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A SHARP SMOOTHNESS OF THE CONJUGATION OF CLASS P-HOMEOMORPHISMS TO DIFFEOMORPHISMS

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Abstract

Let f be a class P-homeomorphism of the circle. We prove that there exists a piecewise analytic homeomorphism that conjugate f to a one-class P with prescribed break points lying on pairwise distinct orbits. As a consequence, we give a sharp estimate for the smoothness of a conjugation of class P-homeomorphism f of the circle satisfying the (D)-property (i.e. the product of f-jumps in the break points contained in a same orbit is trivial), to diffeomorphism. When f does not satisfy the (D)-property the conjugating homeomorphism is never a class P and even more it is not absolutely continuous function when the total product of f-jumps in all the break points is non-trivial.

1. Introduction

Denote by $S^1 = \mathbf{R}/\mathbf{Z}$ the circle and $p : \mathbf{R} \to S^1$ the canonical projection. Let f be an orientation preserving homeomorphism of S^1. The homeomorphism f admits a lift $\tilde{f} : \mathbf{R} \to \mathbf{R}$ that is an increasing homeomorphism of $\mathbf{R}$ such that $p \circ \tilde{f} = f \circ p$. Conversely, the projection of such a homeomorphism of $\mathbf{R}$ is an orientation preserving homeomorphism of S^1. The rotation number of a homeomorphism f of S^1 is defined as $\rho(f) = \lim_{n\to+\infty} \dfrac{\tilde{f}^n(x) - x}{n}$ (mod 1), $x \in \mathbf{R}$. This limit exists and is independent of the choice of the point x and the lift $\tilde{f}$ of f. For example, if $R_\alpha : x \mapsto x + \alpha$ (mod 1) is the rotation by angle α then it is obvious that $\rho(R_\alpha) = \alpha$ (mod 1). From the definition, $\rho(h \circ f \circ h^{-1}) = \rho(f)$ holds for any orientation preserving homeomorphism h of S^1. Recall that a homeomorphism f of S^1 is called a diffeomorphism (resp. a C^r-diffeomorphism ($r \geq 1$)) if it differentiable (resp. of class C^r) whose inverse f^{-1} is also differentiable (resp. of class C^r). Assuming f is a C^r-diffeomorphism ($r \geq 2$) and $\rho(f)$ is irrational, Denjoy ([3]) proved that: every C^r-diffeomorphism f ($r \geq 2$) of S^1 with irrational rotation number $\rho(f)$ is topologically conjugate to the rotation

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$R_{\rho(f)}$. This means that there exists an orientation preserving homeomorphism h of S^1 such that $f = h^{-1} \circ R_{\rho(f)} \circ h$. Denjoy noted that this result can be extended (with the same proof) to a large class of circle homeomorphisms: *the class P* (see [5], Chapter VI) and in particular for piecewise linear (PL) circle homeomorphisms.

Definition 1.1. An orientation preserving homeomorphism f of S^1 is called a *class P*-homeomorphism if it is derivable except at finitely many points, the so called *break points* of f, at which left and right derivatives (denoted, respectively, by Df_- and Df_+) exist and such that the derivative $\mathrm{Df} : S^1 \to \mathbf{R}_+^*$ has the following properties:

- There exist two constants $0 < a < b < +\infty$ such that: $a < \mathrm{Df}(x) < b$, for every x where Df exists,
- $a < \mathrm{Df}_+(c) < b$ and $a < \mathrm{Df}_-(c) < b$ at the break points c.
- $\log \mathrm{Df}$ has bounded variation on S^1 (i.e. the total variation of $\log \mathrm{Df}$ is finite).

We pointed out that the third condition implies the two ones. Also notice that if f is a class P-homeomorphism of S^1 which is C^1 on S^1 then f is a C^1-diffeomorphism of S^1.

Definition 1.2. An orientation preserving homeomorphism f of S^1 is called piecewise linear (PL-*homeomorphism*) if f is derivable except at finitely many break points $(c_i)_{0 \leq i \leq p}$ of S^1 such that the derivative Df is constant on each $]c_i, c_{i+1}[$.

Among the simplest examples of class P-homeomorphisms, we mention:

- C^2-diffeomorphisms,
- Piecewise linear PL-homeomorphisms which are not C^2-diffeomorphisms.

