B}_1}$ and $\sim\mu_2$ of the sets $\{Y_{2,\Box}\}_{\Box\in\mathbb{B}_2}$. Because (3.38) holds for a large fraction of cubes, and because there are only dyadically many choices of μ_1, μ_2, $|Y'| \approx |Y|$. Following the proof of Theorem 3.5 further, up to Equation (3.37), we see that for each i,

$$\left\|e^{it\Delta} f_i\right\|_{L^6(Y)} \lessapprox E^{O(1)} R^{-1/6} \left[\mu_i^2 \sigma_{i,\square}^{-2} \sum_{\square \in \mathbb{B}_i} \|f_{i,\square}\|_{L^2}^6\right]^{1/6}. \tag{3.39}$$

Finally, we give a geometric estimate for μ_1 and μ_2 that takes advantage of the bilinear structure. If $\square_1 \in \mathbb{B}_1$ and $\square_2 \in \mathbb{B}_2$, then the angle between their long axes is ~ 1. Therefore, their intersection is contained in a ball of radius $\sim R^{3/4}$, and so $Y_{\square_1} \cap Y_{\square_2}$ contains $\lessapprox \sigma_{1,\square}\sigma_{2,\square}$ different $R^{1/2}$-balls (see Figure 3.3).

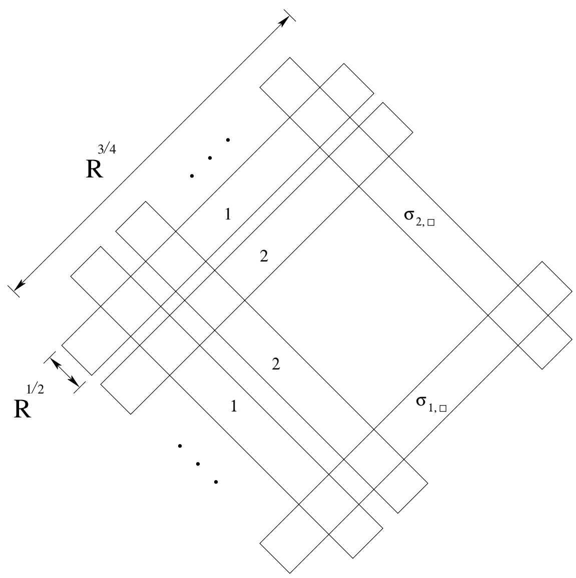

Figure 3.3: at most $O(\sigma_{1,\square}\sigma_{2,\square})$ cubes created by two transversal families of rectangular boxes

For each of the $\approx N$ cubes Q_j in Y', for each i, the cube Q_j lies in $\sim \mu_i$ of the sets $\{Y_{\square_i}\}_{\square_i \in \mathbb{B}_i}$. Therefore,

$$N \prod_{i=1}^{2} \mu_i \lessapprox \prod_{i=1}^{2} \sigma_{i,\square} |\mathbb{B}_i|. \tag{3.40}$$

Starting with (3.39) and inserting this estimate, we see that

$$\prod_{i=1}^{2} \left\|e^{it\Delta} f_i\right\|_{L^6(Y)}^{1/2} \lessapprox E^{O(1)} R^{-1/6} \prod_{i=1}^{2} \left[\mu_i^2 \sigma_{i,\square}^{-2} \sum_{\square \in \mathbb{B}_i} \|f_{i,\square}\|_{L^2}^6\right]^{\frac{1}{6}\cdot\frac{1}{2}}$$

$$\lessapprox E^{O(1)} R^{-1/6} \prod_{i=1}^{2} \left[N^{-1} |\mathbb{B}_i|^2 \sum_{\square \in \mathbb{B}_i} \|f_{i,\square}\|_{L^2}^6\right]^{\frac{1}{6}\cdot\frac{1}{2}} \lesssim E^{O(1)} R^{-1/6} N^{-1/6} \prod_{i=1}^{2} \|f_i\|_{L^2}^{1/2},$$

as desired.

3.3.4 Schrödinger maximal estimate for the bilinear tangent term

In this subsection, using the bilinear refinement of Strichartz in Theorem 3.6 and parabolic rescaling, we prove the following proposition, which implies Proposition 3.4.

Proposition 3.9 *Suppose that $\xi_0 \in B^2(0,1)$ and that f_1 and f_2 in $L^2(\mathbb{R}^2)$ have Fourier supports in $B(\xi_0, 1/M)$ for some $M \geq 1$. Also suppose that the Fourier supports of f_1 and f_2 are separated by at least $1/(KM)$, where $K = K(\epsilon)$ is a large constant. Suppose that each f_i is concentrated in wave packets from $\mathbf{T}_Z(E)$, where $E \geq R^\delta$ and $Z = Z(P)$ and P is a product of distinct non-singular polynomials. Then*

$$\left\| |e^{it\Delta} f_1|^{1/2} |e^{it\Delta} f_2|^{1/2} \right\|_{L^3_x(B_R) L^\infty_t(0,R)} \lessapprox E^{O(1)} \|f_1\|_{L^2}^{1/2} \|f_2\|_{L^2}^{1/2}. \tag{3.41}$$

Proof: We can assume $M \ll R^{1/2}$, otherwise all wave packets were in the same direction and a direct computation would give us the desired result.

Since f is concentrated in wave packets from $\mathbf{T}_Z(E)$, we decompose $N_{ER^{1/2}}Z$ into balls Q of radius $R^{1/2}$. Let η be a smooth bump function approximating χ_Q. As we saw in the proof of Lemma 3.8, in Equation (3.28), the Fourier support of each function $\eta e^{it\Delta} f_i$ is essentially supported on

$$S^* := \{(\xi_1, \xi_2, \xi_3) : (\xi_1, \xi_2) \in S \text{ and } |\xi_3 + \xi_1^2 + \xi_2^2| \lesssim R^{-1/2}\},$$

where $S \subset B(0,1)$ is a strip of width $ER^{-1/2}$. Since the Fourier support of each f_i is also contained in $B(\xi_0, 1/M)$, the Fourier support of $\eta e^{it\Delta} f_i$ is also essentially contained in $B(\xi_0, \frac{2}{M}) \times \mathbb{R}$. The intersection of S^* with the cylinder $B(\xi_0, \frac{2}{M}) \times \mathbb{R}$ is contained in a rectangle of dimensions $\sim ER^{-1/2} \times 1/M \times 1/M$. We denote this rectangle by $A^*(Q)$. Since the Fourier support of each $\eta e^{it\Delta} f_i$ is contained in $A^*(Q)$, $|\eta e^{it\Delta} f_i|$ is morally constant on dual rectangles with dimensions $M \times M \times E^{-1} R^{1/2}$. We tile Q with such dual rectangles, which we denote $A_k(Q)$. The projection of each dual rectangle $A_k(Q)$ to the x-plane is an $M \times E^{-1} R^{1/2}$-rectangle.

Suppose that $\sup_{0<t\leq R} |e^{it\Delta} f_1 e^{it\Delta} f_2|^{1/2} \sim H$ on a set $U \subset B(0,R)$. It suffices for us to prove the bound

$$H|U|^{1/3} \lessapprox E^{O(1)} \|f_1\|_{L^2}^{1/2} \|f_2\|_{L^2}^{1/2}. \tag{3.42}$$

We will bound $|U|$ using the rectangles $A_k(Q)$. For the time being, let us suppose that $|\eta e^{it\Delta} f_i|$ is roughly constant on each $A_k(Q)$. This is not quite rigorous, but useful for intuition. On the next page, we will come back to this point and give a rigorous argument.

There must be a collection of dual rectangles $A_k(Q_j)$ whose projections cover U and so that

$$|e^{it\Delta}f_1 e^{it\Delta}f_2|^{1/2} \sim H$$

on each dual rectangle. We let X denote the union of these dual rectangles. Each $M \times M \times E^{-1}R^{1/2}$ rectangle $A_k(Q_j) \subset X$ has a projection with area $ME^{-1}R^{1/2}$, and since these projections cover U, we have the bound

$$|U| \lesssim M^{-1}|X|. \tag{3.43}$$

We can also assume that no two rectangles $A_k(Q_j) \subset X$ have essentially the same projection. This implies that X contains $\lesssim E^{O(1)}R^{1/2}M^{-1}$ rectangles $A_k(Q)$ in each cube Q. So for each cube Q, we get the bound

$$|X \cap Q| \lesssim E^{O(1)}MR. \tag{3.44}$$

We consider the $R^{1/2}$-cubes Q in $B^2(R) \times [0, R]$ that intersect X. We sort these $R^{1/2}$-cubes Q according to the dyadic value of $\left\| |e^{it\Delta}f_1|^{1/2}|e^{it\Delta}f_2|^{1/2} \right\|_{L^6(Q)}$. We can choose a set of of $R^{1/2}$-cubes Q_j, $j = 1, 2, \cdots, N$, so that

$$\left\| |e^{it\Delta}f_1|^{1/2}|e^{it\Delta}f_2|^{1/2} \right\|_{L^6(Q_j)} \text{ is essentially constant in } j, \tag{3.45}$$

and $|X| \lessapprox |X \cap Y|$, where $Y := \bigcup_{j=1}^{N} Q_j$. Using the locally constant property that $|e^{it\Delta}f_1 e^{it\Delta}f_2|^{1/2} \sim H$ on each rectangle $A_k(Q_j) \subset X$, we see that

$$H|X|^{1/6} \lessapprox E^{O(1)} \left\| |e^{it\Delta}f_1|^{1/2}|e^{it\Delta}f_2|^{1/2} \right\|_{L^6(Y)}. \tag{3.46}$$

Since $|X \cap Q_j| \lesssim E^{O(1)}MR$ for each cube Q_j, $j = 1, ...N$, we see that $|X| \lessapprox |X \cap Y| \lesssim E^{O(1)}MNR$. Therefore,

$$H|X|^{1/3} \lessapprox E^{O(1)}M^{1/6}N^{1/6}R^{1/6} \left\| |e^{it\Delta}f_1|^{1/2}|e^{it\Delta}f_2|^{1/2} \right\|_{L^6(Y)}. \tag{3.47}$$

Finally, since $|U| \lesssim M^{-1}|X|$, we have

$$H|U|^{1/3} \lessapprox E^{O(1)}M^{-1/6}N^{1/6}R^{1/6} \left\| |e^{it\Delta}f_1|^{1/2}|e^{it\Delta}f_2|^{1/2} \right\|_{L^6(Y)}. \tag{3.48}$$

Therefore, our desired bound (3.42) follows from a generalization of Theorem 3.6, which we now state.

