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A SHORT AND CONSTRUCTIVE PROOF OF TARSKI'S FIXED-POINT THEOREM

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Abstract. I give short and constructive proofs of Tarski's fixed-point theorem, and of Zhou's extension of Tarski's fixed-point theorem to set-valued maps.

1. Introduction

I give short and constructive proofs of two related fixed-point theorems. The first is Tarski's fixed-point theorem: If F is a monotone function on a non-empty complete lattice, the set of fixed points of F forms a non-empty complete lattice. The second is Zhou's [9] extension of Tarski's fixed-point theorem to set-valued functions: If $\varphi : X \to 2^X$ is monotone—when 2^X is endowed with the induced set order—the set of fixed-points of φ forms a non-empty complete lattice. Zhou's extension is important in the theory of games with strategic complementarities (see, for example, [6] or [8]).

When F is continuous as well as monotone, my proof is very simple (see Section 4). The proof when F is continuous is thus useful for teaching game theory—if one wishes to prove a fixed-point theorem, but finds Kakutani's too involved, one can teach Tarski's. [1]

Tarski's [5] original proof is beautiful and elegant, but non-constructive and somewhat uninformative. Cousot and Cousot [1] give a constructive proof of Tarski's fixed-point theorem. But their proof is long and quite involved. I present a simpler, and succinct, proof. On the other hand, Cousot and Cousot obtain certain sub-products from their approach that I do not obtain; I shall only be concerned with Tarski's fixed-point theorem, and its extension to set-valued functions.

The extension to set-valued functions was developed by Smithson [4] and Zhou [9]. Earlier, Vives [7] proved a stronger version of the extension, which applied to games with strict strategic complementarities.

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[1] I have taught Tarski's Theorem with F continuous to Caltech undergraduates.

I give a constructive proof of Zhou's version of the result. Smithson has a weaker monotonicity requirement than Zhou, but Smithson does not obtain a lattice structure of the set of fixed-points. In addition, Smithson needs a continuity assumption.

2. Definitions

An in-depth discussion of the following concepts can be found in [6]. A set X endowed with a partial order $\leq$ is denoted $\langle X, \leq \rangle$; $\langle X, \leq \rangle$ is a *complete lattice* if, for all nonempty $B \subseteq X$, the greatest lower bound $\bigwedge_X B$ and the least upper bound $\bigvee_X B$ exist in X. If $A \subseteq X$, say that A is a *subcomplete sublattice* of $\langle X, \leq \rangle$ if, for all nonempty $B \subseteq A$, $\bigwedge_X B \in A$, and $\bigvee_X B \in A$.

Note that $\langle A, \leq \rangle$ may be a complete lattice, even if A is not a sublattice of $\langle A, \leq \rangle$. So, for $A' \subseteq A$, $\bigvee_A A'$ may differ from $\bigvee_X A'$.

Say that $A \subseteq X$ is smaller than $B \subseteq X$ in the *induced set order* (denoted $A \sqsubseteq B$) if

$$(x \in A, y \in B) \Rightarrow (x \wedge y \in A, x \vee y \in B).$$

The induced set order is a partial order on the set of sublattices of X.

Denote by $\preceq$ the usual linear order on ordinal numbers.

Let $\langle X, \leq_X \rangle$ be a lattice and $\langle Y, \leq_Y \rangle$ be a partially ordered set. A function $F : X \to Y$ is *monotone* if $x \leq_X y$ implies $F(x) \leq_Y F(y)$. Say that a set-valued map $\varphi : X \to 2^X$ is *monotone* if it is monotone when $\varphi(X)$ is ordered by the induced set order.

3. Results

Let $\langle X, \leq \rangle$ be a complete lattice and $F : X \to X$ be monotone. The set of fixed points of F is $\mathcal{E}(F) = \{x \in X : x = F(x)\}$.