Denote by

- $\mathrm{Homeo}_+(S^1)$ the group of orientation-preserving homeomorphisms of S^1.
- $\mathscr{P}(S^1)$ the set of class P-homeomorphisms of S^1, it is a subgroup of $\mathrm{Homeo}_+(S^1)$.
- $\mathrm{PL}(S^1)$ the set of PL-homeomorphisms of S^1, it is a subgroup of $\mathscr{P}(S^1)$ which contains rotations.

In this paper, we are mainly concerned with the sharp estimate for the smoothness of a conjugation of class P-homeomorphism with the (D)-property (see Definition 1.3 and Theorem 1.7) to diffeomorphism. For class P-homeomorphism without the (D)-property, the conjugation is never piecewise C^1 (see Proposition 1.6) and even more, can be a singular function (see Corollary 1.9). Before stating the main result, we need the following notations and definitions.

For $f \in \mathscr{P}(S^1)$ and $x \in S^1$, denote by

- $O_f(x) := \{f^n(x) : n \in \mathbf{Z}\}$ called the *orbit* of x by f.

- $\sigma_f(x) := \dfrac{\mathrm{Df}_-(x)}{\mathrm{Df}_+(x)}$ called the *f-jump* in x.
- $C(f) = \{c_0, c_1, c_2, \dots, c_m\}$ the set of break points of f in S^1.
- $c_{m+1} := c_0$.
- $\pi_{s,O_f(c)}(f) = \prod_{x \in C(f) \cap O_f(c)} \sigma_f(x)$, for every $c \in C(f)$.
- $\pi_s(f)$ the product of f-jumps at the break points of f:

$$\pi_s(f) = \prod_{c \in C(f)} \sigma_f(c).$$

Definition 1.3 ([1]). Let $f \in \mathscr{P}(S^1)$. We say that f has the *(D)-property* if the product of f-jumps in the break points on each orbit is trivial; that is $\pi_{s,O_f(c)}(f) = 1$, for every $c \in C(f)$.

In particular, if f has the (D)-property, then $\pi_s(f) = 1$. Conversely, if all break points belong to the same orbit and $\pi_s(f) = 1$ then f has the (D)-property. We established in ([1], Proposition 2.5) that f has the (D)-property if and only if the number of break points of f^n is bounded by some constant that doesn't depend on n.

Definition 1.4 (Maximal connections). Let $f \in \mathscr{P}(S^1)$ and $c \in C(f)$. A *maximal f-connection* of c is a segment

$$[f^{-p}(c), \dots, f^q(c)] := \{f^s(d) : -p \le s \le q\}$$

of the orbit $O_f(c)$ which contains all the break points of f contained on $O_f(c)$ and such that $f^{-p}(c)$ (resp. $f^q(c)$) is the first (resp. last) break point of f on $O_f(c)$.

We have the following properties:
- Two break points of f are on the same maximal f-connection, if and only if, they are on the same orbit.
- Two distinct maximal f-connections are disjoint.

Notations. Let $f \in \mathscr{P}(S^1)$. We let
- $M_i(f) = [c_i, \dots, f^{N_i}(c_i)]$, $N_i \in \mathbf{N}$, the maximal f-connections of $c_i \in C(f)$, $(0 \le i \le p)$.
- $M(f) = \coprod_{i=0}^{p} M_i(f)$.
  So, we have the decomposition: $C(f) = \coprod_{i=0}^{p} C_i(f)$, where $C_i(f) = C(f) \cap M_i(f)$, $0 \le i \le p$. In particular, $C_i(f) \subset M_i(f)$.
- $N := \max_{0 \le i \le p} N_i$.

Note that if f has the (D)-property then:

$$- \prod_{d \in C_i(f)} \sigma_f(d) = \prod_{d \in M_i(f)} \sigma_f(d) = 1, \quad \text{for every } i = 0, \dots, p.$$

Define

- $\pi_{O_f(c_i)}(f) := \prod_{k=1}^{N_i} \sigma_{f^{N+1}}(f^k(c_i))$, $N_i > 0$. If $N_i = 0$, $\pi_{O_f(c_i)}(f) := 1$. By ([1], Lemma 2.7), we also have: $\pi_{O_f(c_i)}(f) := \prod_{j \in \mathbf{Z}}(\sigma_f(f^j(c_i)))^j$.
- $\pi(f) := \prod_{i=0}^{p} \pi_{O_f(c_i)}(f)$.