Proposition 3.10 *Suppose that f_1 and f_2 are as in Proposition 3.9. Suppose that $Q_1, Q_2, \cdots, Q_N$ are lattice $R^{1/2}$-cubes in $B^3(R)$ so that*

$$\left\| |e^{it\Delta} f_1|^{1/2} |e^{it\Delta} f_2|^{1/2} \right\|_{L^6(Q_j)} \text{ is essentially constant in } j. \tag{3.49}$$

Let Y denote $\bigcup_{j=1}^N Q_j$. Then

$$\left\| |e^{it\Delta} f_1|^{1/2} |e^{it\Delta} f_2|^{1/2} \right\|_{L^6(Y)} \lessapprox E^{O(1)} M^{1/6} N^{-1/6} R^{-1/6} \|f_1\|_{L^2}^{1/2} \|f_2\|_{L^2}^{1/2}. \tag{3.50}$$

If $M = 1$, then f_1 and f_2 have Fourier supports separated by ~ 1, and we can apply Theorem 3.6. We first find $Y' \subset Y$ with $|Y'| \approx |Y|$ so that for each i, $\|e^{it\Delta} f_i\|_{L^6(Q_j)}$ is essentially constant among all $Q_j \subset Y'$. Then we apply Theorem 3.6 to Y' to get (3.50):

$$\left\| |e^{it\Delta} f_1|^{1/2} |e^{it\Delta} f_2|^{1/2} \right\|_{L^6(Y)} \approx$$

$$\approx \left\| |e^{it\Delta} f_1|^{1/2} |e^{it\Delta} f_2|^{1/2} \right\|_{L^6(Y')} \lessapprox E^{O(1)} N^{-1/6} R^{-1/6} \|f_1\|_{L^2}^{1/2} \|f_2\|_{L^2}^{1/2}.$$

For larger M, the Fourier supports of f_1 and f_2 are only separated by $\sim 1/M$, and so we will need to apply parabolic rescaling before we can use Theorem 3.6.

Before we do this parabolic rescaling and prove Proposition 3.10, let us return to the issue of $|e^{it\Delta} f_i|$ being morally roughly constant on each rectangle $A_k(Q)$. We used the locally constant property to justify (3.46) above. We can rigorously prove (3.46) as follows. We mentioned above that each function $\eta_Q e^{it\Delta} f_i$ has Fourier transform essentially supported in a rectangle $A^*(Q)$ of dimensions $\sim ER^{-1/2} \times M^{-1} \times M^{-1}$. So the Fourier transform of their product, $g := \eta_Q^2 e^{it\Delta} f_1 e^{it\Delta} f_2$, is essentially supported in a rectangle with the same orientation and roughly the same dimensions. If $\hat{\psi}$ is designed to be identically 1 on this rectangle, then $g * \psi$ is essentially equal to g. We can choose such a ψ where $|\psi|$ is a rapidly-decaying approximation of $|A_k(Q_j)|^{-1} \chi_{A_k(Q_j)}$. Therefore, we see that

$$\sup_{A_k(Q)} |e^{it\Delta} f_1 e^{it\Delta} f_2| \lesssim R^{O(\delta)} \frac{\int_{R^\delta A_k(Q)} |e^{it\Delta} f_1 e^{it\Delta} f_2|}{|A_k(Q_j)|} + R^{-1000} \|f_1\|_{L^2} \|f_2\|_{L^2}, \tag{3.51}$$

where the second term accounts for the tail of ψ. Since $E \geq R^\delta$, we can assume that $R^\delta A_k(Q) \subset Q$.

We let X be a union of rectangles $A_k(Q_j)$ which each obeys

$$H \lesssim \sup_{A_k(Q_j)} |e^{it\Delta}f_1 e^{it\Delta}f_2|^{1/2}.$$

We can arrange that the projections of $10A_k(Q_j)$ cover U and also that any two rectangles $A_k(Q_j)$ in X have essentially different projections. Because of this covering, we still have $|U| \lesssim M^{-1}|X|$. Now if $H \lesssim R^{-100}\|f_1\|_{L^2}^{1/2}\|f_2\|_{L^2}^{1/2}$, then (3.42) follows trivially. Therefore, (3.51) tells us that for each $A_k(Q_j) \subset X$:

$$\int_{R^\delta A_k(Q)} |e^{it\Delta}f_1 e^{it\Delta}f_2| \gtrsim R^{-O(\delta)}|A_k(Q_j)|H^2.$$

We define Y just as above, and this inequality lets us rigorously justify (3.46):

$$H|X|^{1/6} \approx H|X \cap Y|^{1/6} \lessapprox E^{O(1)} \left\| |e^{it\Delta}f_1|^{1/2}|e^{it\Delta}f_2|^{1/2} \right\|_{L^6(Y)}.$$

It only remains to prove Proposition 3.10.

Proof: For function $f \in L^2$ with Fourier support in $B(\xi_0, 1/M)$, by parabolic rescaling, we have

$$\|e^{it\Delta}f(x)\|_{L^p(B^3(R))} \sim M^{\frac{4}{p}-1}\|e^{ir\Delta}\tilde{f}(y)\|_{L^p(B_{R/M}\times I_{R/M^2})}, \tag{3.52}$$

where $\tilde{f}$ has Fourier support in $B^2(0,1)$, $\|\tilde{f}\|_2 = \|f\|_2$, the new coordinates (y,r) and old coordinates (x,t) are related by

$$\begin{cases} y = x/M - 2t\xi_0/M, \\ r = t/M^2, \end{cases}$$

and $B_{R/M} \times I_{R/M^2}$ is a box of dimension $\sim \frac{R}{M} \times \frac{R}{M} \times \frac{R}{M^2}$, which is the range for (y,r) under the change of variables as above. By (3.52), we have

$$\left\| |e^{it\Delta}f_1|^{1/2}|e^{it\Delta}f_2|^{1/2} \right\|_{L^6(Y)} \sim M^{-1/3} \left\| |e^{ir\Delta}\tilde{f}_1|^{1/2}|e^{ir\Delta}\tilde{f}_2|^{1/2} \right\|_{L^6(\tilde{Y})}, \tag{3.53}$$

where $\tilde{f}_1, \tilde{f}_2$ have $1/K$-separated Fourier supports in $B^2(0,1)$, and $\tilde{Y}$ is a union of N $\frac{\sqrt{R}}{M} \times \frac{\sqrt{R}}{M} \times \frac{\sqrt{R}}{M^2}$-boxes in $B_{R/M} \times I_{R/M^2}$, in correspondence to Y under the change of variables as above.

To use Theorem 3.6 to estimate $\left\| |e^{ir\Delta}\tilde{f}_1|^{1/2}|e^{ir\Delta}\tilde{f}_2|^{1/2} \right\|_{L^6(\tilde{Y})}$, we decompose $B_{R/M} \times I_{R/M^2}$ as a union of $\frac{R}{M^2}$-balls $Q_{k,R/M^2}$, and inside each $Q_{k,R/M^2}$ we consider the $\sqrt{R}/M$-cubes $Q^{(k)}$ that intersect $\tilde{Y}$. First, we sort the balls $Q_{k,R/M^2}$ according to the dyadic values $\|e^{ir\Delta}\tilde{f}_i\|_{L^2(Q_{k,R/M^2})}$, $i=1,2$. Then inside each $Q_{k,R/M^2}$ we sort the cubes $Q^{(k)}$ according to the dyadic values $\|e^{ir\Delta}\tilde{f}_i\|_{L^6(Q^{(k)})}$, $i=1,2$. We can choose balls $Q_{k,R/M^2}$,

$k=1,2,\cdots,\bar{W}$, and inside each $Q_{k,R/M^2}$ we can choose a set of $\sqrt{R}/M$-cubes $Q_j^{(k)}$, $j=1,2,\cdots,N_k$, so that

$$\approx N \text{ boxes in } \tilde{Y} \text{ are contained in } \bigcup_{k=1}^{\bar{W}} \tilde{Y}_k, \tag{3.54}$$

where $\tilde{Y}_k := \bigcup_{j=1}^{N_k} Q_j^{(k)}$, and the following conditions hold:

- (a). For each $i=1,2$, $\|e^{ir\Delta}\tilde{f}_i\|_{L^2(Q_{k,R/M^2})}$ is essentially constant in $k=1,\cdots,\bar{W}$.
- (b). For each $k=1,\cdots,\bar{W}$, for each $i=1,2$, $\|e^{ir\Delta}\tilde{f}_i\|_{L^6(Q_j^{(k)})}$ is essentially constant in $j=1,\cdots,N_k$.
- (c). $\left\||e^{ir\Delta}\tilde{f}_1|^{1/2}|e^{ir\Delta}\tilde{f}_2|^{1/2}\right\|_{L^6(\tilde{Y}_k)}$ is essentially constant in $k=1,\cdots,\bar{W}$.