Lemma 1. *$\langle \mathcal{E}(F), \leq \rangle$ has a smallest element.*

Proof. Let η be an ordinal number with cardinality greater than X, let $\xi = \eta + 1$. Define $f : \xi \to X$ by transfinite recursion as $f(0) = \bigwedge_X X$, and

$$f(\beta) = \bigvee_X \{F(f(\alpha)) : \alpha < \beta\}$$

for $\beta > 0$.

That $(\beta \prec \alpha) \Rightarrow (f(\beta) \leq f(\alpha))$ is immediate from the definition of f. Then, for all $\alpha \in \eta$, it follows that $f(\alpha + 1) = F(f(\alpha))$, as $f(\beta) \leq f(\alpha)$, for all $\beta < \alpha$, and F is monotone. Since η has cardinality greater than X, there is $\gamma \in \eta$ such that $f(\gamma) = f(\gamma + 1)$. Let $\underline{\gamma}$ be the smallest such γ; $\underline{\gamma}$ is well-defined because any set of ordinal numbers has a smallest element (see [3] for an informal introduction to ordinal numbers). Let $\underline{e} = f(\underline{\gamma})$. Then $\underline{e} = F(\underline{e})$. So $\underline{e} \in \mathcal{E}(F)$.

I shall prove that $\underline{e}$ is the smallest element in $\langle \mathcal{E}(F), \leq \rangle$. Let $e \in \mathcal{E}(F)$, and consider the proposition $P_\alpha : f(\alpha) \leq e$. Proposition P_α implies that $f(\alpha+1) = F(f(\alpha)) \leq F(e) = e$. By transfinite induction, then, $\underline{e} \leq e$. □

A version of Lemma 1 is also crucial in [1]'s proof of Tarski's fixed-point theorem. It was apparently first proved in [2]; my proof is more direct than the one in [2].

Theorem 2. *$\langle \mathcal{E}(F), \leq \rangle$ is a non-empty complete lattice.*

Proof. By Lemma 1, $\mathcal{E}(F)$ is nonempty. Let $E \subseteq \mathcal{E}(F)$ be nonempty. I shall find $\bigvee_{\mathcal{E}(F)} E$.

Let $x = \bigvee_X E$, and let $Y = \{z \in X : x \leq z\}$ be the set of upper bounds on E. If $z \in Y$, then, for all $e \in E$, $e \leq F(z)$, as $e = F(e) \leq F(z)$. Thus $F(Y) \subseteq Y$. Let $G = F|_Y$. Then G maps Y into Y, and G is monotone.

By Lemma 1, $\langle \mathcal{E}(G), \leq \rangle$ has a smallest element. By definition of G, this smallest element is $\bigvee_{\mathcal{E}(F)} E$.

The construction of $\bigwedge_{\mathcal{E}(F)} E$ is symmetric. □

My proof is constructive in the sense that it gives a procedure for finding a fixed point—and if E is a collection of fixed points, for finding $\bigvee_{\mathcal{E}(F)} E$ and $\bigwedge_{\mathcal{E}(F)} E$. The proof in [1] is constructive in this sense as well. As both proofs use ordinal numbers, there are notions of constructiveness that neither my proof or [1]'s would satisfy.

Let $\varphi : X \to 2^X$ be a set-valued map such that, for all $x \in X$, $\varphi(x)$ is a non-empty subcomplete sublattice of X. Suppose that φ is monotone. The set of fixed points of φ is $\mathcal{E}(\varphi) = \{x \in X : x \in \varphi(x)\}$.

Lemma 3. *$\langle \mathcal{E}(\varphi), \leq \rangle$ has a smallest element.*

Proof. Let $F(x) = \bigwedge_X \varphi(x)$. Note that, for all x, $F(x) \in \varphi(x)$, and that F is monotone. By Lemma 1, there is a smallest element, say $\underline{e}$ of $\langle \mathcal{E}(F), \leq \rangle$. Note that $\underline{e} = F(\underline{e}) \in \varphi(\underline{e})$, so $\underline{e} \in \mathcal{E}(\varphi)$.