Let $\sigma \in \mathbf{R}_+^* \backslash \{1\}$. We shall introduce the two following basic class P-homeomorphisms. Denote by

- g_σ the orientation preserving homeomorphism of S^1 with lift $\widetilde{g_\sigma} : \mathbf{R} \to \mathbf{R}$ restricted to $[0, 1[$ is given by:

$$\tilde{g}_\sigma(x) = \left(\frac{1-\sigma}{1+\sigma}\right)\left(x^2 + \frac{2\sigma}{1-\sigma}x\right), \quad x \in [0, 1[.$$

We identify g_σ with its lift $\widetilde{g_\sigma}$. Since $g_\sigma(0) = 0$, $g_\sigma(1) = 1$ and $\sigma \neq 1$, $g_\sigma \in \mathscr{P}(S^1)$ has one break point 0 such that $\sigma_{g_\sigma}(0) = \sigma$. Moreover, g_σ is quadratic on $S^1 \backslash \{0\}$.

- h_σ the homeomorphism of S^1 with lift $\widetilde{h_\sigma} : \mathbf{R} \to \mathbf{R}$ restricted to $[0, 1[$ is given by:

$$\widetilde{h_\sigma}(x) = \frac{\sigma^x - 1}{\sigma - 1}, \quad x \in [0, 1[.$$

We identify h_σ with its lift $\widetilde{h_\sigma}$. Then $h_\sigma \in \mathscr{P}(S^1)$ has one break point 0 such that $\sigma_{h_\sigma}(0) = \sigma$. Moreover, h_σ is analytic on $S^1 \backslash \{0\}$.

Definition 1.5. A homeomorphism h of S^1 is called a PQ-homeomorphism (resp. PE-homeomorphism) of S^1 if $h = L \circ u$, where $L \in \mathrm{PL}(S^1)$ and $u = R_c \circ g_\sigma \circ R_c^{-1}$ (resp. $R_c \circ h_\sigma \circ R_c^{-1}$), for some $\sigma \in \mathbf{R}_+^* \backslash \{1\}$ and $c \in S^1$.

When f does not satisfy the (D)-property, there is no rigidity; the conjugating homeomorphism is never a class P:

Proposition 1.6. *Let $f \in \mathscr{P}(S^1)$ with irrational rotation number. If f does not satisfy the (D)-property, then it is not conjugate to a diffeomorphism through a class P-homeomorphism.*

When f satisfies the (D)-property, we are in the position to give our main result.

Theorem 1.7. *Let $f \in \mathscr{P}(S^1)$ with the (D)-property and irrational rotation number. Then:*

(i) *If $\pi(f) \neq 1$, f is conjugate to a diffeomorphism through a* PQ (*resp.* PE)-*homeomorphism* (*but not* PL-*homeomorphism*).
(ii) *If $\pi(f) = 1$, f is conjugate to a diffeomorphism through a* PL-*homeomorphism.*

In particular, for PL-homeomorphism, we obtain:

Corollary 1.8. *Let* $f \in \mathrm{PL}(S^1)$ *with the* (D)*-property and irrational rotation number* α. *Assume that* $\pi(f) = 1$. *Then* f *is conjugate to the rotation* R_α *through a* PL*-homeomorphism.*

Actually, using a recent result due to Adouani [2] and independently Dzhalilov et al. [4], we have the following corollary:

Corollary 1.9. *Let* $f \in \mathscr{P}(S^1)$ *with irrational rotation number. Assume that the derivatives* Df *is absolutely continuous on every continuity interval of* Df. *If* $\pi_s(f) \neq 1$ *then any homeomorphism map* h *conjugating* f *to a diffeomorphism of* S^1 *is a singular function i.e. it is continuous on* S^1 *and* $\mathrm{Dh}(x) = 0$ *a.e. with respect to the Lebesgue measure.*

Remark 1. When $\pi_s(f) = 1$, the homeomorphism map h conjugating f to a diffeomorphism can be either a singular function or absolutely continuous function. Teplinsky gave in [6] an example f of $\mathrm{PL}(S^1)$ with four break points lying on pairwise distinct orbits and irrational rotation number of Roth number (but not of bounded type), that is conjugated to the rigid rotation by an absolutely continuous function. It is obvious that such example satisfies $\pi_s(f) = 1$ and does not satisfy the (D)-property. However, Herman has shown in [5] (although not formulated as a statement) that a map $f \in \mathrm{PL}(S^1)$ with two breaks points lying on distinct orbits and irrational rotation number has singular invariant measure; equivalently the homeomorphism h conjugating f to the rigid rotation is a singular function.