Now by (3.49), (3.54) and the condition (c) as above, for each $1\le k\le \bar{W}$ we have

$$\left\||e^{ir\Delta}\tilde{f}_1|^{1/2}|e^{ir\Delta}\tilde{f}_2|^{1/2}\right\|_{L^6(\tilde{Y})} \lessapprox \bar{W}^{\frac{1}{6}} \left\||e^{ir\Delta}\tilde{f}_1|^{1/2}|e^{ir\Delta}\tilde{f}_2|^{1/2}\right\|_{L^6(\tilde{Y}_k)}.$$

Since tangent-to-variety condition is preserved under parabolic rescaling, we can apply Theorem 3.6 to bound $\left\||e^{ir\Delta}\tilde{f}_1|^{1/2}|e^{ir\Delta}\tilde{f}_2|^{1/2}\right\|_{L^6(\tilde{Y}_k)}$ by

$$\lessapprox E^{O(1)}\left(\frac{R}{M^2}\right)^{-1/6} N_k^{-1/6}\left(\frac{R}{M^2}\right)^{-1/2}\prod_{i=1}^{2}\left\|e^{ir\Delta}\tilde{f}_i\right\|_{L^2(Q_{k,R/M^2})}^{1/2}.$$

By the condition (a) as above and parabolic rescaling (3.52), we have

$$\prod_{i=1}^{2}\left\|e^{ir\Delta}\tilde{f}_i\right\|_{L^2(Q_{k,R/M^2})}^{1/2} \lesssim \bar{W}^{-1/2}\prod_{i=1}^{2}\|e^{ir\Delta}\tilde{f}_i\|_{L^2(B_{R/M}\times I_{R/M^2})}^{1/2}$$

$$\sim \bar{W}^{-1/2}M^{-1}\prod_{i=1}^{2}\|e^{it\Delta}f_i\|_{L^2(B^3(R))}^{1/2} \lesssim \bar{W}^{-1/2}M^{-1}R^{1/2}\prod_{i=1}^{2}\|f_i\|_2^{1/2}.$$

Combining (3.53) and the above estimates for $\left\||e^{ir\Delta}\tilde{f}_1|^{1/2}|e^{ir\Delta}\tilde{f}_2|^{1/2}\right\|_{L^6(\tilde{Y})}$, we get

$$\left\||e^{it\Delta}f_1|^{1/2}|e^{it\Delta}f_2|^{1/2}\right\|_{L^6(Y)} \lessapprox E^{O(1)}\bar{W}^{-1/3}N_k^{-1/6}R^{-1/6}\prod_{i=1}^{2}\|f_i\|_2^{1/2}.$$

The above estimate holds for $\bar{W}$ indexes k's. For each k, there are N_k $\frac{\sqrt{R}}{M}$-cubes in $\tilde{Y}_k$, each $\frac{\sqrt{R}}{M}$-cube contains at most M $\frac{\sqrt{R}}{M}\times\frac{\sqrt{R}}{M}\times\frac{\sqrt{R}}{M^2}$-boxes in $\tilde{Y}$, and there are $\approx N$ $\frac{\sqrt{R}}{M}\times\frac{\sqrt{R}}{M}\times\frac{\sqrt{R}}{M^2}$-boxes in $\tilde{Y}$ that are contained in $\bigcup_{k=1}^{\bar{W}}\tilde{Y}_k$. By pigeonholing there is an index k satisfying

$$N \lessapprox N_k \bar{W} M.$$

Therefore

$$\left\| |e^{it\Delta} f_1|^{1/2} |e^{it\Delta} f_2|^{1/2} \right\|_{L^6(Y)} \lessapprox E^{O(1)} \bar{W}^{-1/6} N^{-1/6} M^{1/6} R^{-1/6} \prod_{i=1}^{2} \|f_i\|_2^{1/2}. \tag{3.55}$$

Since $\bar{W} \geq 1$, this completes the proof of Proposition 3.10.

■

This finishes the proof of Proposition 3.9.

■

Finally, to prove Proposition 3.4, we apply Proposition 3.9 to $f_{j,tang}$ on each ball B_j. We expand $f_{j,tang}$ into wave packets at the scale $\rho = R^{1-\delta}$ on the ball B_j. Because of the definition of $f_{j,tang}$, each wave packet will lie in the $R^{1/2+\delta}$-neighborhood of Z and the angles between the wave packets and the tangent space of Z will be bounded by $R^{-1/2+2\delta}$. For a detailed description of the wave packet decomposition of $f_{j,tang}$ on a smaller ball, see Section 7 of [11]. We define E so that $\rho^{1/2} E = R^{1/2+\delta}$. Since $\rho = R^{1-\delta}$, we get $E = R^{(3/2)\delta}$, and so $E\rho^{-1/2} = R^{-1/2+2\delta}$. Each new wave packet lies in the $E\rho^{1/2}$-neighborhood of Z, and the angles between the wave packets and the tangent space of Z are bounded by $E\rho^{-1/2}$. Therefore, the new wave packets are concentrated in $\mathbf{T}_Z(E)$. Now since $E^{O(1)} = R^{O(\delta)}$, the bound from Proposition 3.9 implies Proposition 3.4.

Chapter 4

Schrödinger maximal estimates in higher dimension

In this chapter, we first display Bourgain's counterexample from [3], which shows that pointwise convergence can fail for $s < \frac{n}{2(n+1)}$. We will see that Bourgain's example fits a scenario where all wave packets are contained in a small neighborhood of a hyperplane. In this special scenario, or more generally, in tangent-to-variety case, where all wave packets are contained in a small neighborhood of a variety, we will prove that the exponent $\frac{n}{2(n+1)}$ is optimal.

4.1 Bourgain's counterexample

Proposition 4.1 *Let $p \geq 1$, $n \geq 2$ and $s < \frac{n}{2(n+1)}$. Then there exists $R_k \to \infty$ and $f_k \in L^2(\mathbb{R}^n)$ with $\operatorname{supp}\widehat{f_k}$ supported in the annulus $|\xi| \sim R_k$, such that $\|f_k\|_2 = 1$ and*

$$\lim_{k\to\infty} R_k^{-s} \left\| \sup_{0<t\leq 1/R_k} |e^{it\Delta} f_k(x)| \right\|_{L^p(B^n(0,1))} = \infty. \tag{4.1}$$

Denote $x = (x_1, \cdots, x_n) = (x_1, x') \in B^n(0,1)$. Let $\varphi : \mathbb{R} \to \mathbb{R}_+, \Phi : \mathbb{R}^{n-1} \to \mathbb{R}_+$ satisfy $\operatorname{supp} \widehat{\varphi} \subset [-1,1]$, $\operatorname{supp} \widehat{\Phi} \subset B^{n-1}(0,1)$, $\widehat{\varphi}, \widehat{\Phi}$ smooth and $\varphi(0) = \Phi(0) = 1$. Set $D = R^{\frac{n+2}{2(n+1)}}$ and define

$$f(x) = e^{iRx_1} \varphi(R^{\frac{1}{2}} x_1) \Phi(x') \prod_{j=2}^{n} \left(\sum_{\frac{R}{2D} < \ell_j < \frac{R}{D}} e^{iD\ell_j x_j} \right) \tag{4.2}$$

where $\ell = (\ell_2, \cdots, \ell_n) \in \mathbb{Z}^{n-1}$. By a direct computation, for $\xi = (\xi_1, \xi') \in \mathbb{R}^n$ we get

$$\widehat{f}(\xi_1, \xi') = R^{-\frac{1}{2}} \widehat{\varphi} \left(R^{-\frac{1}{2}} (\xi_1 - R) \right) \left(\sum_{\ell} \widehat{\Phi}(\xi' - D\ell) \right), \tag{4.3}$$

hence,

$$\|f\|_2 \sim R^{-\frac{1}{4}} \left(\frac{R}{D} \right)^{\frac{n-1}{2}} \quad \text{and} \quad \operatorname{supp} \widehat{f} \subset A(R) = \{\xi : |\xi| \sim R\}. \tag{4.4}$$

Clearly, denoting $e(z) = e^{iz}$,

$$e^{it\Delta}f(x) = \int\int \widehat{\varphi}(\xi_1)\widehat{\Phi}(\xi')\Big\{ \sum_\ell e\left((R+R^{\frac12}\xi_1)x_1 + (\xi'+D\ell)\cdot x' - (R+R^{\frac12}\xi_1)^2 t - |\xi'+D\ell|^2 t\right) \Big\} d\xi_1 d\xi' .$$