I shall prove that $\underline{e}$ is the smallest element in $\langle \mathcal{E}(\varphi), \leq \rangle$. Let $e \in \mathcal{E}(\varphi)$. Let f be as in the proof of Lemma 1. Consider the proposition $P_\alpha : f(\alpha) \leq e$. Proposition P_α implies that $f(\alpha+1) = F(f(\alpha)) \leq F(e) \leq e$, as $F(e) = \bigwedge_X \varphi(e)$, and $e \in \varphi(e)$. By transfinite induction, then, $\underline{e} \leq e$. □

Theorem 4. *$\langle \mathcal{E}(\varphi), \leq \rangle$ is a non-empty complete lattice.*

Proof. Lemma 3 implies that $\mathcal{E}(\varphi)$ is nonempty. I shall prove that it is a complete lattice.

Let $E \subseteq \mathcal{E}(\varphi)$ be nonempty. I shall prove that $\bigvee_{\mathcal{E}(\varphi)} E$ exists.

Let $x = \bigvee_X E$, and let $Y = \{z \in X : x \leq z\}$. Define $\psi : Y \to 2^Y$ by $\psi(z) = Y \cap \varphi(z)$.

First, I show $\psi(z) \neq \emptyset$. Note that $x \leq z$ implies that, for all $e \in E$, there is $\hat{z}_e \in \varphi(z)$ with $e \leq \hat{z}_e$, as $e \in \varphi(e)$ and φ is monotone. But $\varphi(z)$ is subcomplete, and thus

$$x \leq \bigvee_{e \in E} \hat{z}_e \in \varphi(z),$$

so $\psi(z) \neq \emptyset$.

Second, I show that ψ is monotone. Let $z \leq z'$, and fix $y \in \psi(z)$ and $y' \in \psi(z')$. The mapping φ is monotone, so $y \wedge y' \in \varphi(z)$ and $y \vee y' \in \varphi(z')$. For $e \in E$, $e \in \varphi(e)$ implies $e \vee (y \wedge y') \in \varphi(z)$. But $y \wedge y' \in Y$, so $\varphi(z) \ni e \vee (y \wedge y') = y \wedge y'$. Similarly, $e \vee (y \vee y') \in \varphi(z')$ and $y \vee y' \in Y$ implies $y \vee y' \in \psi(z')$. Thus $y \wedge y' \in \psi(z)$ and $y \vee y' \in \psi(z')$.

Third, $\psi(z)$ is a subcomplete sublattice because $\psi(z) = Y \cap \varphi(z)$ and $\varphi(z)$ is a subcomplete sublattice.

Thus, ψ satisfies the hypothesis of Lemma 3. Let $e^* \in \mathcal{E}(\psi)$ be the smallest ψ-fixed point. If $e \in \mathcal{E}(\varphi)$ is an upper bound on E, then $e \in Y$ and thus $e \in \mathcal{E}(\psi)$. Then $e^* \leq e$. But $e^* \in \mathcal{E}(\varphi)$, so $e^* = \bigvee_{\mathcal{E}(\varphi)} E$.

The proof that $\bigwedge_{\mathcal{E}(\varphi)} E$ exists is symmetric. □

4. Continuous F

The proof of Tarski's Theorem is elementary when F is order-continuous, in addition to monotone.

First, the proof of Lemma 1 goes as follows: Let $\underline{x} = \bigwedge_X X$ be the smallest point in X, and let $\{x_n\}$ be the sequence of F-iterates from $\underline{x}$; so $x_n = F(x_{n-1})$ and $x_0 = \underline{x}$. Since F is monotone, $\{x_n\}$ is a monotone sequence, and thus converges to a point $\underline{e}$. The continuity of F implies that $\underline{e}$ is a fixed point, as $x_{2n+1} = F(x_{2n})$, and both $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to $\underline{e}$. Further, if e is any other fixed point of F, $\underline{x} \leq e$, and $x_n \leq e$ implies $x_{n+1} = F(x_n) \leq F(e) = e$. By induction, $\underline{e}$ is the smallest fixed point.

Second, Lemma 1 is used to prove Tarski's Theorem as in the proof of Theorem 2 above.

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