This paper is organized as follows. Section 2 is devoted to the main technical part of the paper; we conjugate any class P-homeomorphism f with several break points through a PQ-homeomorphism (resp. PE-homeomorphism) of S^1 to a class P-homeomorphism with prescribed break points on *pairwise distinct orbits*. In Section 3, we study the case where f satisfies the (D)-property, we prove that it is conjugated through a PQ (resp. PE)-homeomorphism of S^1 to a diffeomorphism. In particular, we study the case where f has two successive break points. Section 4 is devoted to class P-homeomorphism without the (D)-property.

2. Reduction to a class P-homeomorphisms with prescribed points on pairwise distinct orbits

The aim of this section is to prove the following

Theorem 2.1. *Let* $f \in \mathscr{P}(S^1)$ *with irrational rotation number, and let* $(k_0, \dots, k_p) \in \mathbf{Z}^{p+1}$. *Then there exists a PQ-homeomorphism* (*resp. PE-homeomorphism*) $h \in \mathscr{P}(S^1)$ *such that* $F := h \circ f \circ h^{-1} \in \mathscr{P}(S^1)$ *with*

- $C(F) \subset \{h(f^{k_i}(c_i)) = F^{k_i}(h(c_i)); i = 0, 1, \dots, p\}$
- $\sigma_F(F^{k_i}(h(c_i))) = \pi_{s, O_f(c_i)}(f),\ i = 0, 1, \dots, p.$

We need the following lemma, for completeness we present its proof.

Lemma 2.2. *Let* $\sigma_0, \dots, \sigma_n \in \mathbf{R}_+^*$ *such that* $\sigma_0 \times \cdots \times \sigma_n = 1$ *and let* $b_0, \dots, b_n \in S^1$. *Then there exists* $L \in \mathrm{PL}(S^1)$ *with break points* $b_0, \dots, b_n$ *and slopes* $\sigma_L(b_0) = \sigma_0, \dots, \sigma_L(b_n) = \sigma_n$. *In particular,* $\pi_s(L) = 1$.

Proof. We let $b_0 = p(a_0), \dots, b_n = p(a_n)$, where $a_0 < a_1 < \cdots < a_n < a_n + 1$ be real numbers. Given $\tau > 0$, let L_τ be the PL function defined on $[a_0, a_0 + 1]$ as follows:

- $a_0, \dots, a_n$ are the break points of L_τ.
- L_τ is linear on each $[a_j, a_{j+1}]$.
- $L_\tau(a_0) = 0$.
- τ is the slope of L_τ on $[a_0, a_1]$.
- $\sigma_{L_\tau}(a_j) = \sigma_j$ the jump of L_τ in a_j, $j = 0, \dots, n$.

Then the function $\tau \mapsto L_\tau(a_0 + 1)$ is monotone and continuous. Moreover, it satisfies $\lim_{\tau \to 0} L_\tau(a_0 + 1) = 0$ and $\lim_{\tau \to +\infty} L_\tau(a_0 + 1) = +\infty$. Therefore there is $\tau_0 > 0$ such that $L_{\tau_0}(a_0 + 1) = 1$. The PL-homeomorphism L of S^1 is then defined by its lift L_{τ_0} restricted to $[a_0, a_0 + 1]$. □

Proof of Theorem 2.1. Set for $i = 0, \dots, p$ and $k \in \mathbf{Z}$:

$$m_i = \min(0, k_i), \quad n_i = \max(k_i, N_i)$$

$$\sigma(f) = \prod_{i=0}^{p} \prod_{k \in \mathbf{Z}} \sigma_{k,i}(f),$$

where

$$\sigma_{k,i}(f) = \begin{cases} \prod_{j \geq k} a_{j,i}(f), & \text{if } k > k_i \\ \dfrac{1}{\prod_{j<k} a_{j,i}(f)}, & \text{if } k \leq k_i \end{cases}$$

and

$$a_{k,i}(f) = \sigma_f(f^k(c_i)).$$

Then we obtain

$$\sigma(f) = \prod_{i=0}^{p} (\pi_{s, O_f(c_i)}(f))^{-k_i} \prod_{j \in \mathbf{Z}} (a_{j,i}(f))^j$$