Taking $t < \frac{c}{R}, |x| < c$ for suitable constant $c > 0$, one get

$$\left|e^{it\Delta}f(x)\right| \sim \left| \int\int \widehat{\varphi}(\xi_1)\widehat{\Phi}(\xi')\Big\{ \sum_\ell e\left(R^{\frac12}\xi_1 x_1 + D\ell\cdot x' - 2R^{\frac32}\xi_1 t - D^2|\ell|^2 t\right) \Big\} d\xi_1 d\xi' \right|$$

$$\sim \varphi\left(R^{\frac12}(x_1 - 2Rt)\right)\Phi(0)\left|\sum_\ell e\left(D\ell\cdot x' - D^2|\ell|^2 t\right)\right| \tag{4.5}$$

Recall that $\Phi(0) = 1$. In order to ensure that the first factor in (4.5) should be ~ 1, we specify

$$R^{\frac12}|x_1 - 2Rt| < \frac15 \tag{4.6}$$

and denote

$$t = \frac{x_1}{2R} + \tau \quad \text{with} \quad |\tau| < \frac{1}{10}\mathrm{R}^{-\frac32} . \tag{4.7}$$

For this choice of t, the third factor of (4.5) becomes

$$\left|\sum_\ell e\left(D\ell\cdot x' - \frac{D^2}{2R}|\ell|^2 x_1 - D^2|\ell|^2\tau\right)\right|$$

$$= \prod_{j=2}^n \left| \sum_{\frac{R}{2D}<\ell_j<\frac{R}{D}} e\left(\ell_j y_j + \ell_j^2(y_1+s)\right)\right| \tag{4.8}$$

with

$$y' = Dx' (\operatorname{mod} 2\pi) \qquad \mathrm{y}_1 = -\frac{\mathrm{D}^2}{2\mathrm{R}}\mathrm{x}_1 (\operatorname{mod} 2\pi) \tag{4.9}$$

and where $s = -D^2\tau$ is subject to the condition

$$|s| \lesssim D^2 R^{-\frac32} = R^{-\frac{n-1}{2(n+1)}} . \tag{4.10}$$

We view (y_1, y') as a point in the n-torus $\mathbb{T}^n$. Next, define the following subset $\Omega \subset \mathbb{T}^n$

$$\Omega = \bigcup_{q\sim R^{\frac{n-1}{2(n+1)}},a} \left\{ (y_1,y') : \left|y_1 - 2\pi\frac{a_1}{q}\right| < cR^{-\frac{n-1}{2(n+1)}} \quad \text{and} \quad \left|\mathrm{y}' - 2\pi\frac{\mathrm{a}'}{\mathrm{q}}\right| < \mathrm{c}\frac{\mathrm{D}}{\mathrm{R}} \right\} \tag{4.11}$$

with $a = (a_1, a') \,(\mathrm{mod\, q})$ and $(a_1, q) = 1$.

Hence

$$|\Omega| \sim R^{\frac{n-1}{2(n+1)}} R^{n\frac{n-1}{2(n+1)}} R^{-\frac{n-1}{2(n+1)}} \big(\frac{D}{R}\big)^{n-1} \sim 1$$

and we take $x \in B^n(0,1)$ for which y given by (4.9) belongs to Ω:

$$\bar{\Omega} = \left\{ x \in B^n(0,1) : (y_1, y') = (-\frac{D^2}{2R}x_1, Dx') \,(\mathrm{mod}\, 2\pi) \in \Omega \right\},$$

clearly $|\bar{\Omega}| = c_1 > 0$. We evaluate (4.8) for $y \in \Omega$. For $y \in \Omega$, we have

$$\left|y_1 - 2\pi\frac{a_1}{q}\right| < cR^{-\frac{n-1}{2(n+1)}} \quad \text{and} \quad \left|y' - 2\pi\frac{a'}{q}\right| < c\frac{D}{R}$$

for some $q \sim R^{\frac{n-1}{2(n+1)}}$ and $a = (a_1, a') \,(\mathrm{mod\, q})$ with $(a_1, q) = 1$. We set

$$s = 2\pi\frac{a_1}{q} - y_1$$

for which (4.10) holds. Clearly for $j = 2, \cdots, n$, by the quadratic Gauss sum evaluation

$$\left| \sum_{\frac{R}{2D}<\ell_j<\frac{R}{D}} e\left(\ell_j y_j + \ell_j^2(y_1+s)\right) \right| \sim \left| \sum_{\frac{R}{2D}<\ell_j<\frac{R}{D}} e\left(2\pi\frac{a_j}{q}\ell_j + 2\pi\frac{a_1}{q}\ell_j^2\right) \right|$$

$$\sim \frac{R/D}{q} \left| \sum_{\ell_j=0}^{q-1} e\left(2\pi\frac{a_j}{q}\ell_j + 2\pi\frac{a_1}{q}\ell_j^2\right) \right| \sim R^{\frac{1}{2(n+1)}} q^{\frac{1}{2}} \sim R^{\frac{1}{4}}$$

and

$$(4.8) \sim R^{\frac{n-1}{4}}. \tag{4.12}$$

Recalling (4.4), we obtain for any x in the set $\bar{\Omega}$ of measure $c_1 > 0$,

$$\sup_{0<t\le 1/R} \frac{\left|e^{it\Delta} f(x)\right|}{\|f\|_2} \gtrsim R^{\frac{n-1}{4}} R^{\frac{1}{4}} \left(\frac{D}{R}\right)^{\frac{n-1}{2}} = R^{\frac{n}{2(n+1)}}. \tag{4.13}$$

The claim in the Proposition follows.

Remark 4.2 *Let us take a look at Bourgain's example after parabolic rescaling. Recall the parabolic rescaling from Section 2.2. For any function* $f \in L^2(\mathbb{R}^n)$ *with* $\operatorname{supp} \widehat{f} \subset A(R)$*, write* $\xi = R\zeta \in A(R)$*, where* $\zeta \in A(1)$*, then*

$$|e^{it\Delta} f(x)| = R^{n/2} |e^{i\tilde{t}\Delta} g(\tilde{x})| \,, \tag{4.14}$$

where the function $g \in L^2(\mathbb{R}^n)$ *is given by*

$$\widehat{g}(\zeta) = R^{n/2} \widehat{f}(R\zeta) \,, \tag{4.15}$$

and $\|g\|_2 = \|f\|_2$*, and the new coordinates and the old coordinates are related by*

$$\begin{cases} \tilde{x} = Rx \,, \\ \tilde{t} = R^2 t \,. \end{cases} \tag{4.16}$$

Moreover,

$$\left\| \sup_{0<t\le 1/R} |e^{it\Delta} f(x)| \right\|_{L^2(B^n(0,1))} = \left\| \sup_{0<\tilde{t}\le R} |e^{i\tilde{t}\Delta} g(\tilde{x})| \right\|_{L^2(B^n(0,R))} . \tag{4.17}$$

After applying parabolic rescaling to Bourgain's example, we get a function g *given by*

$$\widehat{g}(\zeta_1, \zeta') = R^{\frac{n-1}{2}} \widehat{\varphi} \left(R^{\frac{1}{2}} (\zeta_1 - 1) \right) \left(\sum_{\ell} \widehat{\Phi}(R\zeta' - D\ell) \right) , \tag{4.18}$$

where $\ell = (\ell_2, \cdots, \ell_n) \in \mathbb{Z}^{n-1}$ *with* $\frac{R}{2D} < \ell_j < \frac{R}{D}$ *for each* j*. Form this expression, we see that* $\widehat{g}$ *is supported in an* $R^{-1/2}$*-strip in* $A(1)$*:*

$$\operatorname{supp} \widehat{g} \subset \{(\zeta_1, \zeta') \in A(1) : |\zeta_1 - 1| \le R^{-1/2}\} \,. \tag{4.19}$$

And the absolute value $|e^{i\tilde{t}\Delta} g|$ *is*

$$|e^{i\tilde{t}\Delta} g(\tilde{x})| \sim R^{-n/2} \varphi \left(R^{-\frac{1}{2}} (\tilde{x}_1 - 2\tilde{t}) \right) \Phi(0) \left| \sum_{\ell} e \left(R^{-1} D\ell \cdot \tilde{x}' - R^{-2} D^2 |\ell|^2 \tilde{t} \right) \right| . \tag{4.20}$$

Take hyperplane V *to be* $\tilde{x}_1 - 2\tilde{t} = 0$*. From this expression and the factor* $\varphi(R^{-1/2}(\tilde{x}_1 - 2\tilde{t}))$*, we observe that* $e^{i\tilde{t}\Delta} g(\tilde{x})$ *is essentially concentrated in a* $CR^{1/2}$*-neighborhood of* V *and its absolute value is essentially constant along a transverse direction to* V *for a distance* $cR^{1/2}$*. Moreover, for* $\zeta \in \operatorname{supp} \widehat{g}$*, we have*

$$|(2\zeta,1)\cdot(-1/2,0,\cdots,0,1)| = |-\zeta_1+1| \le R^{-1/2}\,,$$

that is,

$$\mathrm{Angle}((2\zeta,1),V) \lesssim R^{-1/2}\,.$$

In summary, for each $R \ge 1$, there is a function $g \in L^2(\mathbb{R}^n)$ with $\widehat{g}$ supported in $A(1)$, and g is concentrated in wave packets from T_V, where V is a hyperplane, satisfying that

$$\left\| \sup_{0<t\le R} |e^{it\Delta} g| \right\|_{L^2(B^n(0,R))} \gtrsim R^{\frac{n}{2(n+1)}} \|g\|_{L^2}. \tag{4.21}$$

4.2 A sharp Schrödinger maximal estimate in tangent-to-variety case

In this section, we point out that we have a linear local refinement of the Strichartz estimate in general dimension, from which a sharp Schrödinger maximal estimate in tangent-to-variety case follows. We have seen that Bourgain's example fits into the tangent-to-hyperplane scenario. Our result says that in this special scenario, the exponent $\frac{n}{2(n+1)}$ is optimal. As before, R is a sufficiently large parameter.