Indeed, we have

$$a_{k,i}(f) = 1, \quad \text{if } k < 0 \text{ or } k > N_i$$

$$\sigma_{k,i}(f) = 1, \quad \text{if } k < m_i \text{ or } k > n_i$$

$$\begin{aligned}\prod_{k\le k_i}\sigma_{k,i}(f)&=\prod_{k\le k_i}\left(\prod_{j<k}(a_{j,i}(f))^{-1}\right)\\&=\prod_{j<k_i}\left(\prod_{j<k\le k_i}(a_{j,i}(f))^{-1}\right)\\&=\prod_{j<k_i}(a_{j,i}(f))^{j-k_i}\\&=\prod_{p<0}(a_{p+k_i,i}(f))^{p}\end{aligned}$$

Similarly,

$$\prod_{k>k_i}\sigma_{k,i}(f)=\prod_{p\ge 0}(a_{p+k_i,i}(f))^{p}$$

So

$$\begin{aligned}\prod_{k\in\mathbf{Z}}\sigma_{k,i}(f)&=\prod_{p\in\mathbf{Z}}(a_{p+k_i,i}(f))^{p}\\&=\prod_{j\in\mathbf{Z}}(a_{j,i}(f))^{j-k_i}\\&=(\pi_{s,O_f(c_i)}(f))^{-k_i}\prod_{j\in\mathbf{Z}}(a_{j,i}(f))^{j}\end{aligned}$$

Therefore

$$\begin{aligned}\sigma(f)&=\prod_{i=0}^{p}\prod_{k\in\mathbf{Z}}\sigma_{k,i}(f)\\&=\prod_{i=0}^{p}(\pi_{s,O_f(c_i)}(f))^{-k_i}\prod_{j\in\mathbf{Z}}(a_{j,i}(f))^{j}\end{aligned}$$

Now, set

$$b_{k,i}(f)=\frac{\sigma_{k+1,i}(f)}{\sigma_{k,i}(f)}a_{k,i}(f).$$

Then we obtain

$$b_{k,i}(f)=\begin{cases}\pi_{s,O_f(c_i)}(f), & \text{if } k=k_i\\ 1, & \text{otherwise}\end{cases}$$

Indeed:

For $k > k_i$,

$$\begin{aligned}\sigma_{k,i}(f) &= \prod_{j \geq k} a_{j,i}(f)\\ &= a_{k,i}(f) \prod_{j \geq k+1} a_{j,i}(f)\\ &= a_{k,i}(f)\sigma_{k+1,i}(f)\end{aligned}$$

For $k < k_i$,

$$\begin{aligned}\sigma_{k,i}(f) &= \frac{1}{\prod_{j<k} a_{j,i}(f)}\\ &= \frac{a_{k,i}(f)}{\prod_{j<k+1} a_{j,i}(f)}\\ &= a_{k,i}(f)\sigma_{k+1,i}(f)\end{aligned}$$

For $k = k_i$,

$$\begin{aligned}b_{k,i}(f) &= \frac{\sigma_{k+1,i}(f)}{\sigma_{k,i}(f)} a_{k,i}(f)\\ &= \frac{\prod_{j \geq k+1} a_{j,i}(f)}{(\prod_{j<k} a_{j,i}(f))^{-1}} a_{k,i}(f)\\ &= \prod_{j \in \mathbf{Z}} a_{j,i}(f)\\ &= \pi_{s, O_f(c_i)}(f)\end{aligned}$$

We distinguish two cases.

Case 1: $\sigma(f) = 1$. By Lemma 2.2, there exists $L \in \mathrm{PL}(S^1)$ with the following properties:

(i) $L(0) = 0$
(ii) $C(L) \subset \{f^k(c_i) : m_i \leq k \leq n_i, 0 \leq i \leq p\}$
(iii) $\sigma_L(f^k(c_i)) = \sigma_{k,i}(f)$

We let $F = L \circ f \circ L^{-1}$. A priori, the break points of F are:

- The break points of L^{-1}: $L(f^k(c_i))$, $m_i \leq k \leq n_i$, $0 \leq i \leq p$,
- The image by L of break points of f: $L(f^k(c_i))$, $m_i - 1 \leq k \leq n_i$, $0 \leq i \leq p$.

Therefore the possible break points of F are among: $L(f^k(c_i))$, $m_i \leq k \leq n_i$, $0 \leq i \leq p$. Compute the jumps of F in these points:

$$\begin{aligned}\sigma_F(L(f^k(c_i))) &= \frac{\sigma_L(f(f^k(c_i)))\sigma_f(f^k(c_i))}{\sigma_L(f^k(c_i))}\\ &= \frac{\sigma_{k+1,i}(f)a_{k,i}(f)}{\sigma_{k,i}(f)}\\ &= b_{k,i}(f)\\ &= \begin{cases}\pi_{s,O_f(c_i)}(f), & \text{if } k = k_i\\ 1, & \text{otherwise}\end{cases}\end{aligned}$$

We conclude that $C(F) \subset \{L(f^k(c_i)) : 0 \le i \le p\}$ with $\sigma_F(L(f^k(c_i))) = \pi_{s,O_f(c_i)}(f)$.