Theorem 4.3 *Let $n \ge 2$. For any function $f \in L^2(\mathbb{R}^n)$ with Fourier support in $B^n(0,1)$, suppose that f is concentrated in wave packets from $\mathbf{T}_Z(E)$, where $Z = Z(P)$ is defined by a polynomial P of degree $\le D = R^{\epsilon^4}$, and P is a product of distinct non-singular polynomials. Suppose that $Q_1, Q_2, \ldots$ are lattice $R^{1/2}$-cubes in $B^{n+1}(R)$, so that*

$$\|e^{it\Delta} f\|_{L^{2(n+1)/(n-1)}(Q_j)} \text{ is essentially constant in } j.$$

Suppose that these cubes are arranged in horizontal strips of the form $\mathbb{R}\times\cdots\times\mathbb{R}\times\{t_0, t_0+R^{1/2}\}$, and that each strip contains $\sim \sigma$ cubes Q_j. Let Y denote $\bigcup_j Q_j$. Then

$$\|e^{it\Delta} f\|_{L^{2(n+1)/(n-1)}(Y)} \lessapprox E^{O(1)} R^{-\frac{1}{2(n+1)}} \sigma^{-\frac{1}{n+1}} \|f\|_{L^2}. \tag{4.22}$$

Proposition 4.4 *Let $n \ge 2$. For any $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that the following holds for any $R \ge 1$. For any function $f \in L^2(\mathbb{R}^n)$ with Fourier support in $B^n(0,1)$, suppose that f is concentrated* *in wave packets from $\mathbf{T}_Z(E)$, where $Z = Z(P)$ is defined by a polynomial P of degree $\leq D = R^{\epsilon^4}$, and P is a product of distinct non-singular polynomials. Then*

$$\left\| \sup_{0<t\leq R} |e^{it\Delta} f| \right\|_{L^2(B^n(0,R))} \leq C(\epsilon) E^{O(1)} R^{\frac{n}{2(n+1)}+\epsilon} \|f\|_{L^2}. \tag{4.23}$$

Remark 4.5 *In tangent-to-variety case, we have the sharp estimate (4.23) of the L^2-norm of the associated maximal function. The issue is that, when estimating L^2-norm, we have no machinery to reduce the original problem to that special case.*

If we estimate $L^{2(n+1)/n}$-norm instead of L^2, we can indeed reduce the original problem to tangent-to-variety case, using polynomial partitioning. To deal with the exponent $p = \frac{2(n+1)}{n}$, we are working to get a multilinear analogue of Theorem 4.3, as well as thinking about how to go from multilinear maximal estimates to linear ones.

4.2.1 Proof of Theorem 4.3

The proof for general dimension follows the same argument for the case $n = 2$. It uses the Bourgain-Demeter l^2-decoupling theorem, together with induction on the radius and parabolic rescaling. First we recall the decoupling result of Bourgain and Demeter in higher dimension in [4].

Theorem 4.6 (Bourgain-Demeter) *Let $n \geq 2$ and $R \geq 1$. Suppose that the R^{-1}-neighborhood of the unit paraboloid in $\mathbb{R}^n$ is divided into $R^{(n-1)/2}$ disjoint rectangular boxes τ, each with dimensions $R^{-1/2} \times \cdots \times R^{-1/2} \times R^{-1}$. Suppose $\widehat{F}_\tau$ is supported in τ and $F = \sum_\tau F_\tau$. Then for any $\varepsilon > 0$,*

$$\|F\|_{L^{2(n+1)/(n-1)}(\mathbb{R}^n)} \leq C_\varepsilon R^\epsilon \left(\sum_\tau \|F_\tau\|^2_{L^{2(n+1)/(n-1)}(\mathbb{R}^n)} \right)^{1/2}.$$

If $E \geq R^{1/4}$ (or any fixed power of R), then the estimate (4.22) is trivial because of the factor $E^{O(1)}$. So we assume that $E \leq R^{1/4}$.

To set up the argument, we decompose f as follows. We break the unit ball $B^n(1)$ in frequency space into small balls τ of radius $R^{-1/4}$, and divide the physical space ball $B^n(R)$ into balls B of radius $R^{3/4}$. For each pair (τ, B), we let $f_{\square_{\tau,B}}$ be the function formed by cutting off f on the ball B (with a Schwartz tail) in physical space and the ball τ in Fourier space. We note that $e^{it\Delta} f_{\square_{\tau,B}}$, restricted to $B^{n+1}(R)$, is essentially supported on an $R^{3/4} \times \cdots \times R^{3/4} \times R$-box, which we denote by $\square_{\tau,B}$ (compare the discussion in Section 2.1). The box $\square_{\tau,B}$ is in the direction given by $(2c(\tau), 1)$ and intersects $t = 0$ at a disk centered at $(c(B), 0)$, where $c(\tau)$ and $c(B)$ are the centers of τ and B respectively. For a fixed τ, the different boxes $\Box_{\tau,B}$ tile $B^{n+1}(R)$. In particular, for each τ, a given cube Q_j lies in exactly one box $\Box_{\tau,B}$.

Since f is concentrated in wave packets from $\mathbf{T}_Z(E)$, we only need to consider those $R^{1/2}$-cubes Q_j that are contained in the $ER^{1/2}$-neighborhood of Z. For each such $R^{1/2}$-cube Q_j, we will see that the wave packets that pass through Q_j are nearly coplanar. Because of this, we will be able to apply the n-dimensional decoupling theorem to study $e^{it\Delta}f$ on Q_j:

Lemma 4.7 *Suppose that f has Fourier support in $B^n(0,1)$ and is concentrated in wave packets from $\mathbf{T}_Z(E)$, where $E \le R^{1/4}$ and $Z = Z(P)$ is a finite union of non-singular varieties. Suppose that an $R^{1/2}$-cube Q is in $N_{ER^{1/2}}(Z)$. Then we have the decoupling bound*

$$\|e^{it\Delta}f\|_{L^{2(n+1)/(n-1)}(Q)} \lessapprox \left(\sum_{\Box} \|e^{it\Delta}f_{\Box}\|^2_{L^{2(n+1)/(n-1)}(10Q)}\right)^{1/2} + R^{-1000n}\|f\|_{L^2}. \tag{4.24}$$

Remark: The parameter R is assumed to be sufficiently large. The $R^{-1000n}\|f\|_{L^2}$ is a negligibly small term which covers minor contributions coming from the tails of the Fourier transforms of smooth functions. We will neglect this term in the sequel.

Proof: Observe that $Q \subset N_{ER^{1/2}}Z$ implies that there exists a non-singular point $z_0 \in Z \cap N_{ER^{1/2}}Q$. Thus for each wave packet $T_{\theta,\nu}$ that intersects Q, we have $z_0 \in Z \cap N_{2ER^{1/2}}(T_{\theta,\nu})$. By the definition of $\mathbf{T}_Z(E)$ we get the angle bound

$$\mathrm{Angle}(G_0(\theta), T_{z_0}[Z(P)]) \le ER^{-1/2}\,. \tag{4.25}$$

We recall from Section 2.1 that $G_0(\theta) = (2c(\theta), 1)$. Suppose that $T_{z_0}Z$ is the plane given by $a_1x_1 + \cdots + a_nx_2 + bt = 0$, with $a_1^2 + \cdots + a_n^2 + b^2 = 1$. The angle condition above restricts the location of θ as follows:

$$|2a \cdot c(\theta) + b| \lesssim ER^{-1/2}. \tag{4.26}$$

We note that each tube $T_{\theta,\nu}$ makes an angle $\gtrsim 1$ with the plane $t = 0$, because $\theta \subset B^n(0,1)$. We can assume that there are some tubes $T_{\theta,\nu}$ tangent to $T_{z_0}Z$, and so $|a| \gtrsim 1$. Therefore, (4.26) confines θ to a strip of width $\sim ER^{-1/2}$ inside of $B^n(0,1)$. We denote this strip by $S \subset B^n(0,1)$.

Let $\mathbf{T}_{Z,Q}(E)$ be the set of (θ,ν) in $\mathbf{T}_Z(E)$ for which each $T_{\theta,\nu}$ intersects Q. For each (θ,ν) in $\mathbf{T}_{Z,Q}(E)$, θ obeys (4.26), and so $\theta \subset S$. Let η be a smooth bump function which approximates χ_Q. We note that $\eta e^{it\Delta}f$ is essentially equal to

$$\sum_{(\theta,\nu)\in \mathbf{T}_{Z,Q}(E)} \eta e^{it\Delta} f_{\theta,\nu}.$$

Therefore, the Fourier transform of the localized solution $\eta e^{it\Delta} f$ is essentially supported in

$$S^* := \{(\xi_1, \cdots, \xi_n, \xi_{n+1}) : (\xi_1, \cdots, \xi_n) \in S \text{ and } |\xi_{n+1} + \xi_1^2 + \cdots + \xi_n^2| \lesssim R^{-1/2}\}. \tag{4.27}$$

(The contribution of the not essential parts is covered by the negligible term $R^{-1000n}\|f\|_{L^2}$ in the statement of the Lemma.)