Case 2: $\sigma(f) \ne 1$. Set $\sigma = \sigma(f)$ and define $u = R_c \circ g_\sigma \circ R_c^{-1}$ (resp. $u = R_c \circ h_\sigma \circ R_c^{-1}$), where $c = f^{N_0+1}(c_0)$. Then u is a particular PQ-homeomorphism (resp. PE-homeomorphism) with one break point c such that: $\sigma_u(c) = \sigma$. We let $F = u \circ f \circ u^{-1}$. A priori, the break points of F are:

– The break point of $u^{-1} : u(f^{N_0+1}(c_0))$

– The image by u of break points of f: $u(f^k(c_i))$, $0 \le k \le N_i$, $0 \le i \le p$

– The image by $u \circ f^{-1}$ of the break point of $u : u(f^{N_0}(c_0))$

Therefore the possible break points of F are among $u(f^k(c_i))$, $0 \le k \le N_i$, $1 \le i \le p$, and $u(f^k(c_0))$, $0 \le k \le N_0 + 1$.

Compute the jumps of F in these points:

For $0 \le k \le N_i$, $1 \le i \le p$,

$$\begin{aligned}\sigma_F(u(f^k(c_i))) &= \frac{\sigma_L(f(f^k(c_i)))\sigma_f(f^k(c_i))}{\sigma_u(f^k(c_i))}\\ &= a_{k,i}(f)\\ \sigma_F(u(f^{N_0}(c_0))) &= \frac{\sigma_u(f(f^{N_0+1}(c_0)))\sigma_f(f^{N_0}(c_0))}{\sigma_u(f^{N_0}(c_0))}\\ &= \frac{\sigma(f)a_{N_0,0}(f)}{1}\\ &= \sigma(f)a_{N_0,0}(f)\\ \sigma_F(u(f^{N_0+1}(c_0))) &= \frac{\sigma_u(f(f^{N_0+2}(c_0)))\sigma_f(f^{N_0+1}(c_0))}{\sigma_u(f^{N_0+1}(c_0))}\\ &= \frac{1 \times 1}{\sigma(f)}\\ &= \frac{1}{\sigma(f)}\\ \sigma_F(u(f^k(c_0))) &= a_{k,0}(f), \quad 0 \le k < N_0\end{aligned}$$

Let $a_{k,i}(F) := \sigma_F(F^k(d_i)) = \sigma_F(u(f^k(c_i)))$, where $d_i = u(c_i)$, for $0 \leq k \leq N_0$, $0 \leq i \leq p$. Then,

$$\sigma_{k,i}(F) = \begin{cases} \prod_{j \geq k} a_{j,i}(F), & \text{if } k > k_i \\ \dfrac{1}{\prod_{j<k} a_{j,i}(F)}, & \text{if } k \leq k_i \end{cases}$$

For $1 \leq i \leq p$,

$$\begin{aligned} \prod_{k \in \mathbf{Z}} \sigma_{k,i}(F) &= (\pi_{s,\, O_F(d_i)}(F))^{-k_i} \prod_{j \in \mathbf{Z}} (a_{j,i}(F))^j \\ &= (\pi_{s,\, O_f(c_i)}(f))^{-k_i} \prod_{j \in \mathbf{Z}} (a_{j,i}(f))^j \\ &= \prod_{k \in \mathbf{Z}} \sigma_{k,i}(f) \end{aligned}$$

$$\begin{aligned} \prod_{k \in \mathbf{Z}} \sigma_{k,0}(F) &= (\pi_{s,\, O_F(d_0)}(F))^{-k_0} \prod_{j \in \mathbf{Z}} (a_{j,0}(F))^j \\ &= (\pi_{s,\, O_f(c_0)}(f))^{-k_0} (a_{N_0,0}(F))^{N_0} (a_{N_0+1,0}(F))^{N_0+1} \\ &\quad \times \prod_{j \in \mathbf{Z}, j \neq N_0, j \neq N_0+1} (a_{j,0}(F))^j \\ &= (\pi_{s,\, O_f(c_0)}(f))^{-k_0} (\sigma(f) a_{N_0,0}(f))^{N_0} \left(\frac{a_{N_0+1,0}(f)}{\sigma(f)} \right)^{N_0+1} \\ &\quad \times \prod_{j \in \mathbf{Z}, j \neq N_0, j \neq N_0+1} (a_{j,0}(f))^j \\ &= \frac{1}{\sigma(f)} (\pi_{s,\, O_f(c_0)}(f))^{-k_0} \prod_{j \in \mathbf{Z}} (a_{j,0}(f))^j \\ &= \frac{1}{\sigma(f)} \prod_{j \in \mathbf{Z}} \sigma_{j,0}(f) \end{aligned}$$