After a rotation in the $(x_1, \cdots, x_n)$-plane we can suppose that the strip S is defined by

$$a_1 \le \xi_1 \le a_1 + ER^{-1/2},$$

for some $a_1 \in [-1, 1]$. We note that at each point $(\xi_1, \cdots, \xi_n) \in S$,

$$\partial_1 \left(-\xi_1^2 - \cdots - \xi_n^2\right) = -2a_1 + O(ER^{-1/2}). \tag{4.28}$$

Let $v \in \mathbb{R}^{n+1}$ be the vector

$$v = (1, 0, \cdots, 0, -2a_1).$$

Let Π be an n-plane perpendicular to v. Because $E \le R^{1/4}$, we claim that the projection of S^* onto Π lies in the $\sim R^{-1/2}$-neighborhood of a paraboloid. We can see this as follows. Let

$$S^*_{core} := \{(\xi_1, \cdots, \xi_n, \xi_{n+1}) : \xi_1 = a_1, |(\xi_2, \cdots, \xi_n)| \le 1, \xi_{n+1} = -\xi_1^2 - \cdots - \xi_n^2\}.$$

The set S^*_{core} is a paraboloid, and its projection onto Π is also a paraboloid. We claim that the projection of S^* to Π lies in the $\sim R^{-1/2}$-neighborhood of this paraboloid. If $(\xi_1, \cdots, \xi_n, \xi_{n+1}) \in S^*$, then (4.28) tells us that

$$(-\xi_1^2 - \cdots - \xi_n^2) = -a_1^2 - \xi_2^2 - \cdots - \xi_n^2 - 2a_1(\xi_1 - a_1) + O(ER^{-1/2} \cdot |\xi_1 - a_1|).$$

Therefore,

$$(\xi_1, \xi_2, \cdots, \xi_{n+1}) = (a_1, \xi_2, \cdots, \xi_n, -a_1^2 - \xi_2^2 - \cdots - \xi_n^2) + (\xi_1 - a_1)v + O(ER^{-1/2}|\xi_1 - a_1| + R^{-1/2}).$$

The first term on the right-hand side lies is S^*_{core}. Since Π is perpendicular to v, the projection to Π kills the second term on the right-hand side. So the distance from the projection of ξ to the projection of S^*_{core} is at most

$$ER^{-1/2}|\xi_1 - a_1| + R^{-1/2} \lesssim E^2 R^{-1} + R^{-1/2} \sim R^{-1/2}.$$

Therefore, if we restrict $\eta e^{it\Delta} f$ to Π, the resulting n-dimensional function has Fourier support in the $\sim R^{-1/2}$-neighborhood of a paraboloid.

We consider the decomposition $f = \sum_{(\tau,B):\square_{\tau,B}\cap Q\neq\emptyset} f_{\square_{\tau,B}}$. Suppose that $e^{it\Delta} f_{\square_{\tau,B}}$ contributes to $\|e^{it\Delta} f\|_{L^{2(n+1)/(n-1)}(Q)}$, there must be a wave packet $T_{\theta,\nu}$ that intersects the $R^{1/2}$-cube Q with $\theta \subset \tau$, and so $\tau \cap S$ must be non-empty. Also, for a given τ, there is only one B so that $\square_{\tau,B} \cap Q$ is non-empty. Also, the Fourier support of $\eta e^{it\Delta} f_{\square_{\tau,B}}$ lies in $S^* \cap (\tau \times \mathbb{R})$, by the same argument we used above for $\eta e^{it\Delta} f$. The projection onto Π of $S^* \cap (\tau \times \mathbb{R})$ is an $R^{-1/4} \times \cdots R^{-1/4} \times R^{-1/2}$ rectangular box. The union of these boxes over all τ intersecting S is the $R^{-1/2}$-neighborhood of a parabola. Therefore, we have the hypotheses to apply the n-dimensional decoupling theorem, Theorem 4.6, which gives:

$$\|\eta e^{it\Delta} f\|_{L^{2(n+1)/(n-1)}(\Pi)} \lessapprox \left(\sum_{\square} \|\eta e^{it\Delta} f_\square\|^2_{L^{2(n+1)/(n-1)}(\Pi)} \right)^{1/2}.$$

Now we integrate in the direction perpendicular to Π and apply Fubini and Minkowski to get

$$\|\eta e^{it\Delta} f\|_{L^{2(n+1)/(n-1)}(\mathbb{R}^{n+1})} \lessapprox \left(\sum_{\square} \|\eta e^{it\Delta} f_\square\|^2_{L^{2(n+1)/(n-1)}(\mathbb{R}^{n+1})} \right)^{1/2}.$$

This implies the desired conclusion.

■

Next, by induction on the radius R, we will show that each function $f_\square$ obeys a version of Theorem 4.3. Here is the statement. Suppose that $S_1, S_2, ...$ are $R^{1/2} \times \cdots \times R^{1/2} \times R^{3/4}$-tubes in $\square$ (running parallel to the long axis of $\square$), and that

$$\|e^{it\Delta} f_\square\|_{L^{2(n+1)/(n-1)}(S_j)} \text{ is essentially constant in } j.$$

Suppose that these tubes are arranged into $R^{3/4}$-strips running parallel to the short axes of $\square$ and that each such strip contains $\sim \sigma_\square$ tubes S_j. Let $Y_\square$ denote $\cup_j S_j$. Then

$$\|e^{it\Delta} f_\square\|_{L^{2(n+1)/(n-1)}(Y_\square)} \lessapprox E^{O(1)} R^{-\frac{1}{4(n+1)}} R^{-\frac{1}{4(n+1)}} \sigma_\square^{-\frac{1}{n+1}} \|f_\square\|_{L^2}. \tag{4.29}$$

This inequality follows by doing a parabolic rescaling and then using Theorem 4.3 at scale $R^{1/2}$, which we can assume holds by induction on R. We write down the details of this parabolic rescaling, and in particular we will check that the tangent-to-variety condition is preserved under parabolic rescaling. For each $R^{-1/4}$-ball τ in $B^n(1)$, we write $\xi = \xi_0 + R^{-1/4}\zeta \in \tau$, then

$$|e^{it\Delta} f_\tau(x)| = R^{-n/8} |e^{i\tilde{t}\Delta} g(\tilde{x})|$$

for some function g with Fourier support in $B^n(1)$ and $\|g\|_2 = \|f_\tau\|_2$, where the new coordinates $(\tilde{x}, \tilde{t})$ are related to the old coordinates (x, t) by

$$\begin{cases} \tilde{x} = R^{-1/4}x - 2tR^{-1/4}\xi_0\,, \\ \tilde{t} = R^{-1/2}t\,. \end{cases} \tag{4.30}$$

Therefore

$$\|e^{it\Delta} f_\Box(x)\|_{L^{2(n+1)/(n-1)}(Y_\Box)} = R^{-\frac{1}{4(n+1)}} \|e^{i\tilde{t}\Delta} g(\tilde{x})\|_{L^{2(n+1)/(n-1)}(\tilde{Y})},$$

where $\tilde{Y}$ is the image of $Y_\Box$ under the new coordinates. Note that $\tilde{Y}$ is a union of $R^{1/4}$-cubes inside an $R^{1/2}$-cube. These $R^{1/4}$-cubes are arranged in $R^{1/4}$-horizontal strips, and each strip contains $\sim \sigma_\Box$ $R^{1/4}$-cubes. Moreover, by the relation (4.30), we see that each wave packet T, at scale R, of dimensions $R^{1/2+\delta} \times \cdots \times R^{1/2+\delta} \times R$ in the old coordinates is mapped to a corresponding wave packet $\tilde{T}$, at scale $R^{1/2}$, of dimensions $R^{1/4+\delta} \times \cdots \times R^{1/4+\delta} \times R^{1/2}$ in the new coordinates. The variety $Z(P)$ corresponds to a new variety $Z(Q)$, given by the relation $Q(\tilde{x}, \tilde{t}) = Q(R^{-1/4}x - 2tR^{-1/4}\xi_0, R^{-1/2}t) = P(x, t)$. We claim that, under the above correspondence, if the wave packet T at scale R is $ER^{-1/2}$-tangent to $Z(P)$, then the wave packet $\tilde{T}$ at scale $R^{1/2}$ is $ER^{-1/4}$-tangent to $Z(Q)$ in the new coordinates.

By the relation (4.30), the distance condition $T \subset N_{ER^{1/2}}Z(P)$ implies that $\tilde{T} \subset N_{ER^{1/4}}Z(Q)$. Given the direction $(2\xi, 1)$ of T, the angle condition

$$\text{Angle}((2\xi, 1), T_{z_0}[Z(P)]) \leq ER^{-1/2}$$

is equivalent to

$$\frac{|(2\xi, 1) \cdot (P_x(x_0, t_0), P_t(x_0, t_0))|}{|(P_x(x_0, t_0), P_t(x_0, t_0))|} \lesssim ER^{-1/2}, \tag{4.31}$$

where $z_0 = (x_0, t_0)$. Note that the direction of the corresponding wave packet $\tilde{T}$ is given by $(2\zeta, 1)$, where

ξ and ζ are related by $\xi = \xi_0 + R^{-1/4}\zeta$. Let $\tilde{z}_0 = (\tilde{x}_0, \tilde{t}_0)$ denote the point corresponding to z_0. Using the relations

$$P_x = R^{-1/4} Q_{\tilde{x}}, \quad P_t = -2R^{-1/4}\xi_0 \cdot Q_{\tilde{x}} + R^{-1/2} Q_{\tilde{t}}\,,$$

after some computation, (4.31) yields that

$$\frac{|(2\zeta, 1) \cdot (Q_{\tilde{x}}(\tilde{x}_0, \tilde{t}_0), Q_{\tilde{t}}(\tilde{x}_0, \tilde{t}_0))|}{|(Q_{\tilde{x}}(\tilde{x}_0, \tilde{t}_0), Q_{\tilde{t}}(\tilde{x}_0, \tilde{t}_0))|} \lesssim ER^{-1/4},$$

which implies that

$$\text{Angle}((2\zeta, 1), \tilde{T}_{\tilde{z}_0}[Z(Q)]) \leq ER^{-1/4}.$$

Therefore the tangent-to-variety condition is preserved under parabolic rescaling and the induction on radius is justified.