Therefore

$$\begin{aligned} \sigma(F) &= \prod_{i=0}^{p} \prod_{j \in \mathbf{Z}} \sigma_{j,i}(F) \\ &= \frac{1}{\sigma(f)} \prod_{i=0}^{p} \prod_{j \in \mathbf{Z}} \sigma_{j,i}(f) \\ &= 1 \end{aligned}$$

We conclude that $F \in \mathscr{P}(S^1)$ that satisfies $\sigma(F) = 1$ and with maximal F-connections

$$M_0(F) = [u(c_0), \dots, F^{N_0+1}(u(c_0))] \quad \text{and} \quad M_i(F) = [u(c_i), \dots, F^{N_i}(u(c_i))],$$

for $1 \le i \le p$.

Then, by the case 1, there exists $L \in \mathrm{PL}(S^1)$ that conjugates F to a class P-homeomorphism $G = L \circ F \circ L^{-1}$ with $C(G) \subset \{G^{k_i}[L(u(c_i))] : 0 \le i \le p\}$ and $\sigma_G(G^{k_i}([L(u(c_i))])) = \pi_{s, O_f(c_i)}(f)$, $0 \le i \le p$. Moreover $h := L \circ u$ is a *PQ*-homeomorphism (resp. *PE*-homeomorphism) that conjugates f to G with $C(G) \subset \{G^{k_i}(h(c_i)) : 0 \le i \le p\}$ and $\sigma_G(G^{k_i}(h(c_i))) = \pi_{s, O_f(c_i)}(f)$, $0 \le i \le p$. This completes the proof. □

Corollary 2.3. *Let $f \in \mathscr{P}(S^1)$ with irrational rotation number. Then, there exists $h \in \mathscr{P}(S^1)$ such that: $F = h \circ f \circ h^{-1} \in \mathscr{P}(S^1)$ with $C(F) \subset \{h(c_0), \dots, h(c_p)\}$, where $c_0, \dots, c_p \in C(f)$ are on pairwise distinct orbits. Moreover $\sigma_F(h(c_i)) = \pi_{s, O_f(c_i)}(f)$, $i = 0, 1, \dots, p$.*

Proof. Take $k_i = 0$ for all i in Theorem 2.1. So we get $F = h \circ f \circ h^{-1} \in \mathscr{P}(S^1)$ with $C(F) \subset \{h(c_0), \dots, h(c_p)\}$, where $c_0, \dots, c_p \in C(f)$ are on pairwise distinct orbits. □

3. Class *P*-homeomorphisms with the (D)-property

3.1. Proof of Theorem 1.7.

Lemma 3.1. *Let $f \in \mathscr{P}(S^1)$ with irrational rotation number. If f has the (D)-property then $\sigma(f) = \pi(f)$.*

Proof. We have

$$\sigma(f) = \prod_{i=0}^{p} (\pi_{s, O_f(c_i)}(f))^{-k_i} \prod_{j \in \mathbf{Z}} (a_{j,i}(f))^j.$$

Since $\pi_{s, O_f(c_i)} = 1$ and $\prod_{j \in \mathbf{Z}} (a_{j,i}(f))^j = \pi(f)$, for every $i = 0, \dots, p$, thus $\sigma(f) = \pi(f)$. □

Proof of Theorem 1.7. From the Corollary 2.3, it follows that $F : h \circ f \circ h^{-1}$ is a diffeomorphism since $\sigma_F(h(c_i)) = \pi_{s, O_f(c_i)}(f) = 1$, $i = 0, 1, \dots, p$. Now by the proof of Theorem 2.1, h is a PL-homeomorphism if $\sigma(f) = 1$ and a PQ (resp. PE)-homeomorphism if $\sigma(f) \ne 1$. We conclude by the Lemma 3.1. □

Remark 2. The PE (resp. PQ)-homeomorphism $h = L \circ u$ that conjugates f to a diffeomorphism can be chosen so that its rotation number is 0. Indeed, let $u = R_c \circ g_\sigma \circ R_c^{-1}$ (resp. $R_c \circ h_\sigma \circ R_c^{-1}$), for some $\sigma \in \mathbf{R}_+^* \backslash \{1\}$ and $c \in S^1$. Set

$d = R_c(0)$ and choose $L \in \mathrm{PL}(S^1)$ such that $L(d) = d$. Then $h(d) = d$ and so h has a rotation number 0.