We have now established inequality (4.29). To apply this inequality, we need to identify a good choice of $Y_\square$. We do this by some dyadic pigeonholing. For each $\square$, we apply the following algorithm to regroup tubes in $\square$.

1. We sort those $R^{1/2} \times \cdots \times R^{1/2} \times R^{3/4}$-tubes S's contained in the box $\square$ according to the order of magnitude of $\|e^{it\Delta} f_\square\|_{L^{2(n+1)/(n-1)}(S)}$, which we denote λ. For each dyadic number λ, we use $\mathbb{S}_\lambda$ to stand for the collection of tubes $S \subset \square$ with $\|e^{it\Delta} f_\square\|_{L^{2(n+1)/(n-1)}(S)} \sim \lambda$.

2. For each λ, we sort the tubes $S \in \mathbb{S}_\lambda$ by looking at the number of such tubes in an $R^{3/4}$-strip. For any dyadic number η, we let $\mathbb{S}_{\lambda,\eta}$ be the set of tubes $S \in \mathbb{S}_\lambda$ so that the number of tubes of $\mathbb{S}_\lambda$ in the $R^{3/4}$-strip containing S is $\sim \eta$.

Let $Y_{\square,\lambda,\eta}$ be the union of the tubes in $\mathbb{S}_{\lambda,\eta}$. Then we represent

$$e^{it\Delta} f = \sum_{\lambda,\eta} \left(\sum_\square e^{it\Delta} f_\square \cdot \chi_{Y_{\square,\lambda,\eta}} \right).$$

Since there are $O(\log R)$ choices for each of λ, η, by pigeonholing, we can choose λ, η so that

$$\|e^{it\Delta} f\|_{L^{2(n+1)/(n-1)}(Q_j)} \lesssim (\log R)^2 \Big\| \sum_\square e^{it\Delta} f_\square \cdot \chi_{Y_{\square,\lambda,\eta}} \Big\|_{L^{2(n+1)/(n-1)}(Q_j)} \tag{4.32}$$

holds for a fraction ≈ 1 of all cubes Q_j in Y. We need this uniform choice of (λ, η), which is independent of Q_j, because later we will sum over all Q_j and arrive at $\|e^{it\Delta} f_\square\|_{L^{2(n+1)/(n-1)}(Y_{\square,\lambda,\eta})}$.

We fix λ and η for the rest of the proof. Let $Y_\Box$ stand for the abbreviation of $Y_{\Box,\lambda,\eta}$. We note that $Y_\Box$ obeys the hypotheses for our inductive estimate (4.29), with $\sigma_\Box$ being the value of η that we have fixed.

The following geometric estimate will play a crucial role in our proof. Each set $Y_\Box$ contains $\lesssim \sigma_\Box$ tubes in each strip parallel to the short axes of $\Box$. Since the angle between the short axes of $\Box$ and the x-axes is bounded away from $\pi/2$, it follows that $Y_\Box$ contains $\lesssim \sigma_\Box$ cubes Q_j in any $R^{1/2}$-horizontal row. Therefore,

$$|Y_\Box \cap Y| \lesssim \frac{\sigma_\Box}{\sigma}|Y|. \tag{4.33}$$

Next we sort the the boxes $\Box$ according to the dyadic size of $\|f_\Box\|_{L^2}$. We can restrict matters to $\lesssim \log R$ choices of this dyadic size, and so we can choose a set of $\Box$'s, $\mathbb{B}$, so that $\|f_\Box\|_{L^2}$ is essentially constant for $\Box \in \mathbb{B}$ and

$$\|e^{it\Delta}f\|_{L^{2(n+1)/(n-1)}(Q_j)} \lessapprox \|\sum_{\Box\in\mathbb{B}} e^{it\Delta}f_\Box \cdot \chi_{Y_\Box}\|_{L^{2(n+1)/(n-1)}(Q_j)} \tag{4.34}$$

for a fraction ≈ 1 of cubes Q_j in Y.

Finally we sort the cubes $Q_j \subset Y$ according to the number of $Y_\Box$ that contain them. We let $Y' \subset Y$ be a set of cubes Q_j which obey (4.34) and which each lies in $\sim \mu$ of the sets $\{Y_\Box\}_{\Box\in\mathbb{B}}$. Because (4.34) holds for a large fraction of cubes, and because there are only dyadically many choices of μ, $|Y'| \approx |Y|$. By the equation (4.33), we see that

$$|Y_\Box \cap Y'| \leq |Y_\Box \cap Y| \lessapprox \frac{\sigma_\Box}{\sigma}|Y| \approx \frac{\sigma_\Box}{\sigma}|Y'|.$$

Therefore, the multiplicity μ is bounded by

$$\mu \lessapprox \frac{\sigma_\Box}{\sigma}|\mathbb{B}|. \tag{4.35}$$

We now are ready to combine all our ingredients and finish our proof. For each $Q_j \subset Y'$, we have

$$\|e^{it\Delta}f\|_{L^{2(n+1)/(n-1)}(Q_j)} \lessapprox \left\|\sum_{\Box\in\mathbb{B}} e^{it\Delta}f_\Box \cdot \chi_{Y_\Box}\right\|_{L^{2(n+1)/(n-1)}(Q_j)}.$$

Now we apply Lemma 4.7 to the function $\sum_{\Box\in\mathbb{B}, Q_j\subset Y_\Box} f_\Box$ to bound the right hand side by

$$\lessapprox \left(\sum_{\Box\in\mathbb{B}, Q_j\subset Y_\Box} \left\|e^{it\Delta}f_\Box\right\|^2_{L^{2(n+1)/(n-1)}(Q_j)}\right)^{1/2}.$$

Since the number of $Y_\Box$ containing Q_j is $\sim \mu$, we can apply Hölder to get

$$\left\| \sum_{\Box \in \mathbb{B}} e^{it\Delta} f_\Box \cdot \chi_{Y_\Box} \right\|_{L^{2(n+1)/(n-1)}(Q_j)} \lessapprox \mu^{\frac{1}{n+1}} \left(\sum_{\Box \in \mathbb{B}, Q_j \subset Y_\Box} \| e^{it\Delta} f_\Box \|_{L^{2(n+1)/(n-1)}(Q_j)}^{2(n+1)/(n-1)} \right)^{(n-1)/2(n+1)} .$$

Now we raise to the $2(n+1)/(n-1)$-th power and sum over $Q_j \subset Y'$ to get

$$\| e^{it\Delta} f \|_{L^{2(n+1)/(n-1)}(Y')}^{2(n+1)/(n-1)} \lessapprox \mu^{\frac{2}{n-1}} \sum_{\Box \in \mathbb{B}} \| e^{it\Delta} f_\Box \|_{L^{2(n+1)/(n-1)}(Y_\Box)}^{2(n+1)/(n-1)} .$$

Since $|Y'| \gtrapprox |Y|$, and since each cube $Q_j \subset Y$ makes an equal contribution to $\|e^{it\Delta} f\|_{L^{2(n+1)/(n-1)}(Y)}$, we see that $\|e^{it\Delta} f\|_{L^{2(n+1)/(n-1)}(Y)} \approx \|e^{it\Delta} f\|_{L^{2(n+1)/(n-1)}(Y')}$ and so

$$\| e^{it\Delta} f \|_{L^{2(n+1)/(n-1)}(Y)}^{2(n+1)/(n-1)} \lessapprox \mu^{\frac{2}{n-1}} \sum_{\Box \in \mathbb{B}} \| e^{it\Delta} f_\Box \|_{L^{2(n+1)/(n-1)}(Y_\Box)}^{2(n+1)/(n-1)} .$$

By a parabolic rescaling and applying our inductive hypothesis (4.29) at scale $R^{1/2}$ to the right-hand side, we see that

$$\| e^{it\Delta} f \|_{L^{2(n+1)/(n-1)}(Y)}^{2(n+1)/(n-1)} \lessapprox E^{O(1)} R^{-\frac{1}{n-1}} \mu^{\frac{2}{n-1}} \sigma_\Box^{-\frac{2}{n-1}} \sum_{\Box \in \mathbb{B}} \| f_\Box \|_{L^2}^{2(n+1)/(n-1)}. \tag{4.36}$$

Plugging in our bound for μ in (4.35), this is bounded by

$$\lesssim E^{O(1)} R^{-\frac{1}{n-1}} \sigma^{-\frac{2}{n-1}} |\mathbb{B}|^{\frac{2}{n-1}} \sum_{\Box \in \mathbb{B}} \| f_\Box \|_{L^2}^{2(n+1)/(n-1)}.$$

Now since $\|f_\Box\|_{L^2}$ is essentially constant among all $\Box \in \mathbb{B}$, the last expression is

$$\sim E^{O(1)} R^{-\frac{1}{n-1}} \sigma^{-\frac{2}{n-1}} (\sum_{\Box \in \mathbb{B}} \| f_\Box \|_{L^2}^2)^{(n+1)/(n-1)} \leq E^{O(1)} R^{-\frac{1}{n-1}} \sigma^{-\frac{2}{n-1}} \| f \|_{L^2}^{2(n+1)/(n-1)}.$$

Taking the $2(n+1)/(n-1)$-th root, we obtain our desired bound:

$$\| e^{it\Delta} f \|_{L^{2(n+1)/(n-1)}(Y)} \lessapprox E^{O(1)} R^{-\frac{1}{2(n+1)}} \sigma^{-\frac{1}{n+1}} \| f \|_{L^2}.$$

This closes the induction on radius and completes the proof.

4.2.2 Proof of Proposition 4.4

Since f is concentrated in wave packets from $\mathbf{T}_Z(E)$, we decompose $N_{ER^{1/2}}Z$ into balls Q of radius $R^{1/2}$. Let η be a smooth bump function approximating χ_Q. As we saw in the proof of Lemma 4.7, in Equation (4.27), the Fourier support of each function $\eta e^{it\Delta}f$ is essentially supported on

$$S^* := \{(\xi_1, \cdots, \xi_n, \xi_{n+1}) : (\xi_1, \cdots, \xi_n) \in S \text{ and } |\xi_{n+1} + \xi_1^2 + \cdots + \xi_n^2| \lesssim R^{-1/2}\},$$

where $S \subset B^n(0,1)$ is a strip of width $ER^{-1/2}$. Therefore, the Fourier support of $\eta e^{it\Delta}f$ is contained in a rectangle of dimensions $\sim ER^{-1/2} \times 1 \times \cdots \times 1$. We denote this rectangle by $A^*(Q)$. Since the Fourier support of each $\eta e^{it\Delta}f_i$ is contained in $A^*(Q)$, $|\eta e^{it\Delta}f_i|$ is morally constant on dual rectangles with dimensions $1 \times \cdots \times 1 \times E^{-1}R^{1/2}$. We tile Q with such dual rectangles, which we denote $A_k(Q)$. The projection of each dual rectangle $A_k(Q)$ to the x-plane is an n-dimensional $1 \times \cdots \times 1 \times E^{-1}R^{1/2}$-rectangle.

Suppose that $\sup_{0<t\le R} |e^{it\Delta}f| \sim H$ on a set $U \subset B^n(0,R)$. It suffices for us to prove the bound

$$H|U|^{1/2} \lessapprox E^{O(1)} R^{\frac{n}{2(n+1)}} \|f\|_2. \tag{4.37}$$

We will bound $|U|$ using the rectangles $A_k(Q)$. For the time being, let us suppose that $|\eta e^{it\Delta}f|$ is roughly constant on each $A_k(Q)$. This is not quite rigorous, but useful for intuition. On the next page, we will come back to this point and give a rigorous argument.

There must be a collection of dual rectangles $A_k(Q_j)$ whose projections cover U and so that $|e^{it\Delta}f| \sim H$ on each dual rectangle. We let X denote the union of these dual rectangles. Each $1 \times \cdots \times 1 \times E^{-1}R^{1/2}$ rectangle $A_k(Q_j) \subset X$ has a projection with area $E^{-1}R^{1/2}$, and since these projections cover U, we have the bound

$$|U| \lesssim |X|. \tag{4.38}$$

We can also assume that no two rectangles $A_k(Q_j) \subset X$ have essentially the same projection. This implies that X contains $\lesssim E^{O(1)}R^{(n-1)/2}$ rectangles $A_k(Q)$ in each cube Q. So for each cube Q, we get the bound

$$|X \cap Q| \lesssim E^{O(1)} R^{n/2}. \tag{4.39}$$

We consider the $R^{1/2}$-cubes Q in $B^n(R) \times [0,R]$ that intersect X. We sort these $R^{1/2}$-cubes Q according to the dyadic value of $\left\|e^{it\Delta}f\right\|_{L^{2(n+1)/(n-1)}(Q)}$. We can choose a set of of $R^{1/2}$-cubes Q_j, $j = 1, 2, \cdots, N$, so that

$$\left\|e^{it\Delta} f\right\|_{L^{2(n+1)/(n-1)}(Q_j)} \text{ is essentially constant in } j, \tag{4.40}$$

and $|X| \lessapprox |X \cap Y|$, where $Y := \bigcup_{j=1}^{N} Q_j$. Using the locally constant property that $|e^{it\Delta} f| \sim H$ on each rectangle $A_k(Q_j) \subset X$, we see that

$$H|X|^{\frac{n-1}{2(n+1)}} \lessapprox E^{O(1)} \left\|e^{it\Delta} f\right\|_{L^{2(n+1)/(n-1)}(Y)} . \tag{4.41}$$

Since $|X \cap Q_j| \lesssim E^{O(1)} R^{n/2}$ for each cube Q_j, $j = 1, ...N$, we see that $|X| \lessapprox |X \cap Y| \lesssim E^{O(1)} N R^{n/2}$. Therefore,

$$H|X|^{1/2} \lessapprox E^{O(1)} N^{\frac{1}{n+1}} R^{\frac{n}{2(n+1)}} \left\|e^{it\Delta} f\right\|_{L^{2(n+1)/(n-1)}(Y)} . \tag{4.42}$$

Since $|U| \lesssim |X|$, we have

$$H|U|^{1/2} \lessapprox E^{O(1)} N^{\frac{1}{n+1}} R^{\frac{n}{2(n+1)}} \left\|e^{it\Delta} f\right\|_{L^{2(n+1)/(n-1)}(Y)} . \tag{4.43}$$

Finally, we can find a sub-collection of $R^{1/2}$-cubes $Y' \subset Y$ with $|Y'| \approx |Y|$ so that the cubes in Y' are arranged in horizontal strips of the form $\mathbb{R} \times \cdots \mathbb{R} \times [t_0, t_0 + R^{1/2}]$, and that each strip contains $\sim \sigma$ cubes in Y'. Note that

$$\sigma \gtrapprox \frac{N}{R^{1/2}} . \tag{4.44}$$

By Theorem 4.3, we get

$$\left\|e^{it\Delta} f\right\|_{L^{2(n+1)/(n-1)}(Y)} \approx \left\|e^{it\Delta} f\right\|_{L^{2(n+1)/(n-1)}(Y')}$$

$$\lessapprox E^{O(1)} R^{-\frac{1}{2(n+1)}} \left(\frac{N}{R^{1/2}}\right)^{-\frac{1}{n+1}} \|f\|_2 ,$$

combining this bound with (4.43), we have

$$H|U|^{1/2} \lessapprox E^{O(1)} R^{\frac{n}{2(n+1)}} \|f\|_2 ,$$

and our desired bound (4.37) follows.

It only remains to deal with the issue of $|e^{it\Delta}f|$ being morally roughly constant on each rectangle $A_k(Q)$. We used the locally constant property to justify (4.41) above. We can rigorously prove (4.41) as follows. We mentioned above that the function $\eta_Q e^{it\Delta}f$ has Fourier transform essentially supported in a rectangle $A^*(Q)$ of dimensions $\sim ER^{-1/2}\times 1\times\cdots\times 1$. If $\hat{\psi}$ is designed to be identically 1 on this rectangle, then $(\eta_Q e^{it\Delta}f)*\psi$ is essentially equal to $\eta_Q e^{it\Delta}f$. We can choose such a ψ where $|\psi|$ is a rapidly-decaying approximation of $|A_k(Q_j)|^{-1}\chi_{A_k(Q_j)}$. Therefore, we see that

$$\sup_{A_k(Q)}|e^{it\Delta}f| \lesssim R^{O(\delta)}\frac{\int_{R^\delta A_k(Q)}|e^{it\Delta}f|}{|A_k(Q_j)|} + R^{-1000n}\|f\|_{L^2}, \tag{4.45}$$

where the second term accounts for the tail of ψ. Since $E\geq R^\delta$, we can assume that $R^\delta A_k(Q)\subset Q$.

We let X be a union of rectangles $A_k(Q_j)$ which each obeys

$$H\lesssim \sup_{A_k(Q_j)}|e^{it\Delta}f|.$$

We can arrange that the projections of $10A_k(Q_j)$ cover U and also that any two rectangles $A_k(Q_j)$ in X have essentially different projections. Because of this covering, we still have $|U|\lesssim |X|$. Now if $H\lesssim R^{-100n}\|f\|_{L^2}$, then (4.37) follows trivially. Therefore, (4.45) tells us that for each $A_k(Q_j)\subset X$:

$$\int_{R^\delta A_k(Q)}|e^{it\Delta}f|\gtrsim R^{-O(\delta)}|A_k(Q_j)|H.$$

We define Y just as above, and this inequality lets us rigorously justify (4.41):

$$H|X|^{\frac{n-1}{2(n+1)}}\approx H|X\cap Y|^{\frac{n-1}{2(n+1)}}\lessapprox E^{O(1)}\left\|e^{it\Delta}f\right\|_{L^{2(n+1)/(n-1)}(Y)}.$$

This completes the proof.

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