3.2. Case of two break points. Let $f \in \mathscr{P}(S^1)$ with irrational rotation number α and with two break points b and $f(b)$. Assume that f satisfies the (D)-property. We give a direct conjugation h from f to a diffeomorphism. This conjugation h is different from that constructed in the proof of Theorem 2.1. We let $b' = f(b)$ and $\sigma = \pi(f)^{-1}$. Define $h := R_{b'} \circ h_\sigma^{-1} \circ (R_{b'})^{-1}$. Then h is a PE-homeomorphism with one break point b' such that: $\sigma_h(b') = \sigma^{-1}$.

Proposition 3.2. *Let $f \in \mathscr{P}(S^1)$ with two break points b and $f(b)$ and irrational rotation number. Then $\pi(f) \neq 1$ and $F := h \circ f \circ h^{-1} \in \mathscr{P}(S^1)$ with $C(F) \subset \{h(b)\}$ such that $\sigma_F(h(b)) = \pi_{s,O_f(b)}(f)$.*

In particular if f satisfies the (D)-property then F is a diffeomorphism.

Proof. One has $\pi(f) = \sigma_f(b') \neq 1$, where $b' = f(b)$. We let $F = h \circ f \circ h^{-1}$. Then

$$\sigma_F(h(b')) = \frac{\sigma_h(f(b'))\sigma_f(b')}{\sigma_h(b')}.$$

As $\sigma_h(f(b')) = 1$ and $\sigma_f(b') = \sigma^{-1}$, so $\sigma_F(h(b')) = 1$.

On the other hand, we have:

$$\sigma_F(h(b)) = \frac{\sigma_h(b')\sigma_f(b)}{\sigma_h(b)}.$$

As $\sigma_h(b') = \sigma^{-1}$, $\sigma_h(b) = 1$ and $\sigma_f(b) = \dfrac{\pi_{s,O_f(b)}(f)}{\sigma^{-1}}$ then $\sigma_F(h(b)) = \pi_{s,O_f(b)}(f)$ and $C(F) \subset \{h(b)\}$.

In particular, if f satisfies the (D)-property then $\pi_{s,O_f(b)}(f) = 1$. So F has no break points and F is a diffeomorphism. □

Corollary 3.3. *Let $f \in \mathrm{PL}(S^1)$ with irrational rotation number α and with two break points b and $f(b)$. Then $h \circ f \circ h^{-1}$ is the rotation R_α.*

Proof. By hypothesis, we see that f satisfies the (D)-property. One can check that $h_\sigma^{-1} \circ ((R_{b'})^{-1} \circ f \circ R_{b'}) \circ h_\sigma = R_\alpha$ and therefore $h \circ f \circ h^{-1} = R_\alpha$. □

4. Class P-homeomorphisms without the (D)-property

Proof of Proposition 1.6. Suppose there is $h \in \mathrm{P}(S^1)$ that conjugates f to a diffeomorphism F: $f = h \circ F \circ h^{-1}$. Let $c \in S^1$. We have for every $j \in \mathbf{Z}$:

$$\sigma_f(f^j(h(c))) = \frac{\sigma_h(F(F^j(c)))}{\sigma_h(F^j(c))}\sigma_F(F^j(c)) = \frac{\sigma_h(F(F^j(c)))}{\sigma_h(F^j(c))}.$$

Hence

$$\pi_{s, O_f(h(c))}(f) = \prod_{j \in \mathbf{Z}} \sigma_f(f^j(h(c))) = \frac{\prod_{j\in\mathbf{Z}} \sigma_h(F^{j+1}(c))}{\prod_{j\in\mathbf{Z}} \sigma_h(F^j(c))} = 1.$$

We conclude that f has the (D)-property. Notice that we also have

$$\pi(f) = \pi_s(h)^{-1}. \qquad \square$$

Proof of Corollary 1.9. This follows directly from ([2], Main Theorem) since $\pi_s(F) = 1$ for any diffeomorphism F of S^1. ☐

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